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SOME SHARP INEQUALITIES INVOLVING SEIFFERT AND OTHER MEANS AND THEIR CONCISE PROOFS

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Abstract. In the paper, by establishing the monotonicity of some functions involving the sine and cosine functions, the authors provide a unified and concise proof of some known inequalities and find some new sharp inequalities involving the Seiffert, contra-harmonic, centroidal, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a \neq b$.

1. Introduction

The quantities

$$C(a,b) = \frac{a^2+b^2}{a+b}, \qquad \overline{C}(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}, \tag{1.1}$$

$$A(a,b) = \frac{a+b}{2}, \qquad G(a,b) = \sqrt{ab}, \tag{1.2}$$

$$H(a,b) = \frac{2ab}{a+b}, \qquad S(a,b) = \sqrt{\frac{a^2+b^2}{2}} \tag{1.3}$$

are called in the literature the contra-harmonic, centroidal, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a \neq b$.

For $a,b > 0$ with $a \neq b$, the first Seiffert mean $P(a,b)$ was defined in [24] by

$$P(a,b) = \frac{a-b}{4\arctan\sqrt{\frac{a}{b}} - \pi}. \tag{1.4}$$

Its equivalent form

$$P(a,b) = \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)} \tag{1.5}$$

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was given by [19, Eq. (2.4)]. For $a,b>0$ with $a\neq b$, the second Seiffert mean $T(a,b)$ was introduced in [23] by

$$T(a,b)=\frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}. \tag{1.6}$$

Recently, the following double inequalities involving the Seiffert, contra-harmonic, centroidal, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a\neq b$ were obtained.

PROPOSITION 1.1. ([5, Theorem 2.1]) *The double inequality*

$$\alpha A(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta A(a,b)+(1-\beta)H(a,b) \tag{1.7}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\frac{2}{\pi}$ and $\beta\geqslant\frac{5}{6}$.

PROPOSITION 1.2. ([17, Theorem 2.2]) *The double inequality*

$$\alpha C(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta C(a,b)+(1-\beta)H(a,b) \tag{1.8}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\frac{1}{\pi}$ and $\beta\geqslant\frac{5}{12}$.

PROPOSITION 1.3. ([10, Theorem 2.1]) *The double inequality*

$$\alpha\overline{C}(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta\overline{C}(a,b)+(1-\beta)H(a,b) \tag{1.9}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\frac{3}{2\pi}$ and $\beta\geqslant\frac{5}{8}$.

PROPOSITION 1.4. ([6, Theorem 2.1]) *The double inequality*

$$\alpha S(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta S(a,b)+(1-\beta)A(a,b) \tag{1.10}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant\frac{4-\pi}{(\sqrt{2}-1)\pi}$ and $\beta\geqslant\frac{2}{3}$.

For more information on this topic, please refer to [4, 7, 8, 9, 11, 12, 14, 15, 25, 26].

We point out that all the proofs of Propositions 1.1 to 1.4 are very complicated and tedious.

In this paper, by establishing the monotonicity of some functions involving the sine and cosine functions, we provide a unified and concise proof of Propositions 1.1 to 1.3, supply a concise proof of Proposition 1.4, and find some new sharp inequalities involving the Seiffert, contra-harmonic, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a\neq b$.

2. Lemmas

For establishing the monotonicity of some functions involving the sine and cosine functions, we need some lemmas below.

LEMMA 2.1. *The Bernoulli numbers B_{2n} for $n \in \mathbb{N}$ have the property*

$$(-1)^{n-1}B_{2n} = |B_{2n}|, \tag{2.1}$$

where the Bernoulli numbers B_i for $i \geqslant 0$ are defined by

$$\frac{x}{e^x-1} = \sum_{i=0}^{\infty} \frac{B_i}{i!}x^i = 1 - \frac{x}{2} + \sum_{i=1}^{\infty} B_{2i}\frac{x^{2i}}{(2i)!}, \quad |x| < 2\pi. \tag{2.2}$$

Proof. In [3, p. 16 and p. 56], it is listed that for $q \geqslant 1$

$$\zeta(2q) = (-1)^{q-1}\frac{(2\pi)^{2q}}{(2q)!}\frac{B_{2q}}{2}, \tag{2.3}$$

where ζ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.4}$$

From (2.3), the formula (2.1) follows. □

LEMMA 2.2. *For $0 < |x| < \pi$, we have*

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n-1}. \tag{2.5}$$

Proof. This is an easy consequence of combining the equality

$$\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2\left(2^{2n-1}-1\right)B_{2n}}{(2n)!}x^{2n-1}, \tag{2.6}$$

see [1, p. 75, 4.3.68], with Lemma 2.1. □

LEMMA 2.3. ([1, p. 75, 4.3.70]) *For $0 < |x| < \pi$,*

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1}. \tag{2.7}$$

LEMMA 2.4. *For $0 < |x| < \pi$,*

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|}{(2n)!}x^{2(n-1)}. \tag{2.8}$$

Proof. Since

$$\frac{1}{\sin^2 x} = \csc^2 x = -\frac{\mathrm{d}}{\mathrm{d}x}(\cot x),$$

the formula (2.8) follows from differentiating (2.7). □

LEMMA 2.5. *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x) \neq 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) in (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing) in (a,b).*

The above Lemma 2.5 can be found, for example, in [2, p. 292, Lemma 1], [13, p. 57, Lemma 2.3], [20, p. 92, Lemma 1], [21, p. 161, Lemma 2.3], [22, Lemma 2.9] and closely related references therein.

3. Monotonicity of some functions involving sine and cosine

For providing a unified and concise proof of Propositions 1.1 to 1.3, supplying a concise proof of Proposition 1.4, and finding some new sharp inequalities involving the Seiffert, contra-harmonic, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a \neq b$, we need the following monotonicity of some functions involving the sine and cosine functions, which can be proved by making use of Lemmas 2.2 to 2.5.

THEOREM 3.1. *For $x \in (0,\pi)$, the function*

$$h_1(x) = \frac{\frac{\sin x}{x} - \cos^2 x}{\sin^2 x} \tag{3.1}$$

is strictly decreasing and has the limits

$$\lim_{x\to 0^+} h_1(x) = \frac{5}{6} \quad \text{and} \quad \lim_{x\to\pi^-} h_1(x) = -\infty. \tag{3.2}$$

Proof. It is easy to see that

$$h_1(x) = \frac{1}{x\sin x} - \frac{1}{\sin^2 x} + 1$$

for $x \in (0,\pi)$. By using (2.5) and (2.8), we have

$$\begin{aligned} h_1(x) &= \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-2} - \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2} + 1 \\ &= \sum_{n=1}^{\infty} \frac{(1-n)2^{2n+1}-2}{(2n)!}|B_{2n}|x^{2n-2} + 1. \end{aligned}$$

So the function $h_1(x)$ is strictly decreasing on $(0,\pi)$.

The two limits in (3.2) come from L'Hôspital rule and standard arguments. The proof of Theorem 3.1 is complete. □

THEOREM 3.2. *For $x \in (0, 2\pi)$, the function*

$$h_2(x) = \frac{\sin x - x\cos x}{x(1-\cos x)} \tag{3.3}$$

is strictly decreasing and has the limits

$$\lim_{x\to 0^+} h_2(x) = \frac{2}{3} \quad \text{and} \quad \lim_{x\to(2\pi)^-} h_2(x) = -\infty. \tag{3.4}$$

Proof. Let

$$f_1(x) = \sin x - x\cos x \quad \text{and} \quad f_2(x) = x(1-\cos x).$$

Then

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x\sin x}{1-\cos x + x\sin x} = \left(1 + \frac{1-\cos x}{x\sin x}\right)^{-1} = \left(1 + \frac{\tan\frac{x}{2}}{x}\right)^{-1}.$$

Since

$$\frac{f_2'(x)}{f_1'(x)} = \frac{1-\cos x + x\sin x}{x\sin x} = 1 + \frac{1-\cos x}{x\sin x} = 1 + \frac{\tan\frac{x}{2}}{x}$$

is increasing on both $(0,\pi)$ and $(\pi, 2\pi)$, the function $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on both $(0,\pi)$ and $(\pi, 2\pi)$. Hence, by virtue of Lemma 2.5 and the continuity of $h_2(x)$ at $x = \pi$, it follows that the function $h_2(x)$ is strictly decreasing on $(0, 2\pi)$.

Two limits in (3.4) may be derived from L'Hôspital rule and standard arguments. The proof of Theorem 3.2 is complete. □

THEOREM 3.3. *For $x \in (0,\pi)$, the function*

$$h_3(x) = \frac{x - \sin x\cos x}{x\sin^2 x} \tag{3.5}$$

is strictly increasing and satisfies

$$\lim_{x\to 0^+} h_3(x) = \frac{2}{3} \quad \text{and} \quad \lim_{x\to\pi^-} h_3(x) = \infty. \tag{3.6}$$

Proof. The function $h_3(x)$ may be rewritten as

$$h_3(x) = \frac{1}{\sin^2 x} - \frac{\cot x}{x}$$

for $x \in (0,\pi)$. By using (2.8) and (2.7), we have

$$h_3(x) = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2} - \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-2}$$

$$= \sum_{n=1}^{\infty} \frac{n2^{2n+1}}{(2n)!} |B_{2n}| x^{2n-2}.$$

So the function $h_3(x)$ is strictly increasing on $(0, \pi)$.

The limits in (3.6) may be concluded from L'Hôspital rule and standard arguments. The proof of Theorem 3.3 is complete. □

THEOREM 3.4. *For $x \in (0, \pi)$, the function*

$$h_4(x) = \frac{(x - \sin x)\cos x}{x - \sin x \cos x} \tag{3.7}$$

is strictly decreasing, with

$$\lim_{x \to 0^+} h_4(x) = \frac{1}{4} \quad \text{and} \quad \lim_{x \to \pi^-} h_4(x) = -1. \tag{3.8}$$

Proof. It is obvious that

$$h_4(x) = 1 - \frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x) = x(1 - \cos x) \quad \text{and} \quad f_2(x) = x - \sin x \cos x.$$

Easy computations give

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1 - \cos x + x \sin x}{2 \sin^2 x} \triangleq \frac{f_3(x)}{f_4(x)}$$

and

$$\frac{f_3'(x)}{f_4'(x)} = \frac{2 \sin x + x \cos x}{4 \sin x \cos x} = \frac{1}{2 \cos x} + \frac{x}{4 \sin x}.$$

Since $\frac{1}{\cos x}$ and $\frac{x}{\sin x}$ are increasing on both $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$, the function $\frac{f_3'(x)}{f_4'(x)}$ is strictly increasing on both $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. Hence, By Lemma 2.5 and the continuity of $h_4(x)$ at $x = \frac{\pi}{2}$, we see that $h_4(x)$ is is strictly decreasing on $(0, \pi)$.

The limits in (3.8) can be deduced from L'Hôspital rule and standard arguments. The proof of Theorem 3.4 is complete. □

4. Unified and concise proofs of Propositions 1.1 to 1.4

Now we are in a position to provide a unified and concise proof of Propositions 1.1 to 1.3 and to supply a concise proof of Proposition 1.4.

A unified and concise proof of Propositions 1.1 *to* 1.3. It is not difficult to see that the inequalities (1.7), (1.8), and (1.9) can be rearranged respectively and equivalently as

$$\alpha < \frac{P(a,b) - H(a,b)}{A(a,b) - H(a,b)} < \beta, \tag{4.1}$$

$$\alpha < \frac{P(a,b)-H(a,b)}{C(a,b)-H(a,b)} < \beta, \tag{4.2}$$

and

$$\alpha < \frac{P(a,b)-H(a,b)}{\overline{C}(a,b)-H(a,b)} < \beta. \tag{4.3}$$

The denominators in (4.1), (4.2), and (4.3) meet

$$2[A(a,b)-H(a,b)] = \frac{2}{3}\left[\overline{C}(a,b)-H(a,b)\right] = C(a,b)-H(a,b) = \frac{(a-b)^2}{a+b}. \tag{4.4}$$

This implies that Propositions 1.1 to 1.3 are identical up to a scalar. So it is sufficient to verify one of the three Propositions 1.1 to 1.3, that is, to prove one of the three inequalities (4.1), (4.2), and (4.3).

Without loss of generality, we assume that $a > b > 0$. Let $x = \frac{a}{b} > 1$. Then the inequality (4.1) becomes

$$\frac{P(a,b)-H(a,b)}{A(a,b)-H(a,b)} = \frac{\frac{x-1}{2\arcsin\frac{x-1}{x+1}} - \frac{2x}{1+x}}{\frac{x+1}{2} - \frac{2x}{1+x}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{P(a,b)-H(a,b)}{A(a,b)-H(a,b)} = \frac{\frac{t}{\arcsin t} - \left(1-t^2\right)}{t^2}.$$

Let $t = \sin\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$. Then

$$\frac{P(a,b)-H(a,b)}{A(a,b)-H(a,b)} = \frac{\frac{\sin\theta}{\theta} - \cos^2\theta}{\sin^2\theta}.$$

By Theorem 3.1 and $h_1\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$, Propositions 1.1 to 1.3 thus follow. □

A concise proof of Proposition 1.4. The inequality (1.10) may be rewritten as

$$\alpha < \frac{T(a,b)-A(a,b)}{S(a,b)-A(a,b)} < \beta.$$

Without loss of generality, we assume that $a > b > 0$. Let $x = \frac{a}{b} > 1$. Then

$$\frac{T(a,b)-A(a,b)}{S(a,b)-A(a,b)} = \frac{\frac{x-1}{2\arctan\frac{x-1}{x+1}} - \frac{x+1}{2}}{\sqrt{\frac{x^2+1}{2}} - \frac{x+1}{2}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{T(a,b)-A(a,b)}{S(a,b)-A(a,b)} = \frac{\frac{t}{\arctan t} - 1}{\sqrt{1+t^2} - 1}.$$

Let $t = \tan\theta$ for $\theta \in \left(0, \frac{\pi}{4}\right)$. Then

$$\frac{T(a,b)-A(a,b)}{S(a,b)-A(a,b)} = \frac{\frac{\tan\theta}{\theta}-1}{\frac{1}{\cos\theta}-1} = \frac{\sin\theta-\theta\cos\theta}{\theta(1-\cos\theta)}.$$

By Theorem 3.2 and $h_2\left(\frac{\pi}{4}\right) = \frac{4-\pi}{(\sqrt{2}-1)\pi}$, we obtain Proposition 1.4. $\square$

5. New inequalities involving Seiffert and other means

In this section we will find some new sharp inequalities involving the Seiffert, contra-harmonic, arithmetic, geometric, harmonic, and root-square means of two positive real numbers a and b with $a \neq b$.

THEOREM 5.1. *The double inequality*

$$\alpha C(a,b) + (1-\alpha)H(a,b) < T(a,b) < \beta C(a,b) + (1-\beta)H(a,b) \tag{5.1}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant \frac{2}{\pi}$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. The double inequality (5.1) is the same as

$$\alpha < \frac{T(a,b)-H(a,b)}{C(a,b)-H(a,b)} < \beta.$$

Without loss of generality, we assume that $a>b>0$. Let $x = \frac{a}{b} > 1$. Then

$$\frac{T(a,b)-H(a,b)}{C(a,b)-H(a,b)} = \frac{\frac{x-1}{2\arctan\frac{x-1}{x+1}} - \frac{2x}{x+1}}{\frac{x^2+1}{x+1} - \frac{2x}{x+1}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{T(a,b)-H(a,b)}{C(a,b)-H(a,b)} = \frac{\frac{t}{\arctan t} - 1 + t^2}{2t^2}.$$

Let $t = \tan\theta$ for $\theta \in \left(0, \frac{\pi}{4}\right)$. Then

$$\frac{T(a,b)-H(a,b)}{C(a,b)-H(a,b)} = \frac{\frac{\tan\theta}{\theta} - 1 + \tan^2\theta}{2\tan^2\theta} = 1 - \frac{\theta - \sin\theta\cos\theta}{2\theta\sin^2\theta}.$$

By Theorem 3.3 and $h_3\left(\frac{\pi}{4}\right) = 2 - \frac{4}{\pi}$, we obtain Theorem 5.1. $\square$

THEOREM 5.2. *The double inequality*

$$\alpha C(a,b) + (1-\alpha)T(a,b) < S(a,b) < \beta C(a,b) + (1-\beta)T(a,b) \tag{5.2}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant \frac{\pi-2\sqrt{2}}{(\pi-2)\sqrt{2}}$ *and* $\beta \geqslant \frac{1}{4}$.

Proof. It is sufficient to show

$$\alpha < \frac{S(a,b)-T(a,b)}{C(a,b)-T(a,b)} < \beta.$$

Without loss of generality, we assume $a > b > 0$. Let $x = \frac{a}{b} > 1$. Then

$$\frac{S(a,b)-T(a,b)}{C(a,b)-T(a,b)} = \frac{\sqrt{\frac{x^2+1}{2}} - \frac{x-1}{2\arctan\frac{x-1}{x+1}}}{\frac{x^2+1}{x+1} - \frac{x-1}{2\arctan\frac{x-1}{x+1}}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{S(a,b)-T(a,b)}{C(a,b)-T(a,b)} = \frac{\sqrt{1+t^2} - \frac{t}{\arctan t}}{1+t^2 - \frac{t}{\arctan t}}.$$

Let $t = \tan\theta$ for $\theta \in \left(0, \frac{\pi}{4}\right)$. Then

$$\frac{\frac{1}{\cos\theta} - \frac{\tan\theta}{\theta}}{\frac{1}{\cos^2\theta} - \frac{\tan\theta}{\theta}} = \frac{\cos\theta(\theta - \sin\theta)}{\theta - \sin\theta\cos\theta}.$$

By Theorem 3.4 and $h_4\left(\frac{\pi}{4}\right) = \frac{\pi - 2\sqrt{2}}{\sqrt{2}\pi - 2\sqrt{2}}$, we obtain Theorem 5.2. $\square$

THEOREM 5.3. *The double inequality*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{5.3}$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leqslant \frac{2}{\pi}$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. The inequality (5.3) is equivalent to

$$\alpha < \frac{P(a,b)-G(a,b)}{A(a,b)-G(a,b)} < \beta.$$

Without loss of generality, assume $a > b > 0$. Let $x = \frac{a}{b}$. Then $x > 1$ and

$$\frac{P(a,b)-G(a,b)}{A(a,b)-G(a,b)} = \frac{\frac{x-1}{2\arcsin\frac{x-1}{x+1}} - \sqrt{x}}{\frac{x+1}{2} - \sqrt{x}}.$$

Let $t = \frac{x-1}{x+1}$. Then $t \in (0,1)$ and

$$\frac{P(a,b)-G(a,b)}{A(a,b)-G(a,b)} = \frac{\frac{t}{\arcsin t} - \sqrt{1-t^2}}{1 - \sqrt{1-t^2}}.$$

Let $t = \sin\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$. Then

$$\frac{P(a,b)-G(a,b)}{A(a,b)-G(a,b)} = \frac{\frac{\sin\theta}{\theta} - \cos\theta}{1-\cos\theta} = \frac{\sin\theta - \theta\cos\theta}{\theta(1-\cos\theta)}. \tag{5.4}$$

By Theorem 3.2 and $h_2\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$, we obtain Theorem 5.3. $\square$

REMARK 5.1. In [18], the double inequality

$$\frac{1}{2}[A(a,b)+G(a,b)] < P(a,b) < \frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)$$

for all $a,b>0$ with $a\neq b$, a special case of Theorem 5.3 for $\alpha=\frac{1}{2}$ and $\beta=\frac{2}{3}$, was given.

REMARK 5.2. This is a revised version of the preprint [16].

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