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ON SOME HERMITE–HADAMARD–FEJÉR INEQUALITIES FOR (k,h)–CONVEX FUNCTIONS

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Abstract. We introduce the class of (k,h)-convex functions defined on k-convex domains, and we prove some new inequalities of Hermite-Hadamard and Fejér type for such mappings. This generalizes results given for h-convex functions in [1, 17], and for s-Orlicz convex mappings in [4].

1. Introduction

Let $f : I \to \mathbb{R}$ be a convex function defined on a real interval I and fix $a,b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \cdot \int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as the Hermite-Hadamard inequality for convex functions (see [12] for the historical background). In [8] Fejér gave the important generalization of the inequality (1):

$$f\left(\frac{a+b}{2}\right) \cdot \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \cdot \int_a^b g(x)\,dx, \tag{2}$$

which holds if f is convex, and g is nonnegative and symmetric with respect to the point $(a+b)/2$. For various modifications of (1) and (2), see e.g. [5] and the references given there.

In the paper [18] by Varošanec, the so called h-convex functions were introduced with the following definition.

DEFINITION 1.1. Let I be a real interval and let $h : (0,1) \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. A nonnegative function $f : I \to \mathbb{R}$ is then called h-convex if, for all $x,y \in I$ and $t \in (0,1)$, we have

$$f\big(tx+(1-t)y\big) \leqslant h(t)f(x)+h(1-t)f(y). \tag{3}$$

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It is evident that this notion generalizes the concepts of classical convexity (for $h(t)=t$, see e.g. [9, 14]), s-Breckner convexity (for $h(t)=t^s$ with some $s\in(0,1)$, see [2, 7]), P-functions (for $h(t)=1$, see [13]) and Godunova-Levin functions (for $h(t)=t^{-1}$, see [6]).

In the recent paper [1] by Bombardelli and Varošanec, the following Hermite-Hadamard-Fejér inequalities for h-convex functions were obtained (the existence of integrals is assumed in both formulas).

PROPOSITION 1.2. *Let $f:[a,b]\to\mathbb{R}$ be h-convex and let $g:[a,b]\to\mathbb{R}$, $g\geqslant 0$ be symmetric with respect to $(a+b)/2$. Then*

$$\frac{1}{b-a}\cdot\int_a^b f(t)g(t)\,dt\leqslant[f(a)+f(b)]\cdot\int_0^1 h(t)\cdot g\big(ta+(1-t)b\big)\,dt. \tag{4}$$

PROPOSITION 1.3. *Let h be defined on $[0,\max\{1,b-a\}]$ and let $f\colon[a,b]\to\mathbb{R}$ be h-convex. Moreover, assume that $g:[a,b]\to\mathbb{R}$, $g\geqslant 0$ is symmetric with respect to $(a+b)/2$ and $\int_a^b g(t)\,dt>0$. Then*

$$f\left(\frac{a+b}{2}\right)\leqslant C\int_a^b f(t)g(t)\,dt, \tag{5}$$

where $C=\dfrac{2h(1/2)}{\int_a^b g(t)\,dt}$.

In [17] Sarikaya, Set and Özdemir proved another version of the Fejér inequality for h-convex functions.

PROPOSITION 1.4. *Let $f\colon[a,b]\to\mathbb{R}$ be h-convex and integrable, $h(1/2)>0$, and assume that $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric with respect to $(a+b)/2$. Then*

$$\begin{aligned}\frac{1}{2h(1/2)}\cdot f\left(\frac{a+b}{2}\right)\cdot\int_a^b g(x)\,dx&\leqslant\int_a^b f(x)g(x)\,dx\\&\leqslant\frac{f(a)+f(b)}{2}\cdot[h(t)+h(1-t)]\cdot\int_a^b g(x)\,dx\end{aligned} \tag{6}$$

for all $t\in(0,1)$.

In the most recent paper [10], Maksa and Palés introduced even more general notion of convexity. More precisely, (α,β,a,b)-convex functions are defined as solutions f of the functional inequality

$$f\big(\alpha(t)x+\beta(t)y\big)\leqslant a(t)f(x)+b(t)f(y), \tag{7}$$

where $\emptyset\neq T\subset[0,1]$ and $\alpha,\beta,a,b:T\to\mathbb{R}$ are given functions.

In our note we define and study the basic properties of (k,h)-convex functions with k-convex domains (see Definitions 2.1 and 2.4). Such mappings satisfy the inequality

(7) with $T=(0,1)$ and $\alpha(t)=k(t)$, $\beta(t)=k(1-t)$, $a(t)=h(t)$, $b(t)=h(1-t)$. In particular, we will see that (k,h)-convexity is a generalization of s-Orlicz convexity (see [3, 7]), subadditivity (see e.g. [11, 16]) and h-convexity.

Moreover, as our main results, we prove two inequalities of Hermite-Hadamard-Fejér type for (k,h)-convex functions (Theorems 3.1 and 3.5), and we apply them to various classes of mappings.

2. Preliminaries

Here we define the classes of k-convex sets and (k,h)-convex functions, and we discuss some properties of these concepts.

DEFINITION 2.1. Let $k:(0,1)\to\mathbb{R}$ be a given function. Then a subset D of a real linear space X will be called k-convex if $k(t)x+k(1-t)y\in D$ for all $x,y\in D$ and $t\in(0,1)$.

Let us point out that the definition given above, for conveniently chosen functions k, produces various families of well-known sets. This is shown, in part, by

EXAMPLE 2.2. 1. Our definition agrees with the one of classical convexity for $k(t)=t$.

2. If $k(t)=t^{1/p}$ with $p\in(0,1)$, then D is k-convex if and only if it is p-convex (see e.g. [15]).

3. For $s>0$ and $k(t)=t^{1/s}$, the family of k-convex sets is equal to the class of s-Orlicz convex sets, as defined by Dragomir and Fitzpatrick in [3].

4. If $k(t)=1$ for all t, then D is k-convex if and only if $(D,+)$ is a semigroup.

5. For $k(t)=1/2$, our definition generates the family of all midconvex subsets of X.

6. Let k be defined by the formula

$$k(t)=\begin{cases}2t & \text{for } t<1/2\\ 0 & \text{for } t\geqslant 1/2.\end{cases} \tag{8}$$

Then D is a k-convex set if and only if it is starshaped with respect to 0, i.e. $tx\in D$ for all $t\in[0,1]$ and $x\in D$. The proof of this fact is contained in Example 2.5.5.

Next, we present some basic facts on k-convex subsets of linear spaces.

REMARK 2.3. 1. Every linear subspace Y of X is k-convex in X. An affine subspace, however, may not be a k-convex set.

2. If $k(t)\geqslant 0$ for all t, then every pointed convex cone $K\subset X$, i.e. a set which is closed under linear combinations with nonnegative coefficients, is k-convex.

3. For any pair of k-convex sets $C,D\subset X$ and for every $\alpha\in\mathbb{R}$, the sets $C+D$ and αD are also k-convex.

4. If $\{D_\alpha\}_{\alpha\in A}$ is a family of k-convex sets, then their intersection $\bigcap_{\alpha\in A} D_\alpha$ is also k-convex.

5. If all sets $D_1 \subset D_2 \subset D_3 \subset \dots$ are k-convex, then their union $\bigcup_{n\in\mathbb{N}} D_\alpha$ is also k-convex.

6. Assume that X is a metric linear space and $D \subset X$ is a k-convex set. Then its closure cl D is also k-convex.

Now we are ready to give a definition of (k,h)-convexity, which will be essential in the next section.

DEFINITION 2.4. Let $k,h:(0,1)\to\mathbb{R}$ be two given functions and suppose that $D\subset X$ is a k-convex set. Then a function $f:D\to\mathbb{R}$ is (k,h)-convex if, for all $x,y\in D$ and $t\in(0,1)$,

$$f\big(k(t)x+k(1-t)y\big) \leqslant h(t)f(x)+h(1-t)f(y). \tag{9}$$

If (9) can be replaced with the corresponding equality, f will be called (k,h)-affine (more general functions of this type are subject of the paper [10]).

Again, this definition coincides with the previously introduced terminology in many important cases, some of which are listed below.

EXAMPLE 2.5. 1. For $k(t)=t$, the notion of (k,h)-convexity agrees with the one of h-convexity, given by (3) (without the additional assumption of nonnegativity).

In particular, for suitable functions h, the condition (9) produces the families of convex functions, s-Breckner convex functions, P-functions and Godunova-Levin functions.

2. If $s>0$, $k(t)=t^{1/s}$ and $h(t)=t$, then f is (k,h)-convex if and only if it is s-Orlicz convex.

3. For $k(t)=h(t)=1$, the class of (k,h)-convex functions consists of all mappings which are subadditive.

4. If $k(t)=h(t)=1/2$ for all t, then (9) produces the family of Jensen-convex functions.

5. Let k be given by (8). Then f is a (k,k)-convex function if and only if it is starshaped, i.e. $f(tx)\leqslant tf(x)$ for all $t\in[0,1]$ and $x\in D$.

To see this, fix $x,y\in D$ and choose $t\in(0,1)$. Then, assuming that f is (k,k)-convex, we get

$$f(tx)=f\big(k(t/2)x+k(1-t/2)x\big)\leqslant k(t/2)f(x)+k(1-t/2)f(x)=tf(x)$$

and

$$f(0)=f\big(k(1/2)x+k(1/2)x\big)\leqslant k(1/2)f(x)+k(1/2)f(x)=0.$$

On the other hand, if f is starshaped, we obtain

$$f\big(k(t)x+k(1-t)y\big)=\begin{cases} f(2t\cdot x)\leqslant 2t\cdot f(x) & \text{for } t\in(0,1/2),\\ f(0)\leqslant 0 & \text{for } t=1/2,\\ f\big((2-2t)\cdot y\big)\leqslant(2-2t)\cdot f(y) & \text{for } t\in(1/2,1),\end{cases}$$

and so (9) holds for all t, with $h=k$.

Many of the well-known properties of convex functions may be similarly applied to (k,h)-convex mappings. In particular, we have

REMARK 2.6. 1. If $f,g:D\to\mathbb{R}$ are (k,h)-convex functions and $c\geqslant 0$, then $f+g$, cf are also (k,h)-convex.

2. Suppose that $h\geqslant 0$ and let $\{f_i\}_{i\in I}$ be a family of (k,h)-convex functions defined on D. Then it is easy to check that the function $f=\sup\limits_{i\in I} f_i$ also satisfies (9) for all x,y and t.

3. Let f be a (k,h)-convex function with $h(t)=t$, and define the sublevel set $f^c=\{x\in D: f(x)\leqslant c\}$. Then f^c is a k-convex set for every $c\in\mathbb{R}$.

Indeed, for $x,y\in f^c$ and $t\in(0,1)$ we get

$$f\big(k(t)x+k(1-t)y\big)\leqslant t\cdot f(x)+(1-t)\cdot f(y)\leqslant tc+(1-t)c=c.$$

4. If f is a (k,k)-convex function with $k\geqslant 0$, then the epigraph of f, i.e. the set $\operatorname{epi} f=\big\{(x,y)\in X\times\mathbb{R}: x\in D,\, y\geqslant f(x)\big\}$, is k-convex.

This follows from the inequality

$$f\big(k(t)x_1+k(1-t)x_2\big)\leqslant k(t)\cdot f(x_1)+k(1-t)\cdot f(x_2)\leqslant k(t)y_1+k(1-t)y_2,$$

valid for $(x_1,y_1),(x_2,y_2)\in\operatorname{epi} f$ and $t\in(0,1)$.

5. Suppose that the epigraph of f is k-convex. Then f is a (k,k)-convex function.

Indeed, since $P_1=(x,f(x))$ and $P_2=(y,f(y))$ are elements of the epigraph, we have $k(t)\cdot P_1+k(1-t)\cdot P_2\in\operatorname{epi} f$, which gives

$$f\big(k(t)x+k(1-t)y\big)\leqslant k(t)f(x)+k(1-t)f(y).$$

6. If D is a k-convex subset of X and $f:D\to\mathbb{R}$ is a (k,h)-affine function, then an easy verification shows that the image $f(D)$ of f is h-convex in $\mathbb{R}$.

7. Assume that $f_1:D_1\to\mathbb{R}$ is (k,h)-convex, $f_2:D_2\to\mathbb{R}$ is (h,h)-convex and nondecreasing, and $f_1(D_1)\subset D_2$. Then $f=f_2\circ f_1$ is a (k,h)-convex function.

Finally, let us observe that every nonnegative and (k,h_1)-convex function is also (k,h_2)-convex for all $h_2\geqslant h_1$.

3. Main Results

In this section we prove some new inequalities of Hermite-Hadamard and Fejér types for (k,h)-convex functions. From now on, we suppose that D is a k-convex subset of $\mathbb{R}$ and that all integrals considered below exist.

THEOREM 3.1. (The first Fejér inequality for (k,h)-convex functions) *Let $f: D \to \mathbb{R}$ be a (k,h)-convex function with $h(1/2) > 0$, fix $a < b$ such that $[a,b] \subset D$ and let $g: [a,b] \to \mathbb{R}$ be a nonnegative function which is symmetric with respect to $(a+b)/2$. Then*

$$\frac{f\big(k(1/2)\cdot(a+b)\big)}{2\cdot h(1/2)}\cdot\int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx. \tag{10}$$

Proof. Writing (9) with $t = 1/2$, $x = wa + (1-w)b$ and $y = (1-w)a + wb$, we get

$$\begin{aligned} f\big(k(1/2)\cdot(a+b)\big) &= f\big(k(1/2)x + k(1/2)y\big) \\ &\leqslant h(1/2)\cdot\big[f\big(wa+(1-w)b\big) + f\big((1-w)a+wb\big)\big]. \end{aligned} \tag{11}$$

We may now multiply both sides of (11) by $g(x) = g(y)$, and then integrate it with respect to w, getting

$$\begin{aligned} &f\big(k(1/2)\cdot(a+b)\big)\cdot\int_0^1 g\big(wa+(1-w)b\big)\,dw \\ &\quad\leqslant h(1/2)\cdot\Big[\int_0^1 f\big(wa+(1-w)b\big)\cdot g\big(wa+(1-w)b\big)\,dw \\ &\qquad + \int_0^1 f\big((1-w)a+wb\big)\cdot g\big((1-w)a+wb\big)\,dw\Big]. \end{aligned}$$

This implies

$$f\big(k(1/2)\cdot(a+b)\big)\cdot\frac{1}{b-a}\cdot\int_a^b g(x)dx \leqslant h(1/2)\cdot 2\cdot\frac{1}{b-a}\cdot\int_a^b f(x)g(x)\,dx,$$

and (10) follows. □

If we assume that $g(t) = 1$ for all $t \in (0,1)$, from (10) we obtain the first inequality of Hermite-Hadamard type for (k,h)-convex functions.

COROLLARY 3.2. *Let $f: D \to \mathbb{R}$ be a (k,h)-convex function with $h(1/2) > 0$ and choose $a < b$ such that $[a,b] \subset D$. Then*

$$\frac{f\big(k(1/2)\cdot(a+b)\big)}{2\cdot h(1/2)} \leqslant \frac{1}{b-a}\cdot\int_a^b f(x)\,dx. \tag{12}$$

Moreover, writing (10) with $k(t) = t^{1/s}$ and $h(t) = t$, we get

COROLLARY 3.3. *Suppose that $f : D \to \mathbb{R}$ is an s-Orlicz convex function and that a, b, g satisfy the assumptions of Theorem 3.1. Then*

$$f\left(\frac{a+b}{2^{1/s}}\right) \cdot \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx. \tag{13}$$

REMARK 3.4. 1. If we apply (10) to an h-convex function f, we obtain (5), which is also the left-hand side of (6).

2. The condition (13) for $g = 1$ gives the inequality

$$f\left(\frac{a+b}{2^{1/s}}\right) \leqslant \frac{1}{b-a} \cdot \int_a^b f(x)\,dx,$$

which was proved in [4].

3. By Theorem 3.1, for every subadditive function f the following inequality of Fejér type is valid:

$$\frac{f(a+b)}{2} \cdot \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx.$$

In particular, for $g = 1$ we get the Hermite-Hadamard inequality

$$\frac{f(a+b)}{2} \leqslant \frac{1}{b-a} \cdot \int_a^b f(x)\,dx.$$

4. For Jensen-convex functions, from (10) and (12) we recover the left-hand sides of the classical inequalities (2) and (1), respectively.

THEOREM 3.5. (The second Fejér inequality for (k,h)-convex functions) *Assume that $f : D \to \mathbb{R}$ is a (k,h)-convex function with $h(1/2) > 0$, $a, b \in D$, $a < b$ and $g : [a,b] \to \mathbb{R}$ is a nonnegative function, symmetric with respect to $(a+b)/2$. Then*

$$\begin{aligned}
\frac{1}{2h(1/2)} \cdot \int_0^1 & f\big(k(1/2)\cdot[k(t)+k(1-t)]\cdot(a+b)\big)\cdot g\big(ta+(1-t)b\big)\,dt \\
&\leqslant \int_0^1 f\big(k(t)a+k(1-t)b\big)\cdot g\big(ta+(1-t)b\big)\,dt \\
&\leqslant [f(a)+f(b)]\cdot \int_0^1 h(t)\cdot g\big(ta+(1-t)b\big)\,dt.
\end{aligned} \tag{14}$$

Proof. By (9) with $x = k(w)a + k(1-w)b$, $y = k(1-w)a + k(w)b$ and $t = 1/2$, we have

$$\begin{aligned}
f\big(k(1/2)\cdot[k(w)+k(1-w)]\cdot(a+b)\big) &= f\big(k(1/2)x + k(1/2)y\big) \\
&\leqslant h(1/2)\cdot\big[f\big(k(w)a+k(1-w)b\big) + f\big(k(1-w)a+k(w)b\big)\big].
\end{aligned} \tag{15}$$

As in the proof of the previous theorem, we multiply both sides of (15) by $g\big(wa+(1-w)b\big)=g\big((1-w)a+wb\big)$, and we integrate the new inequality over $(0,1)$, getting

$$\begin{aligned}\int_0^1 f\big(k(1/2)\cdot[k(w)+k(1-w)]\cdot(a+b)\big)\cdot g\big(wa+(1-w)b\big)\,dw \\ \leqslant h(1/2)\cdot\Big[\int_0^1 f\big(k(w)a+k(1-w)b\big)\cdot g\big(wa+(1-w)b\big)\,dw \\ +\int_0^1 f\big(k(1-w)a+k(w)b\big)\cdot g\big((1-w)a+wb\big)\,dw\Big] \\ =2h(1/2)\cdot\int_0^1 f\big(k(t)a+k(1-t)b\big)\cdot g\big(ta+(1-t)b\big)\,dt.\end{aligned}$$

From this we obtain the first desired inequality.

To prove the second one, we need to use the definition of (k,h)-convexity with $x=a$ and $y=b$. Namely, we have

$$f\big(k(t)a+k(1-t)b\big)\leqslant h(t)f(a)+h(1-t)f(b),$$

which, by symmetry of g, implies

$$\begin{aligned}\int_0^1 f\big(k(t)a+k(1-t)b\big)\cdot g\big(ta+(1-t)b\big)\,dt \\ \leqslant f(a)\int_0^1 h(t)\cdot g\big(ta+(1-t)b\big)\,dt+f(b)\int_0^1 h(1-t)\cdot g\big((1-t)a+tb\big)\,dt \\ =[f(a)+f(b)]\cdot\int_0^1 h(t)\cdot g\big(ta+(1-t)b\big)\,dt,\end{aligned}$$

and the proof is complete. □

As a corollary, we obtain the second Hermite-Hadamard inequality for (k,h)-convex functions.

COROLLARY 3.6. *Let $f:D\to\mathbb{R}$ be a (k,h)-convex function, $h(1/2)>0$ and choose $a,b\in D$ such that $a<b$. Then*

$$\begin{aligned}\frac{1}{2h(1/2)}\cdot\int_0^1 f\big(k(1/2)\cdot[k(t)+k(1-t)]\cdot(a+b)\big)\,dt \\ \leqslant\int_0^1 f\big(k(t)a+k(1-t)b\big)\,dt\leqslant[f(a)+f(b)]\cdot\int_0^1 h(t)\,dt.\end{aligned}\tag{16}$$

We also get the following version of the Fejér inequality for s-Orlicz convex functions.

COROLLARY 3.7. *Suppose that $f:D\to\mathbb{R}$ is an s-Orlicz-convex function and* *that a,b,g satisfy the assumptions of Theorem 3.5. Then*

$$\begin{aligned}\int_0^1 f\left(\frac{1}{2^{1/s}}\cdot\left[t^{1/s}+(1-t)^{1/s}\right]\cdot(a+b)\right)\cdot g\big(ta+(1-t)b\big)\,dt\\ \leqslant \int_0^1 f\big(t^{1/s}a+(1-t)^{1/s}b\big)\cdot g\big(ta+(1-t)b\big)\,dt\\ \leqslant [f(a)+f(b)]\cdot\int_0^1 t\cdot g\big(ta+(1-t)b\big)\,dt. \qquad (17)\end{aligned}$$

REMARK 3.8. 1. Applying (14) to an h-convex function f, we obtain the inequalities (4) and (5).

2. If f is an s-Orlicz convex function and $g=1$, then (17) becomes

$$\begin{aligned}\int_0^1 f\left(\frac{1}{2^{1/s}}\cdot\left[t^{1/s}+(1-t)^{1/s}\right]\cdot(a+b)\right)dt\\ \leqslant \int_0^1 f\big(t^{1/s}a+(1-t)^{1/s}b\big)\,dt \leqslant \frac{f(a)+f(b)}{2},\end{aligned}$$

and thus we recover another result from [4].

3. If f is starshaped and $a,b\neq 0$, then the right-hand side of (16) has the form

$$\frac{1}{a}\cdot\int_0^a f(t)\,dt+\frac{1}{b}\cdot\int_0^b f(t)\,dt\leqslant\frac{f(a)+f(b)}{2},$$

which can also be derived from [5, Theorem 196] with $m=0$.

4. For convex functions, from (16) and (14) we get the classical inequalities (1) and (2), respectively.

REFERENCES

[1] M. BOMBARDELLI AND S. VAROŠANEC, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl. **58**, 9 (2009), 1869–1877.

[2] W. W. BRECKNER, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. (Beograd) **23(37)** (1978), 13–20.

[3] S. S. DRAGOMIR AND S. FITZPATRICK, *s-Orlicz convex functions in linear spaces and Jensen's discrete inequality*, J. Math. Anal. Appl. **210**, 2 (1997), 419–439.

[4] S. S. DRAGOMIR AND S. FITZPATRICK, *Hadamard's inequality for s-convex functions in the first sense and applications*, Demonstratio Math. **31**, 3 (1998), 633–642.

[5] S. S. DRAGOMIR AND C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
(Online: `http://rgmia.vu.edu.au/monographs/`)

[6] E. K. GODUNOVA AND V. I. LEVIN, *Neravenstwa dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkcii*, in: Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, 138–142.

[7] H. HUDZIK AND L. MALIGRANDA, *Some remarks on s-convex functions*, Aequationes Math. **48**, 1 (1994), 100–111.

[8] L. FEJÉR, *Über die Fourierreihen, II*, Math. Naturwiss. Anz. Ungar. Akad. Wiss. **24** (1906), 369–390.

[9] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality. Second edition*, Birkhäuser Verlag, Basel, 2009.
[10] G. MAKSA, ZS. PÁLES, *The equality case in some recent convexity inequalities*, Opuscula Math. **31**, 2 (2011), 269–277.
[11] J. MATKOWSKI AND T. ŚWIĄTKOWSKI, *On subadditive functions*, Proc. Amer. Math. Soc. **119** (1993), 187–197.
[12] D. S. MITRINOVIĆ AND I. B. LACKOVIĆ, *Hermite and convexity*, Aequationes Math. **28**, 3 (1985), 229–232.
[13] C. E. M. PEARCE AND A. M. RUBINOV, *P-functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240**, 1 (1999), 92–104.
[14] A. W. ROBERTS AND D. E. VARBERG, *Convex Functions*, Pure and Applied Mathematics, vol. 57, Academic Press, New York–London, 1973.
[15] S. ROLEWICZ, *Metric Linear Spaces. Second edition*, PWN, Warszawa, 1984.
[16] R. A. ROSENBAUM, *Sub-additive functions*, Duke Math J. **17** (1950), 227–247.
[17] M. Z. SARIKAYA, E. SET AND M. E. ÖZDEMIR, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Math. Univ. Comenian. **79**, 2 (2010), 265–272.
[18] S. VAROŠANEC, *On h-convexity*, J. Math. Anal. Appl. **326**, 1 (2007), 303–311.

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