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INEQUALITIES FOR WEIGHTED SUMS OF POWERS AND THEIR APPLICATIONS

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Abstract. Two inequalities for weighted sums of powers are established. Applications to Jacobian elliptic functions and Legendre's elliptic integrals of the first kind are presented. Some known and new inequalities for circular and hyperbolic functions are obtained. Applications to certain iterative means including Gauss' arithmetic-geometric mean and Schwab-Borchardt mean are included.

1. Introduction

In recent years several researchers have obtained new inequalities for the trigonometric and hyperbolic functions. See [5], [6], [10], [16], [17] and the references therein. Also, there is an interest in the study of inequalities involving other classes of elementary functions and the the higher transcendental functions as well. In this paper we demonstrate that some inequalities involving Jacobian elliptic functions, Legendre's elliptic integrals of the first kind, circular and hyperbolic functions, Gauss' arithmetic-geometric mean and Schwab-Borchardt mean all follow from two inequalities proven in this paper.

This paper is a continuation of the author's earlier investigations reported in [5, 6, 7, 10], and is organized as follows. Assumptions and lemmas are given in Section 2. Two inequalities (3.1) and (3.8) for weighted sums of powers of two numbers are established in Section 3. Applications of these results to inequalities for functions and means mentioned in the previous paragraph are given in Section 4.

Research presented in this paper was partially inspired by the following result of S.-H. Wu and H.M. Srivastava [16]:

THEOREM 1.1. *Let* $0 < x < \frac{\pi}{2}, \lambda > 0, \mu > 0$ *and* $p \leqslant \frac{2q\mu}{\lambda}$. *Then for* $q > 0$ *or* $q \leqslant min\left\{-\frac{\lambda}{\mu}, -1\right\}$, *the following inequality*

$$1 < \frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q$$

holds true.

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2. Preliminaries

In what follows the letters u and v will stand for two positive numbers which satisfy the following conditions

$$\min(u,v) < 1 < \max(u,v), \tag{2.1}$$

$$1 < u^{\alpha} v^{\beta}, \tag{2.2}$$

and

$$1 < \frac{\alpha}{\alpha+\beta}\frac{1}{u} + \frac{\beta}{\alpha+\beta}\frac{1}{v}, \tag{2.3}$$

where the last two inequalities must be satisfied for some positive numbers α and β.

With $w_1 = \alpha/(\alpha+\beta)$ and $w_2 = \beta/(\alpha+\beta)$ one can easily verify that the conditions (2.2) and (2.3) can be combined and written as a two-sided inequality

$$H(w_1,w_2;u,v) < 1 < G(w_1,w_2;u,v),$$

where H and G are the weighted harmonic and geometric means, respectively, of u and v.

For later use let us record two results.

LEMMA 2.1. ([10]) *Let r and s be positive unequal numbers. If $1 < rs$, then*

$$\frac{1}{r} + \frac{1}{s} < r + s.$$

LEMMA 2.2. ([6]) *Let r and s be the same as in Lemma 2.1 and let the positive number γ and δ satisfy $\gamma + \delta = 1$. If*

$$1 < \gamma\frac{1}{r} + \delta\frac{1}{s} < \gamma r + \delta s, \tag{2.4}$$

then for $p \geqslant 1$

$$1 < \gamma\frac{1}{r^p} + \delta\frac{1}{s^p} < \gamma r^p + \delta s^p. \tag{2.5}$$

The second inequality in (2.5) holds true if $p > 0$.

3. Inequalities for Weighted Sums of Powers

The goal of this section is to establish two inequalities which involve two positive numbers u and v. In what follows we will assume that they satisfy conditions (2.1)–(2.3).

Our first result reads as follows.

THEOREM 3.1. *Let $\lambda > 0$ and $\mu > 0$. If $u < 1 < v$, then*

$$1 < \frac{\lambda}{\lambda+\mu}u^p + \frac{\mu}{\lambda+\mu}v^q \tag{3.1}$$

if either

$$q > 0 \qquad \textit{and} \qquad p \leqslant q\frac{\alpha\mu}{\beta\lambda} \tag{3.2}$$

or if

$$p \leqslant q \leqslant -1 \qquad \textit{and} \qquad \beta\lambda \geqslant \alpha\mu. \tag{3.3}$$

If $v < 1 < u$*, then the inequality* (3.1) *holds true if either*

$$p > 0 \qquad \textit{and} \qquad q \leqslant p\frac{\beta\lambda}{\alpha\mu} \tag{3.4}$$

or if

$$q \leqslant p \leqslant -1 \qquad \textit{and} \qquad \alpha\mu \geqslant \beta\lambda. \tag{3.5}$$

Proof. We shall establish (3.1) only when $u < 1 < v$. Let

$$d = \frac{\lambda}{\lambda+\mu}u^p + \frac{\mu}{\lambda+\mu}v^q.$$

Application of the inequality of weighted arithmetic and geometric means gives

$$d^{\lambda+\mu} \geqslant u^{p\lambda}v^{q\mu}. \tag{3.6}$$

It follows from (2.2) that $v > u^{-\alpha/\beta}$. Assume that $q > 0$. Then $v^{q\mu} > u^{-q\mu\alpha/\beta}$. This and (3.6) give

$$d^{\lambda+\mu} > u^{p\lambda}u^{-q\alpha\mu/\beta} = u^{p\lambda-q\alpha\mu/\beta} \geqslant 1,$$

where the last inequality follows from $0 < u < 1$ and the second condition in (3.2).

We shall establish now the inequality (3.1) when conditions (3.3) are satisfied. Since $0 < u < 1$ and $p \leqslant q < 0$, $u^p \geqslant u^q$. Again, let d stand for the right side of (3.1). Then

$$d \geqslant \frac{\lambda}{\lambda+\mu}u^q + \frac{\mu}{\lambda+\mu}v^q.$$

Let us write this inequality in the form

$$d \geqslant \frac{\lambda}{\lambda+\mu}\left(\frac{1}{u}\right)^{-q} + \frac{\mu}{\lambda+\mu}\left(\frac{1}{v}\right)^{-q}. \tag{3.7}$$

It follows from (2.3) that

$$\frac{1}{u} > 1 + \frac{\beta}{\alpha}\left(1 - \frac{1}{v}\right) = 1 + \frac{\beta}{\alpha}(1-a),$$

where $a = 1/v$. Thus $0 < a < 1$. Since $-q > 0$,

$$\left(\frac{1}{u}\right)^{-q} > \left(1 + \frac{\beta}{\alpha}(1-a)\right)^{-q}.$$

This in conjunction with (3.7) yields

$$d > \frac{\lambda}{\lambda+\mu}\left(1+\frac{\beta}{\alpha}(1-a)\right)^{-q} + \frac{\mu}{\lambda+\mu}a^{-q} =: f(a).$$

Differentiation of $f(a)$ gives

$$f'(a) = \frac{q\mu}{\lambda+\mu}\left(1+\frac{\beta}{\alpha}(1-a)\right)^{-q-1} a^{-q-1}\left[\frac{\beta\lambda}{\alpha\mu}a^{q+1} - \left(1+\frac{\beta}{\alpha}(1-a)\right)^{q+1}\right] \leqslant 0,$$

where the last inequality follows from

$$\frac{\beta\lambda}{\alpha\mu}a^{q+1} \geqslant 1 \geqslant \left(1+\frac{\beta}{\alpha}(1-a)\right)^{q+1}.$$

Thus the function $f(a)$ is decreasing on the interval $(0,1)$. This in conjunction with $\lim_{a\to 1^-} f(a) = 1$ yields $d > 1$. This completes the proof when $u < 1 < v$. The assertion when $v < 1 < u$ can be established in a similar way. We omit further details. The proof is complete. □

We shall now prove the following.

THEOREM 3.2. *Let* $\alpha \geqslant 1$ *and* $\beta \geqslant 1$. *Then for* $p \geqslant 1$

$$2 < \left(\frac{1}{u}\right)^{\alpha p} + \left(\frac{1}{v}\right)^{\beta p} < u^{\alpha p} + v^{\beta p}. \tag{3.8}$$

The second inequality in (3.8) *holds true for* $p > 0$.

Proof. First we shall prove (3.8) when $p = 1$. Assume that $v < 1 < u$. Then (2.3) can be written as

$$\frac{1}{v} > 1 + \frac{\alpha}{\beta}\left(1 - \frac{1}{u}\right).$$

This in turn implies that

$$\begin{aligned}\left(\frac{1}{u}\right)^{\alpha} + \left(\frac{1}{v}\right)^{\beta} &> \left(\frac{1}{u}\right)^{\alpha} + \left(1+\frac{\alpha}{\beta}(1-a)\right)^{\beta} \\ &= a^{\alpha} + \left(1+\frac{\alpha}{\beta}(1-a)\right)^{\beta} =: f(a),\end{aligned}$$

where $a = 1/u$. Thus $0 < a < 1$. We shall demonstrate now that $f(a) > 2$ holds for all $a \in (0,1)$. Differentiation of $f(a)$ with respect to a gives

$$f'(a) = \alpha\left[a^{\alpha-1} - \left(1+\frac{\alpha}{\beta}(1-a)\right)^{\beta-1}\right].$$

Since $0<a<1<1+(\alpha/\beta)(1-a)$ and $\alpha,\beta\geqslant 1$, $f'(a)<0$ for $0<a<1$. Taking into account that $\lim_{a\to 1^-} f(a)=2$, we conclude that $f(a)>2$. This completes the proof of the first inequality in (3.8). In order to establish the second one we apply Lemma 2.1 with $r=u^\alpha$ and $s=v^\beta$. In order to establish (3.8) for values of p as stated in Theorem 3.2 we apply Lemma 2.2 with $\gamma=\delta=1/2$ and r and s as defined at the end of the proof when $p=1$. The case $u<1<v$ can be established in an analogous manner. This completes the proof. $\square$

4. Applications

In this section we present several applications of two theorems established in the previous section. We begin with

4.1. Applications to Jacobian elliptic functions

Let $0<k<1$ be the modulus of Legendre's complete elliptic integral of the first kind

$$K=\int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}. \tag{4.1}$$

In what follows we will assume that the variable x satisfies $0<|x|\leqslant K$. The Jacobian version of Legendre's incomplete elliptic integral of the first kind is

$$x=\int_0^{sn(x)} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \tag{4.2}$$

where $sn(x)\equiv sn(x,k)\equiv sn$ is one of the twelve Jacobian elliptic functions (see, e.g., [11], [4], [14]). Other Jacobian elliptic functions used in this paper are

$$sc=\frac{sn}{cn}, \qquad sd=\frac{sn}{dn},$$

where cn and dn are subordinate functions of sn and they satisfy fundamental identities

$$sn^2+cn^2=1, \qquad k^2sn^2+dn^2=1 \tag{4.3}$$

(see, e.g., [11, Ch. 22]).

For later use we define

$$u=\frac{sn(x)}{x}, \qquad v=\frac{sc(x)}{x\,dn(x)}. \tag{4.4}$$

It is known that $0<u<1<v$.

We need the following.

PROPOSITION 4.1. *Let u and v be defined in (4.4). Then*

$$1<u^2v \tag{4.5}$$

and

$$1 < \frac{2}{3}\frac{1}{u} + \frac{1}{3}\frac{1}{v}. \tag{4.6}$$

Proof. Inequality (4.5) is established in [5, (3.12)]. For the proof of (4.6) it suffices to show that the second inequality in

$$3 < \frac{x}{sn(x)} + \frac{x}{sc(x)} + \frac{x}{sd(x)} < 2\frac{x}{sn(x)} + \frac{x\,dn(x)}{sc(x)} \tag{4.7}$$

holds true where the first one is established in [5, (3.11)]. The second inequality in (4.7) can be written as

$$\frac{x}{sn(x)}\left(1 + cn(x) + dn(x)\right) < \frac{x}{sn(x)}\left(2 + cn(x)dn(x)\right). \tag{4.8}$$

Thus in order to obtain the second inequality in (4.7) we have to show that

$$cn(x) + dn(x) < 1 + cn(x)dn(x). \tag{4.9}$$

Using (4.3) we obtain

$$(cn(x) + dn(x))^2 = 2 - (1+k^2)sn^2(x) + 2cn(x)dn(x)$$

and also that

$$(1 + cn(x)dn(x))^2 = 2 - (1+k^2)sn^2(x) + 2cn(x)dn(x) + k^2 sn^4(x).$$

Comparing the right sides of the last two expressions we see that

$$(cn(x) + dn(x))^2 < (1 + cn(x)dn(x))^2.$$

Since $cn(x) + dn(x) > 0$ and $1 + cn(x)dn(x) > 0$ for $0 \leqslant |x| \leqslant K$, the inequality (4.9) follows. This in turn implies the validity of the inequality (4.8). Hence the assertion follows. The proof is complete. □

With u and v as defined in (4.4) we see, using Proposition 4.1, Theorem 3.1 and Theorem 3.2, that the following results are valid.

COROLLARY 4.2. *Let $\lambda > 0$ and $\mu > 0$. If either $q > 0$ and $p \leqslant 2q\frac{\mu}{\lambda}$ or if $p \leqslant q \leqslant -1$ and $\lambda \geqslant 2\mu$, then*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{sn(x)}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{sc(x)}{x\,dn(x)}\right)^q. \tag{4.10}$$

If $p \geqslant 1$, then

$$2 < \left(\frac{x}{sn(x)}\right)^{2p} + \left(\frac{x\,dn(x)}{sc(x)}\right)^p < \left(\frac{sn(x)}{x}\right)^{2p} + \left(\frac{sc(x)}{x\,dn(x)}\right)^p, \tag{4.11}$$

where the second inequality in (4.11) is valid provided $p > 0$.

The second inequality in (4.11) has been established in [5, Theorem 3.3].

4.2. Applications to Legendre's elliptic integral F

Let $0 < \phi < \pi/2$ and let $0 < k < 1$. Legendre's incomplete elliptic integral of the first kind is defined by

$$F(\phi,k) = \int_0^{\sin\phi} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}.$$

It is well known that with $x = F(\phi,k)$ one has $sn(x) = \sin\phi$, $cn(x) = \cos\phi$, $dn(x) = \sqrt{1-k^2 sin^2\phi} =: \Delta$, $sc(x) = \tan\phi$ and $sd(x) = \frac{\sin\phi}{\Delta}$.

Writing for brevity $F \equiv F(\phi,k)$ we obtain the following using (4.10) and (4.11).

COROLLARY 4.3. *With the assumptions for* p,q,λ *and* μ *as in Corollary 4.2 the following inequalities*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{\sin\phi}{F}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{\tan\phi}{\Delta F}\right)^q \tag{4.12}$$

and

$$2 < \left(\frac{F}{\sin\phi}\right)^{2p} + \left(\frac{\Delta F}{\tan\phi}\right)^p < \left(\frac{\sin\phi}{F}\right)^{2p} + \left(\frac{\tan\phi}{\Delta F}\right)^p \tag{4.13}$$

are valid.

4.3. Applications to circular and hyperbolic functions

Some known inequalities involving either circular or hyperbolic functions can be obtained from (4.10) and (4.11). Letting $k \to 0^+$ we have $sn \to \sin$, $sc \to \tan$, $dn \to 1$ and $K \to \pi/2$. This yields the following.

COROLLARY 4.4. *With the assumptions for* p,q,λ *and* μ *as in Corollary 4.2 one has*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^q \tag{4.14}$$

and

$$2 < \left(\frac{x}{\sin x}\right)^{2p} + \left(\frac{x}{\tan x}\right)^p < \left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^p \tag{4.15}$$

($0 < |x| \leqslant \pi/2$*).*

Letting in (4.14) $p = 2$, $q = 1$, $\lambda = \mu = 1$ we obtain Wilker's inequality [15]. The same inequality becomes Huygen's inequality [3] when $p = q = 1$, $\lambda = 2$ and $\mu = 1$. Inequalities (4.15) appear in [5], [10], [17]. For related results the interested reader is referred to [16], [6] and the references therein.

The counterparts of inequalities (4.14) and (4.15) for the hyperbolic functions also follow from (4.10) and (4.11). Taking into account that $k \to 1^-$, implies $sn \to \tanh$, $sc \to \sinh$, $dn \to \text{sech}$ (see, e.g., [11, Ch. 22] we obtain the desired results. We omit further details.

The remaining two subsections of this section deal with bivariate iterative means of two positive numbers x and y ($x \neq y$).

4.4. Applications to Gauss' arithmetic-geometric mean

Throughout the sequel Gauss' arithmetic-geometric mean of x and y will be denoted by $AGM(x,y)$ or for brevity, by AGM. It is an iterative mean, i.e.,

$$AGM = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where

$$x_0 = x, \;\; y_0 = y, \;\; x_{n+1} = \frac{x_n + y_n}{2}, \;\; y_{n+1} = \sqrt{x_n y_n}$$

($n \geqslant 0$). See, e.g., [1], [2]. It is symmetric and homogeneous of degree 1 in its variables. It is well-known that

$$(xy)^{1/2} < AGM < \frac{x+y}{2}.$$

Assume that $y < x$. Letting $u = AGM/x$ and $v = AGM/y$ we see that $u < 1 < v$. Moreover, numbers u and v satisfy conditions (2.2) and (2.3) with $\alpha = \beta = 1$. The following result is an immediate consequence of Theorem 3.1 and Theorem 3.2.

COROLLARY 4.5. *Assume that $\lambda > 0$ and $\mu > 0$ and let $y < x$. If either $q > 0$ and $p \leqslant q\mu/\lambda$ or if $p \leqslant q \leqslant -1$ and $\lambda \geqslant \mu$, then*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{AGM}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{AGM}{y}\right)^q. \tag{4.16}$$

If $x < y$, then the inequality (4.16) holds true if either $p > 0$ and $q \leqslant p\lambda/\mu$ or if $q \leqslant p \leqslant -1$ and $\mu \geqslant \lambda$.

If $p \geqslant 1$, then

$$2 < \left(\frac{x}{AGM}\right)^p + \left(\frac{y}{AGM}\right)^p < \left(\frac{AGM}{x}\right)^p + \left(\frac{AGM}{y}\right)^p, \tag{4.17}$$

where the second inequality in (4.17) is valid provided $p > 0$.

Inequalities (4.16) and (4.17) imply inequalities involving Legendre's complete elliptic integral K (see (4.1)). Let $k' = \sqrt{1-k^2}$ be the complementary modulus. Using the formula which connects AGM and K:

$$AGM(1,k') = \frac{\pi}{2K}$$

(see, e.g., [1]) together with (4.16) and (4.17) one can obtain inequalities involving integral K. We omit further details.

4.5. Applications to the Schwab-Borchardt mean

The Schwab-Borchardt mean of $x \geqslant 0$ and $y > 0$, in the sequel, will be denoted by $SB(x,y)$ or simply by SB. It is another iterative mean, i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where now

$$x_0 = x,\ \ y_0 = y,\ \ x_{n+1} = \frac{x_n + y_n}{2},\ \ y_{n+1} = \sqrt{x_{n+1} y_n}$$

$(n = 0, 1, \ldots)$. See, e.g., [1], [2]. It is worth mentioning that the mean under discussion is not symmetric in its variables, i.e. $SB(x,y) \neq SB(y,x)$ if $x \neq y$. This mean has been studied extensively in [8], [9], and [7]. For future reference, let us record one result [8, Thm. 3.1]:

$$(xy^2)^{1/3} < SB(x,y) < \frac{x+2y}{3}. \tag{4.18}$$

Let $u = SB/x$ and $v = SB/y$. If $y < x$, then $y < SB < x$ and $u < 1 < v$. It follows from (4.18) that inequalities (2.2) and (2.3) are satisfied with $\alpha = 1$ and $\beta = 2$. The following result is an immediate consequence of Theorem 3.1.

COROLLARY 4.6. *Assume that $\lambda > 0$ and $\mu > 0$. If $y < x$, then the inequality*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{SB}{x}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{SB}{y}\right)^q \tag{4.19}$$

holds true if either

$$q > 0 \text{ and } p \leqslant q\frac{\mu}{2\lambda} \tag{4.20}$$

or

$$p \leqslant q \leqslant -1 \text{ and } \mu \leqslant 2\lambda. \tag{4.21}$$

If $x < y$, then the inequality (4.19) is satisfied if either

$$p > 0 \text{ and } q \leqslant p\frac{2\lambda}{\mu} \tag{4.22}$$

or

$$q \leqslant p \leqslant -1 \text{ and } \mu \geqslant 2\lambda. \tag{4.23}$$

A two-sided inequality involving mean SB, which also follows from Theorem 3.2, is established in [7].

To the end of this subsection we will deal with inequalities for the logarithmic mean and the first Seiffert mean. Recall that the logarithmic mean of two positive numbers x and y is defined as

$$L(x,y) \equiv L = \frac{x-y}{\log x - \log y}$$

$(x \neq y)$. The first Seiffert mean of x and y is given by [12]

$$P(x,y) \equiv P = A\frac{z}{\sin^{-1} z},$$

where A is the unweighted arithmetic mean of x and y and $z = (x-y)/(x+y)$. It is known that

$$L = SB(A,G) \tag{4.24}$$

and

$$P = SB(G,A) \tag{4.25}$$

(see [8]). Here G stands for the unweighted geometric mean of x and y. Assuming that $\lambda > 0$ and $\mu > 0$ we obtain using (4.24) and Corollary 4.6 that

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{L}{A}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{L}{G}\right)^q \tag{4.26}$$

if either conditions (4.20) or (4.21) are satisfied. Similarly, using (4.25) together with Corollary 4.6 we arrive at

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{P}{G}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{P}{A}\right)^q \tag{4.27}$$

which is valid if either conditions (4.22) or (4.23) are satisfied.

Two other iterative means which can be derived from the Schwab-Borchardt mean are the second Seiffert mean [13]

$$T(x,y) \equiv T = A\frac{z}{\tan^{-1} z}$$

and

$$M(x,y) \equiv M = A\frac{z}{\sinh^{-1} z}$$

(see [8]). It has been demonstrated in [8] that

$$T = SB(A,Q) \text{ and } M = SB(Q,A),$$

where Q stands for the unweighted power mean of order 2 of x and y. Means M and T are comparable and they satisfy $A < M < T < Q$. Inequalities similar to (4.26) and (4.27) can be easily obtained using Corollary 4.6. For more inequalities involving the Schwab-Borchardt mean and four subordinate means L, P, T and M the interested reader is referred to [7].

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