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CHARACTERIZATION OF THE INTERMEDIATE VALUES OF THE TRIANGLE INEQUALITY

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Abstract. We shall present a norm inequality which gives the all intermediate values of the triangle inequality. As an application of it, we shall prove two kinds of norm inequalities in [8] and [12].

1. Introduction

There are quite a lot of researches concerning norm inequalities under various settings. The typical one is as follows.

PROBLEM 1.1. *Let* $(X, \|\cdot\|)$ *be a normed space. Suppose that, for* $x, y \in X$, *a norm inequality* $\|x\| \leqslant \|y\|$ *holds. Construct a positive value* C *with respect to* x *and* y *satisfying* $\|x\| + C \leqslant \|y\|$.

Note that Problem 1.1 is the same to find the intermediate value between 0 and $\|y\| - \|x\|$. We are interested in this problem to the triangle inequality. The (generalized) triangle inequality, namely

$$\left\| \sum_{j=1}^{n} x_j \right\| \leqslant \sum_{j=1}^{n} \|x_j\|,$$

where $x_1, x_2, \cdots, x_n$, are elements in a Banach space $(X, \|\cdot\|)$ is one of the most fundamental norm inequalities in analysis. This inequality has attracted the attention of a number of authors, and many interesting refinements and reverse inequalities of it have been obtained (cf. [1, 2, 3, 9, 10, 11, 14]). For the triangle inequality, we consider the following problem.

PROBLEM 1.2. *Characterize all the intermediate value* C *which satisfies*

$$0 \leqslant C \leqslant \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|$$

by using $x_1, x_2, \cdots, x_n \in X$.

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When $n = 2$ Hudzik and Landes [6] proved the following inequality which gives a partial solution of Problem 1.2.

THEOREM 1.3. ([6, Lemma 1]) *For all nonzero elements x, y in X,*

$$0 \leqslant \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \leqslant \|x\| + \|y\| - \|x+y\|$$

Kato, Saito and Tamura [8] extended this result for an arbitrary number of finitely many nonzero elements $x_1, x_2, \cdots, x_n$ in X to treat the uniform non-ℓ_1^n-ness of Banach spaces as follows.

THEOREM 1.4. ([8, Theorem 1]) *For all nonzero elements $x_1, x_2, \cdots, x_n$ in X,*

$$0 \leqslant \left(n - \left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\|\right) \min_{1 \leqslant j \leqslant n} \|x_j\| \leqslant \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|$$

After that, several authors improved and generalized these inequalities (cf. [4, 5, 7]). Mitani, Saito, Kato and Tamura [12] succeeded in the further extension of Theorem 1.4 as follows.

THEOREM 1.5. ([12, Theorem 1]) *For all nonzero elements $x_1, x_2, \cdots, x_n$ in X,*

$$0 \leqslant \sum_{i=2}^{n} \left(i - \left\|\sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|}\right\|\right) (\|x_i^*\| - \|x_{i+1}^*\|) \leqslant \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|,$$

where $x_1^*, x_2^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ satisfying $\|x_1^*\| \geqslant \|x_2^*\| \geqslant \cdots \geqslant \|x_n^*\|$ and $x_{n+1}^* = 0$.

Since the intermediate value in Theorem 1.5 can be calculated as follows

$$\begin{aligned}
&\sum_{i=2}^{n} \left(i - \left\|\sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|}\right\|\right) (\|x_i^*\| - \|x_{i+1}^*\|) \\
&= \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right) \min_{1 \leqslant j \leqslant n} \|x_j\| + \sum_{i=2}^{n-1} \left(i - \left\|\sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|}\right\|\right) (\|x_i^*\| - \|x_{i+1}^*\|) \\
&\geqslant \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right) \min_{1 \leqslant j \leqslant n} \|x_j\| \geqslant 0,
\end{aligned}$$

if we put

$$(\mathrm{KST}) = \left(n - \left\|\sum_{j=1}^{n} \frac{x_j^*}{\|x_j^*\|}\right\|\right) \min_{1 \leqslant j \leqslant n} \|x_j\|$$

and

$$(\text{MSKT}) = \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|),$$

then we see that

$$0 \leqslant (\text{KST}) \leqslant (\text{MSKT}) \leqslant \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

Hence we know that (KST) and (MSKT) give one solution of Problem 1.2. However, other intermediate values are still unknown.

The aim of this paper is to characterize the all intermediate value in the triangle inequality, and two kinds of inequalities in Theorem 1.4 and 1.5 are concretely expressed as the intermediate value of it.

2. The intermediate values in the triangle inequality ($n = 2, 3$)

In this section, we shall discuss in the case of $n = 2$ and 3. They might be useful to understand a general case.

THEOREM 2.1. (cf. [5, Theorem 2.2]) *For each x, y in a Banach space (or normed space) X, and s, t in $\mathbb{R}$ with $0 \leqslant s, t \leqslant 1$, the following inequality holds*

$$\|sx\| + \|ty\| - \|sx + ty\| \leqslant \|x\| + \|y\| - \|x + y\|.$$

Proof. Since

$$x + y = (sx + ty) + \{(1-s)x + (1-t)y\} \quad \text{and} \quad 1-s, 1-t \geqslant 0,$$

by repeatedly using the triangle inequality, we have

$$\begin{aligned}
\|x + y\| &\leqslant \|sx + ty\| + \|(1-s)x + (1-t)y\| \\
&\leqslant \|sx + ty\| + \|(1-s)x\| + \|(1-t)y\| \\
&= \|sx + ty\| + \|x\| - \|sx\| + \|y\| - \|ty\| \\
&= \|x\| + \|y\| - (\|sx\| + \|ty\| - \|sx + ty\|).
\end{aligned}$$

This completes the proof. □

For x and y in X, putting a function f on a product space $[0,1] \times [0,1]$ as

$$f(s,t) = \|sx\| + \|ty\| - \|sx + ty\| \qquad (\forall (s,t) \in, [0,1] \times [0,1]),$$

then it is clear that f is a continuous function on $[0,1] \times [0,1]$ satisfying

$$f(0,0) = 0 \quad \text{and} \quad f(1,1) = \|x\| + \|y\| - \|x + y\|.$$

Since $[0,1] \times [0,1]$ is connected, by using the intermediate values theorem, we have

COROLLARY 2.2. *Let $x,y\in X$. For each ω with $0\leqslant\omega\leqslant\|x\|+\|y\|-\|x+y\|$, there is (s_0,t_0) in $[0,1]\times[0,1]$ such that*

$$\omega=\|s_0x\|+\|t_0y\|-\|s_0x+t_0y\|.$$

Corollary 2.2 not only gives the solution of Problem 1.2 but also contains Theorem 1.3. Indeed, for any x and y in X and $s,s_0,t,t_0\in\mathbb{R}$ with $0\leqslant s\leqslant s_0\leqslant 1$, $0\leqslant t\leqslant t_0\leqslant 1$, we see that $s_0x,t_0y\in X$, $0\leqslant\frac{s}{s_0}\leqslant 1$ and $0\leqslant\frac{t}{t_0}\leqslant 1$. Hence, by Theorem 2.1, we have

$$\begin{aligned} f(s,t)&=\|sx\|+\|ty\|-\|sx+ty\|\\ &=\left\|\frac{s}{s_0}(s_0x)\right\|+\left\|\frac{t}{t_0}(t_0y)\right\|-\left\|\frac{s}{s_0}(s_0x)+\frac{t}{t_0}(t_0y)\right\|\\ &\leqslant\|s_0x\|+\|t_0y\|-\|s_0x+t_0y\|\\ &=f(s_0,t_0).\end{aligned}$$

Thus f is a non decreasing continuous function on $[0,1]\times[0,1]$, and so if $\|y\|\leqslant\|x\|$, then we have

$$\begin{aligned} 0=f(0,0)\leqslant f(s_1,1)\leqslant f\left(\frac{\|y\|}{\|x\|},1\right)\leqslant f(s_2,1)\\ \leqslant f(1,1)=\|x\|+\|y\|-\|x+y\| \qquad \left(0\leqslant s_1\leqslant\frac{\|y\|}{\|x\|}\leqslant s_2\leqslant 1\right).\end{aligned}$$

Since

$$f\left(\frac{\|y\|}{\|x\|},1\right)=\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\},$$

by Theorem 2.1, we have Theorem 1.3 as

$$0\leqslant\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\}\leqslant\|x\|+\|y\|-\|x+y\|.$$

Next, we consider a geometrical meaning of these inequalities as $X=\mathbb{R}^2$ and $\|y\|\leqslant\|x\|$. Under this setting, Theorem 2.1 shows the relation of the differences of the inequalities concerning two triangles in Figure 1.

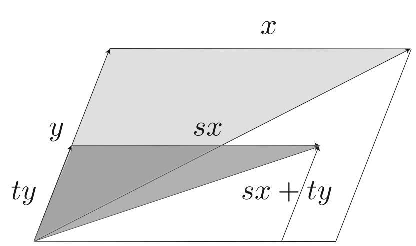

Figure 1: *Theorem* 2.1

Especially, Theorem 2.1 is as follows for $t=1$.

$$\begin{aligned} 0 &\leqslant \|sx\|+\|y\|-\|sx+y\| \\ &= f(s,1) \\ &\leqslant \|x\|+\|y\|-\|x+y\| \qquad (0\leqslant s\leqslant 1). \end{aligned}$$

The following figure is useful to understand this relation.

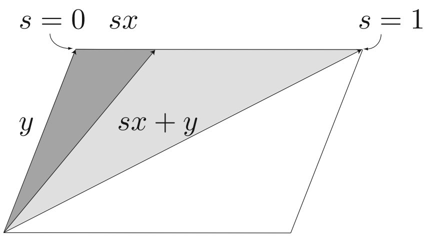

Figure 2: *Case with* $t=1$

In this case, if the value of s is continuously moved from 0 to 1, all the values between $f(0,1)=0$ and $f(1,1)=\|x\|+\|y\|-\|x+y\|$ can be obtained. And, when the value of s is just $\|y\|/\|x\|$, it is Theorem 1.3.

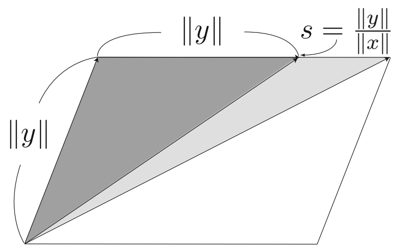

Figure 3: *Theorem* 1.3

Next, we shall discuss it in case of $n=3$.

THEOREM 2.3. *For each* x,y,z *in* X, *and* $\alpha,\beta,\gamma,\lambda,\mu$ *in* $\mathbb{R}$ *with* $0\leqslant\alpha,\beta,\gamma,\lambda,\mu\leqslant 1$, *the following inequality holds*

$$\begin{aligned} &(\|\alpha x\|+\|\beta y\|+\|\gamma z\|-\|\alpha x+\beta y+\gamma z\|) \\ &\quad+(\|\lambda(1-\alpha)x\|+\|\mu(1-\beta)y\|-\|\lambda(1-\alpha)x+\mu(1-\beta)y\|) \\ &\leqslant\|x\|+\|y\|+\|z\|-\|x+y+z\|. \end{aligned}$$

Proof. Put $x_\alpha = (1-\alpha)x,\ y_\beta = (1-\beta)y,\ z_\gamma = (1-\gamma)z$, then

$$\|x_\alpha\| = \|x\| - \|\alpha x\|,\ \|y_\beta\| = \|y\| - \|\beta y\|,\ \|z_\gamma\| = \|z\| - \|\gamma z\|$$

and, by Theorem 2.1, we have

$$\|\lambda x_\alpha\| + \|\mu y_\beta\| - \|\lambda x_\alpha + \mu y_\beta\| \leqslant \|x_\alpha\| + \|y_\beta\| - \|x_\alpha + y_\beta\|.$$

Thus we have

$$\begin{aligned}
&\|x+y+z\| = \|(\alpha x + \beta y + \gamma z) + (x_\alpha + y_\beta + z_\gamma)\| \\
&\leqslant \|\alpha x + \beta y + \gamma z\| + \|x_\alpha + y_\beta + z_\gamma\| \\
&\leqslant \|\alpha x + \beta y + \gamma z\| + \|x_\alpha + y_\beta\| + \|z_\gamma\| \\
&\leqslant \|\alpha x + \beta y + \gamma z\| + \{\|x_\alpha\| + \|y_\beta\| - (\|\lambda x_\alpha\| + \|\mu y_\beta\| - \|\lambda x_\alpha + \mu y_\beta\|)\} + \|z_\gamma\| \\
&= \|x\| + \|y\| + \|z\| - (\|\alpha x\| + \|\beta y\| + \|\gamma z\| - \|\alpha x + \beta y + \gamma z\|) \\
&\qquad - (\|\lambda x_\alpha\| + \|\mu y_\beta\| - \|\lambda x_\alpha + \mu y_\beta\|) \\
&= \|x\| + \|y\| + \|z\| - (\|\alpha x\| + \|\beta y\| + \|\gamma z\| - \|\alpha x + \beta y + \gamma z\|) \\
&\qquad - (\|\lambda(1-\alpha)x\| + \|\mu(1-\beta)y\| - \|\lambda(1-\alpha)x + \mu(1-\beta)y\|).
\end{aligned}$$

This completes the proof. □

As well as in the case of $n=2$, for x,y,z in X, if we put a function g on a product space $\prod_{i=1}^{5}[0,1] = \underbrace{[0,1]\times\cdots\times[0,1]}_{5 \text{ times}}$ as

$$\begin{aligned}
g(\alpha,\beta,\gamma,\lambda,\mu) &= (\|\alpha x\| + \|\beta y\| + \|\gamma z\| - \|\alpha x + \beta y + \gamma z\|) \\
&\quad + (\|\lambda(1-\alpha)x\| + \|\mu(1-\beta)y\| - \|\lambda(1-\alpha)x + \mu(1-\beta)y\|),
\end{aligned} \tag{2.1}$$

then it can easily check that, for each $(\alpha_1,\beta_1,\gamma_1,\lambda_1,\mu_1)$, $(\alpha_2,\beta_2,\gamma_2,\lambda_2,\mu_1)$ in $\prod_{i=1}^{5}[0,1]$,

$$\begin{aligned}
&|g(\alpha_1,\beta_1,\gamma_1,\lambda_1,\mu_1) - g(\alpha_2,\beta_2,\gamma_2,\lambda_2,\mu_1)| \\
&\qquad \leqslant 2(2|\alpha_1-\alpha_2| + |\lambda_1-\lambda_2|)\cdot\|x\| \\
&\qquad\qquad 2(2|\beta_1-\beta_2| + |\mu_1-\mu_2|)\cdot\|y\| + 2|\gamma_1-\gamma_2|\cdot\|z\|,
\end{aligned}$$

and so, g is a continuous function on $\prod_{i=1}^{5}[0,1]$. Moreover, we see that

$$\begin{aligned}
&g(0,0,0,0,0) = 0 \quad \text{and} \\
&g(1,1,1,\lambda,\mu) = \|x\| + \|y\| + \|z\| - \|x+y+z\| \quad (0 \leqslant \lambda,\mu \leqslant 1).
\end{aligned}$$

Thus we have

COROLLARY 2.4. *Let $x,y,z \in X$. For each ω with $0 \leqslant \omega \leqslant \|x\|+\|y\|+\|z\| - \|x+y+z\|$, there exists $(s_1,s_2,s_3,s_4,s_5) \in \prod_{i=1}^{5}[0,1]$ such that*

$$\begin{aligned}
\omega &= (\|s_1x\| + \|s_2y\| + \|s_3z\| - \|s_1x + s_2y + s_3z\|) \\
&\quad + (\|s_4(1-s_1)x\| + \|s_5(1-s_2)y\| - \|s_4(1-s_1)x + s_5(1-s_2)y\|)
\end{aligned}$$

There are infinitely many paths to take the value from 0 to $\|x\|+\|y\|+\|z\|-\|x+y+z\|$ by how to choose variables. We can obtain Theorem 1.4 and 1.5 in the case $n=3$ by choosing it well. Note that Theorem 1.4 and 1.5 are as follows respectively for $n=3$.

THEOREM 2.5. (cf. Theorem 1.4) *For all nonzero elements x,y,z in X with $\|x\| \geqslant \|y\| \geqslant \|z\|$,*

$$0 \leqslant \left(3-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right\|\right)\|z\| \leqslant \|x\|+\|y\|+\|z\|-\|x+y+z\|$$

THEOREM 2.6. (cf. Theorem 1.5) *For all nonzero elements x,y,z in X with $\|x\| \geqslant \|y\| \geqslant \|z\|$,*

$$\begin{aligned} 0 &\leqslant \left(3-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right\|\right)\|z\| + \left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)(\|y\|-\|z\|) \\ &\leqslant \|x\|+\|y\|+\|z\|-\|x+y+z\|. \end{aligned}$$

To obtain these theorems, the following examples give one of the paths.

EXAMPLE 2.7. *We may assume that $\|x\| \neq \|z\|$. For each t with $0 \leqslant t \leqslant \frac{\|z\|}{\|x\|}$, put*

$$s_1^t = t,\ s_2^t = \frac{\|x\|}{\|y\|}t,\ s_3^t = \frac{\|x\|}{\|z\|}t,\ s_4^t = 0,\ s_5^t = 0,$$

and for each t with $\frac{\|z\|}{\|x\|} \leqslant t \leqslant 1$, put

$$s_1^t = t,\ s_2^t = \frac{\|z\|}{\|y\|} + \frac{\|x\|}{\|y\|}\cdot\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\cdot\left(t-\frac{\|z\|}{\|x\|}\right),\ s_3^t = 1,\ s_4^t = 0,\ s_5^t = 0,$$

then we see that $\left(s_1^t,s_2^t,s_3^t,s_4^t,s_5^t\right) \in \prod_{i=1}^5[0,1]$ satisfying

$$\begin{aligned} &\left(s_1^0,s_2^0,s_3^0,s_4^0,s_5^0\right) = (0,0,0,0,0),\ \left(s_1^1,s_2^1,s_3^1,s_4^1,s_5^1\right) = (1,1,1,0,0) \text{ and} \\ &\left(s_1^{\frac{\|z\|}{\|x\|}},s_2^{\frac{\|z\|}{\|x\|}},s_3^{\frac{\|z\|}{\|x\|}},s_4^{\frac{\|z\|}{\|x\|}},s_5^{\frac{\|z\|}{\|x\|}}\right) = \left(\frac{\|z\|}{\|x\|},\frac{\|z\|}{\|y\|},\frac{\|z\|}{\|z\|},0,0\right). \end{aligned}$$

Hence if we define a function f on $[0,1]$ by using g in (2.1) as

$$f(t) = g\left(s_1^t,s_2^t,s_3^t,s_4^t,s_5^t\right),$$

then f is a continuous function on $[0,1]$ such that

$$f(0)=0 \quad \text{and} \quad f(1) = \|x\|+\|y\|+\|z\|-\|x+y+z\|.$$

Moreover

$$f\left(\frac{\|z\|}{\|x\|}\right)=g\left(s_1^{\frac{\|z\|}{\|x\|}},s_2^{\frac{\|z\|}{\|x\|}},s_3^{\frac{\|z\|}{\|x\|}},s_4^{\frac{\|z\|}{\|x\|}},s_5^{\frac{\|z\|}{\|x\|}}\right)=g\left(\frac{\|z\|}{\|x\|},\frac{\|z\|}{\|y\|},\frac{\|z\|}{\|z\|},0,0\right)$$
$$=\left\|\frac{\|z\|}{\|x\|}x\right\|+\left\|\frac{\|z\|}{\|y\|}y\right\|+\left\|\frac{\|z\|}{\|z\|}z\right\|-\left\|\frac{\|z\|}{\|x\|}x+\frac{\|z\|}{\|y\|}y+\frac{\|z\|}{\|z\|}z\right\|$$
$$=\left(3-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}+\frac{z}{\|z\|}\right\|\right)\|z\|.$$

By Theorem 2.3, we see that $f(0)\leqslant f\left(\frac{\|z\|}{\|x\|}\right)\leqslant f(1)$, *and so we have Theorem 2.5.*

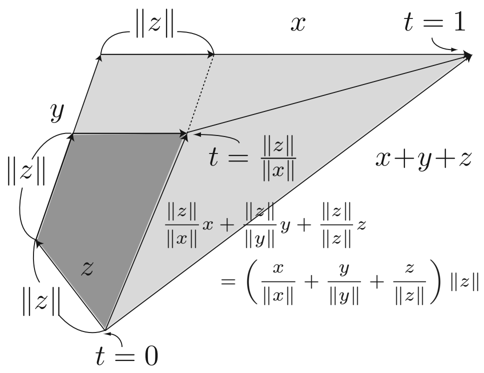

Figure 4: *Example 2.7*

According to Figure 4, Theorem 2.5 gives the relation between quadrangle concerning x,y,z and $x+y+z$ and quadrangle concerning $\frac{\|z\|}{\|x\|}x$, $\frac{\|z\|}{\|y\|}y$, $\frac{\|z\|}{\|z\|}z$ and $\frac{\|z\|}{\|x\|}x+\frac{\|z\|}{\|y\|}y+\frac{\|z\|}{\|z\|}z$.

EXAMPLE 2.8. *We may assume that* $\|x\|\neq\|z\|$. *For each* t *with* $0\leqslant t\leqslant\frac{\|z\|}{\|x\|}$, *put*

$$s_1^t=t,\ s_2^t=\frac{\|x\|}{\|y\|}t,\ s_3^t=\frac{\|x\|}{\|z\|}t,\ s_4^t=\frac{\|x\|}{\|z\|}\cdot\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\cdot t,\ s_5^t=\frac{\|x\|}{\|z\|}t,$$

and for each t *with* $\frac{\|z\|}{\|x\|}\leqslant t\leqslant 1$, *put*

$$s_1^t=t,\ s_2^t=\frac{\|z\|}{\|y\|}+\frac{\|x\|}{\|y\|}\cdot\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\cdot\left(t-\frac{\|z\|}{\|x\|}\right),\ s_3^t=1,$$
$$s_4^t=\frac{\|y\|-\|z\|}{\|x\|-\|z\|}+\frac{(\|x\|-\|y\|)(\|x\|t-\|z\|)}{(\|x\|-\|z\|)^2},\ s_5^t=1,$$

then we see that $\left(s_1^t, s_2^t, s_3^t, s_4^t, s_5^t\right) \in \prod_{i=1}^5 [0,1]$ *satisfying*

$$\left(s_1^0, s_2^0, s_3^0, s_4^0, s_5^0\right) = (0,0,0,0,0), \quad \left(s_1^1, s_2^1, s_3^1, s_4^1, s_5^1\right) = (1,1,1,1,1) \text{ and}$$
$$\left(s_1^{\frac{\|z\|}{\|x\|}}, s_2^{\frac{\|z\|}{\|x\|}}, s_3^{\frac{\|z\|}{\|x\|}}, s_4^{\frac{\|z\|}{\|x\|}}, s_5^{\frac{\|z\|}{\|x\|}}\right) = \left(\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|y\|}, \frac{\|z\|}{\|z\|}, \frac{\|y\|-\|z\|}{\|x\|-\|z\|}, 1\right).$$

Hence if we define a function f *on* $[0,1]$ *by*

$$f(t) = g\left(s_1^t, s_2^t, s_3^t, s_4^t, s_5^t\right),$$

then f *is a continuous function on* $[0,1]$ *such that*

$$f(0) = 0 \quad \text{and} \quad f(1) = \|x\| + \|y\| + \|z\| - \|x+y+z\|.$$

Moreover

$$\begin{aligned}
f\left(\frac{\|z\|}{\|x\|}\right) &= g\left(s_1^{\frac{\|z\|}{\|x\|}}, s_2^{\frac{\|z\|}{\|x\|}}, s_3^{\frac{\|z\|}{\|x\|}}, s_4^{\frac{\|z\|}{\|x\|}}, s_5^{\frac{\|z\|}{\|x\|}}\right) \\
&= g\left(\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|y\|}, \frac{\|z\|}{\|z\|}, \frac{\|y\|-\|z\|}{\|x\|-\|z\|}, 1\right) \\
&= \left(\left\|\frac{\|z\|}{\|x\|}x\right\| + \left\|\frac{\|z\|}{\|y\|}y\right\| + \left\|\frac{\|z\|}{\|z\|}z\right\| - \left\|\frac{\|z\|}{\|x\|}x + \frac{\|z\|}{\|y\|}y + \frac{\|z\|}{\|z\|}z\right\|\right) \\
&\quad + \left\{\left\|\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\left(1 - \frac{\|z\|}{\|x\|}\right)x\right\| + \left\|\left(1 - \frac{\|z\|}{\|y\|}\right)y\right\|\right. \\
&\quad \left. - \left\|\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\left(1 - \frac{\|z\|}{\|x\|}\right)x + \left(1 - \frac{\|z\|}{\|y\|}\right)y\right\|\right\} \\
&= \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| \\
&\quad + \left(\left\|\frac{\|y\|-\|z\|}{\|x\|}\cdot x\right\| + \left\|\frac{\|y\|-\|z\|}{\|y\|}\cdot y\right\| - \left\|\frac{\|y\|-\|z\|}{\|x\|}\cdot x + \frac{\|y\|-\|z\|}{\|y\|}\cdot y\right\|\right) \\
&= \left(3 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{z}{\|z\|}\right\|\right)\|z\| + \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right)(\|y\| - \|z\|).
\end{aligned}$$

Since $f(0) \leqslant f\left(\frac{\|z\|}{\|x\|}\right) \leqslant f(1)$, *we have Theorem* 2.6.

Figure 5 shows that Theorem 2.6 *is a combination of Theorem* 1.3 *and* 2.5. *That is, we obtain Theorem* 2.6, *applying Theorem* 1.3 *for two vectors* $\frac{y}{\|y\|}(\|y\| - \|z\|)$ *and* $\frac{x}{\|x\|}(\|x\| - \|z\|)$ *in addition to Theorem* 2.5.

Furthermore, if we put, for each t, u *with* $0 \leqslant t, u \leqslant \frac{\|z\|}{\|x\|}$,

$$s_1^t = t,\ s_2^t = \frac{\|x\|}{\|y\|}t,\ s_3^t = \frac{\|x\|}{\|z\|}t,\ s_4^u = \frac{\|x\|}{\|z\|}\cdot\frac{\|y\|-\|z\|}{\|x\|-\|z\|}\cdot u,\ s_5^u = \frac{\|x\|}{\|z\|}u,$$

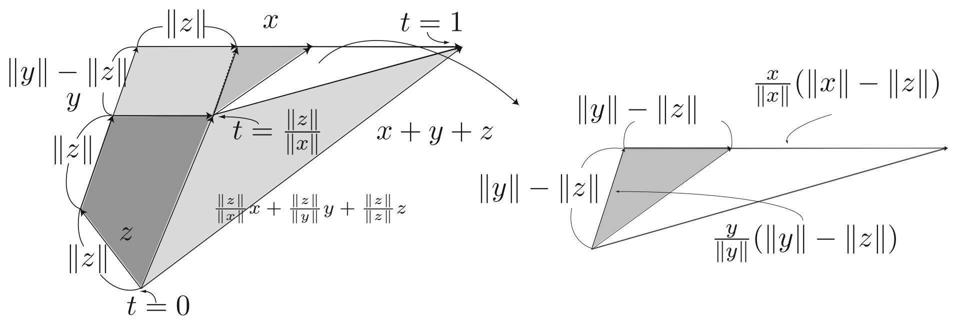

Figure 5: *Example* 2.8

and for each t,u *with* $\frac{\|z\|}{\|x\|} \leqslant t,u \leqslant 1$, *put*

$$s_1^t = t,\ s_2^t = \frac{\|z\|}{\|y\|} + \frac{\|x\|}{\|y\|} \cdot \frac{\|y\|-\|z\|}{\|x\|-\|z\|} \cdot \left(t - \frac{\|z\|}{\|x\|}\right),\ s_3^t = 1,$$

$$s_4^u = \frac{\|y\|-\|z\|}{\|x\|-\|z\|} + \frac{(\|x\|-\|y\|)(\|x\|u-\|z\|)}{(\|x\|-\|z\|)^2},\ s_5^t = 1,$$

then we can define a function h *on* $[0,1]\times[0,1]$ *by*

$$h(t,u) = g(s_1^t, s_2^t, s_3^t, s_4^u, s_5^u).$$

It contains both Theorem 2.5 *and* 2.6 *at the same time, that is, if we take* $(t,u) = \left(\frac{\|z\|}{\|x\|}, 0\right)$, *then we have Theorem* 2.5 *and if we take* $(t,u) = \left(\frac{\|z\|}{\|x\|}, \frac{\|z\|}{\|x\|}\right)$, *then we also have Theorem* 2.6.

3. The intermediate values (general case)

In this section, we discuss a general case. To do it, we need some preparations.

For positive integer $n \geqslant 2$, let $M_n([0,1])$ be the set of all n by n matrices whose all elements belong to the interval $[0,1]$ and L_n denote the set of all lower triangular matrices of $M_n([0,1])$; i.e.,

$$L_n = \left\{ a = (a_{ij}) \in M_n([0,1]) \,\middle|\, a_{ij} = 0\ (i<j) \right\}.$$

Let $1 \leqslant m \leqslant n$. For each $a = (a_{ij})$ in L_n, we set

$$\ell_{1j}^a(m) = a_{1j} \quad \text{and} \quad \ell_{mj}^a(m) = a_{mj} \quad (1 \leqslant j \leqslant m)$$

and if $3 \leqslant n$, then, for each m with $3 \leqslant m \leqslant n$, we put

$$\ell_{ij}^a(m) = a_{ij} \prod_{k=i+1}^{m} (1 - a_{kj}) \quad (2 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant m).$$

LEMMA 3.1. *Keep the notation as above. Let* $a = (a_{ij}),\ b = (b_{ij})$ *in* L_n. *Then the following statements hold.*

(i) *Let* $1 \leqslant m \leqslant n$. *For each* i *with* $1 \leqslant i \leqslant m$,

$$0 \leqslant \ell_{ij}^{a}(m) \leqslant a_{ij} \leqslant 1 \quad (1 \leqslant j \leqslant n),$$

and $n \times n$ *matrix* $\left(\ell_{ij}^{a}(n)\right)$ *belongs to* L_n.

(ii) *Let* $n \geqslant 3$. *For each* m *with* $3 \leqslant m \leqslant n$,

$$\ell_{ij}^{a}(m) = \begin{cases} \ell_{ij}^{a}(m-1)(1-a_{mj}) & (2 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant m-1) \\ a_{mj} & (i = m,\ 1 \leqslant j \leqslant m) \end{cases}.$$

(iii) *For each* i *with* $1 \leqslant i \leqslant n$,

$$|\ell_{ij}^{a}(m) - \ell_{ij}^{b}(m)| \leqslant \sum_{k=j}^{m} |a_{ik} - b_{ik}| \qquad (1 \leqslant j \leqslant m).$$

Proof. We only prove (iii). It is clear in case of $n = 1$ and 2. Let $n \geqslant 3$. It is also clear in case of $i = 1$ and m. By using induction on m, we see that

$$\left| \prod_{k=i+1}^{m} (1-a_{kj}) - \prod_{k=i+1}^{m} (1-b_{kj}) \right| \leqslant \sum_{k=i+1}^{m} \left| a_{kj} - b_{kj} \right|.$$

Thus we have

$$\begin{aligned} |\ell_{ij}^{a}(m) - \ell_{ij}^{b}(m)| &= \left| a_{ij} \prod_{k=i+1}^{m} (1-a_{kj}) - b_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| \\ &\leqslant \left| a_{ij} \prod_{k=i+1}^{m} (1-a_{kj}) - a_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| + \left| a_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) - b_{ij} \prod_{k=i+1}^{m} (1-b_{kj}) \right| \\ &\leqslant \left| \prod_{k=i+1}^{m} (1-a_{kj}) - \prod_{k=i+1}^{m} (1-b_{kj}) \right| + \left| a_{ij} - b_{ij} \right| \\ &\leqslant \sum_{k=i+1}^{m} \left| a_{kj} - b_{kj} \right| + \left| a_{ij} - b_{ij} \right| \\ &\leqslant \sum_{k=i}^{m} \left| a_{ij} - b_{ij} \right|. \end{aligned}$$

This completes the proof. □

Take any $a=(a_{ij})\in L_n$ and fix it. Considering $\left(\ell^a_{ij}(n)\right)$ as the matrix acting on a Banach space $\underbrace{X\oplus X\oplus\cdots\oplus X}_{n\text{ times}}$, we have

$$\begin{pmatrix} \ell^a_{11}(n) & & & \\ \ell^a_{21}(n) & \ell^a_{22}(n) & & \text{\Large 0} \\ \vdots & \vdots & \ddots & \\ \ell^a_{n1}(n) & \dots & \ell^a_{nn-1}(n) & \ell^a_{nn}(n) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \ell^a_{11}(n)x_1 \\ \sum_{j=1}^2 \ell^a_{2j}(n)x_j \\ \vdots \\ \sum_{j=1}^n \ell^a_{nj}(n)x_j \end{pmatrix},$$

where $x_1,x_2,\cdots,x_n\in X$. For each entries, we have the triangle inequalities

$$\left\|\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\| \leqslant \sum_{j=1}^{i}\left\|\ell^a_{ij}(n)x_j\right\| \qquad (1\leqslant i\leqslant n).$$

We revealed the fact that the sum of all differences of the triangle inequalities

$$\left\|\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\| \leqslant \sum_{j=1}^{i}\|\ell^a_{ij}(n)x_j\| \qquad (1\leqslant i\leqslant n)$$

is less than the difference of the triangle inequality

$$\left\|\sum_{j=1}^{n}x_j\right\| \leqslant \sum_{j=1}^{n}\|x_j\|$$

as follows.

THEOREM 3.2. *Let $n\geqslant 2$. With the above notation, take any $a=(a_{ij})$ in L_n. For all elements $x_1,x_2,\cdots,x_n$ in a Banach space X, the following inequalities hold*

$$0\leqslant \sum_{i=1}^{n}\left(\sum_{j=1}^{i}\|\ell^a_{ij}(n)x_j\| - \left\|\sum_{j=1}^{i}\ell^a_{ij}(n)x_j\right\|\right) \leqslant \sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\|.$$

Proof. Let $n=2$. For each $a=(a_{ij})\in L_2$, we note that $\ell^a_{21}(2)=a_{21}$ and $\ell^a_{22}(2)=a_{22}$. Hence, by Theorem 2.1, the following inequality holds; for each x_1,x_2 in X,

$$0\leqslant (\|\ell^a_{21}(2)x_1\|+\|\ell^a_{22}(2)x_2\|-\|\ell^a_{21}(2)x_1+\ell^a_{22}(2)x_2\|)\leqslant \|x_1\|+\|x_2\|-\|x_1+x_2\|.$$

Thus, we have

$$\sum_{i=1}^{2}\left(\sum_{j=1}^{i}\|\ell^a_{ij}(2)x_j\| - \left\|\sum_{j=1}^{i}\ell^a_{ij}(2)x_j\right\|\right) \leqslant \sum_{j=1}^{2}\|x_j\| - \left\|\sum_{j=1}^{2}x_j\right\|.$$

We shall next prove the case $n\geqslant 3$ by induction. To do this, for any $a=(a_{ij})$ in L_n and m with $2\leqslant m\leqslant n$, we only have to prove that the following holds; for all $x_1,x_2,\cdots,x_m$ in X,

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{i}\|\ell^a_{ij}(m)x_j\| - \left\|\sum_{j=1}^{i}\ell^a_{ij}(m)x_j\right\|\right) \leqslant \sum_{j=1}^{m}\|x_j\| - \left\|\sum_{j=1}^{m}x_j\right\|. \tag{a}$$

The case $m=2$ is the same as above. Assume that the inequality (a) holds for m with $2 \leqslant m \leqslant n-1$. For each $x_1, x_2, \cdots, x_m, x_{m+1}$ in X, we have

$$\begin{aligned}
\left\|\sum_{j=1}^{m+1} x_j\right\| &\leqslant \left\|\sum_{j=1}^{m+1} a_{m+1\,j} x_j\right\| + \left\|\sum_{j=1}^{m+1} (1-a_{m+1\,j}) x_j\right\| \\
&= \left\|\sum_{j=1}^{m+1} a_{m+1\,j} x_j\right\| + \left\|\sum_{j=1}^{m} (1-a_{m+1\,j}) x_j + (1-a_{m+1\,m+1}) x_{m+1}\right\| \\
&\leqslant \left\|\sum_{j=1}^{m+1} a_{m+1\,j} x_j\right\| + \left\|\sum_{j=1}^{m} (1-a_{m+1\,j}) x_j\right\| + \|(1-a_{m+1\,m+1}) x_{m+1}\| \\
&= \left\|\sum_{j=1}^{m+1} a_{m+1\,j} x_j\right\| + \left\|\sum_{j=1}^{m} (1-a_{m+1\,j}) x_j\right\| + \|x_{m+1}\| - \|a_{m+1\,m+1} x_{m+1}\| \\
&= \left\|\sum_{j=1}^{m+1} \ell^a_{m+1\,j}(m+1) x_j\right\| + \left\|\sum_{j=1}^{m} (1-a_{m+1\,j}) x_j\right\| + \|x_{m+1}\| - \|a_{m+1\,m+1} x_{m+1}\|.
\end{aligned}$$

Recall that

$$\ell^a_{ij}(m+1) = \ell^a_{ij}(m)(1-a_{m+1\,j}) \qquad (2 \leqslant i \leqslant m).$$

Applying the assumption for m elements $(1-a_{m+1\,1})x_1, (1-a_{m+1\,2})x_2, \cdots, (1-a_{m+1\,m})x_m$ in X, we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{m} (1-a_{m+1\,j}) x_j\right\| \\
&\leqslant \sum_{j=1}^{m} \|(1-a_{m+1\,j}) x_j\| - \sum_{i=1}^{m} \left(\sum_{j=1}^{i} \|\ell^a_{ij}(m)(1-a_{m+1\,j}) x_i\| - \left\|\sum_{j=1}^{i} \ell^a_{ij}(m)(1-a_{m+1\,j}) x_j\right\| \right) \\
&= \sum_{j=1}^{m} \left(\|x_j\| - \|a_{m+1\,j} x_j\|\right) - \sum_{i=1}^{m} \left(\sum_{j=1}^{i} \|\ell^a_{ij}(m+1) x_j\| - \left\|\sum_{j=1}^{i} \ell^a_{ij}(m+1) x_j\right\| \right) \\
&= \sum_{j=1}^{m} \|x_j\| - \sum_{j=1}^{m} \|\ell^a_{m+1\,j}(m+1) x_j\| - \sum_{i=1}^{m} \left(\sum_{j=1}^{i} \|\ell^a_{ij}(m+1) x_j\| - \left\|\sum_{j=1}^{i} \ell^a_{ij}(m+1) x_j\right\| \right).
\end{aligned}$$

Therefore, by using induction, we conclude that

$$\sum_{i=1}^{m+1} \left(\sum_{j=1}^{i} \|\ell^a_{ij}(m) x_j\| - \left\|\sum_{j=1}^{i} \ell^a_{ij}(m) x_j\right\| \right) \leqslant \sum_{j=1}^{m+1} \|x_j\| - \left\|\sum_{j=1}^{m+1} x_j\right\|.$$

This completes the proof. □

Let $x_1, x_2, \cdots, x_n$ in X. For each a in L_n, if we put

$$f(a) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell^a_{ij}(n) x_j\| - \left\|\sum_{j=1}^{i} \ell^a_{ij}(n) x_j\right\| \right),$$

then we see that

$$f(a_0) = 0 \quad \text{and} \quad f(a_1) = \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|,$$

where $a_0, a_1 \in L_n$ with

$$a_0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

Moreover, by Lemma 3.1, we see that, for each $a = (a_{ij}),\, b = (b_{ij}) \in L_n$,

$$\begin{aligned} |f(a) - f(b)| &\leqslant 2 \sum_{i=2}^{n} \sum_{j=1}^{i} |\ell_{ij}^{a}(n) - \ell_{ij}^{b}(n)| \cdot \|x_j\| \\ &\leqslant 2 \max_{1 \leqslant j \leqslant n} \|x_j\| \cdot \sum_{i=2}^{n} \sum_{j=1}^{i} \sum_{k=i}^{n} |a_{ki} - b_{ki}|. \end{aligned}$$

Thus, considering f to a function on $\prod_{i=1}^{n(n+1)/2}[0,1]$, f is continuous on $\prod_{i=1}^{n(n+1)/2}[0,1]$. Therefore, as a solution of Problem 1.2, we obtain the following

COROLLARY 3.3. *Keep the notations as above. Let* $x_1, x_2, \cdots, x_n \in X$. *For each* ω *with* $0 \leqslant \omega \leqslant \sum_{j=1}^{n} \|x_j\| - \left\|\sum_{j=1}^{n} x_j\right\|$, *there exists* a *in* L_n *such that*

$$\omega = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^{a}(n) x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^{a}(n) x_j \right\| \right).$$

Of course, Theorem 3.2 contains Theorem 1.4 and 1.5.

COROLLARY 3.4. ([8, Theorem 1]) *For all nonzero elements* $x_1, x_2, \cdots, x_n$ *in a Banach space* X, *the following inequalities hold*

$$0 \leqslant \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leqslant j \leqslant n} \|x_j\| \leqslant \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|.$$

Proof. We may assume that $\|x_n\| = \min_{1 \leqslant j \leqslant n} \|x_j\|$. If we take $a \in L_n$ as

$$a = \begin{pmatrix} 0 & & & \\ \vdots & \ddots & \mathbf{0} & \\ 0 & \dots & 0 & \\ \frac{\|x_n\|}{\|x_1\|} & \dots & \frac{\|x_n\|}{\|x_{n-1}\|} & \frac{\|x_n\|}{\|x_n\|} \end{pmatrix},$$

then it is clear that

$$\ell_{ij}^a(n) = 0 \ (1 \leqslant i \leqslant n-1) \quad \text{and} \quad \ell_{nj}^a(n) = \frac{\|x_n\|}{\|x_j\|} \quad (1 \leqslant j \leqslant n).$$

In this case, we see that

$$\begin{aligned} f(a) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n)x_j\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n)x_j \right\| \right) \\ &= \sum_{j=1}^{n} \|\ell_{nj}^a(n)x_j\| - \left\| \sum_{j=1}^{n} \ell_{nj}^a(n)x_j \right\| = \sum_{j=1}^{n} \left\| \frac{\|x_n\|}{\|x_j\|} x_j \right\| - \left\| \sum_{j=1}^{n} \frac{\|x_n\|}{\|x_j\|} x_j \right\| \\ &= \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leqslant j \leqslant n} \|x_j\|. \end{aligned}$$

Thus, by Theorem 3.2, we have

$$0 \leqslant \left(n - \left\| \sum_{j=1}^{n} \frac{x_j}{\|x_j\|} \right\| \right) \min_{1 \leqslant j \leqslant n} \|x_j\| \leqslant \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|$$

This completes the proof. □

COROLLARY 3.5. ([12, Theorem 1]) *For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X, the following inequalities hold*

$$0 \leqslant \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|) \leqslant \sum_{j=1}^{n} \|x_j\| - \left\| \sum_{j=1}^{n} x_j \right\|,$$

where $x_1^*, \cdots, x_n^*$ are the rearrangement of $x_1, x_2, \cdots, x_n$ which satisfies $\|x_1^*\| \geqslant \|x_2^*\| \geqslant \cdots \geqslant \|x_n^*\|$ and $x_{n+1}^* = 0$.

Proof. Let us first show in the case $\|x_1^*\| > \|x_2^*\| > \cdots \|x_n^*\|$. If we take $a = (a_{ij}) \in L_n$ as

$$a_{ij} = \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \qquad (1 \leqslant j \leqslant i \leqslant n),$$

then, for all $j \leqslant i$ with $2 \leqslant i \leqslant n$, we have $\ell_{ij}^a(n) = \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|}$. Indeed, when $2 \leqslant j \leqslant n-1$, for all i with $1 \leqslant j \leqslant i$, we have

$$\begin{aligned} \ell_{ij}^a(n) &= a_{ij} \prod_{k=i+1}^{n} (1 - a_{kj}) \\ &= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \prod_{k=i+1}^{n} \left(1 - \frac{\|x_k^*\| - \|x_{k+1}^*\|}{\|x_j^*\| - \|x_{k+1}^*\|} \right) \end{aligned}$$

$$= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \prod_{k=i+1}^{n} \frac{\|x_j^*\| - \|x_k^*\|}{\|x_j^*\| - \|x_{k+1}^*\|}$$
$$= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+1}^*\|} \cdot \frac{\|x_j^*\| - \|x_{i+1}^*\|}{\|x_j^*\| - \|x_{i+2}^*\|} \cdot \frac{\|x_j^*\| - \|x_{i+2}^*\|}{\|x_j^*\| - \|x_{i+3}^*\|} \cdots \frac{\|x_j^*\| - \|x_n^*\|}{\|x_j^*\| - \|x_{n+1}^*\|}$$
$$= \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|}.$$

Moreover, when $j = n$, we see that $\ell_{in}^a(n) = \frac{\|x_n^*\|}{\|x_j^*\|} = \frac{\|x_n^*\| - \|x_{n+1}^*\|}{\|x_j^*\|}$. In this case, we see that

$$f(a) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} \|\ell_{ij}^a(n) x_j^*\| - \left\| \sum_{j=1}^{i} \ell_{ij}^a(n) x_j^* \right\| \right)$$
$$= \sum_{j=2}^{n} \left(\sum_{j=1}^{i} \left\| \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|} x_j^* \right\| - \left\| \sum_{j=1}^{i} \frac{\|x_i^*\| - \|x_{i+1}^*\|}{\|x_j^*\|} x_j^* \right\| \right)$$
$$= \sum_{i=2}^{n} \left\{ \sum_{j=1}^{i} \left\| \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) \right\}$$
$$= \sum_{i=2}^{n} \left\{ \sum_{j=1}^{i} (\|x_i^*\| - \|x_{i+1}^*\|) - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| (\|x_i^*\| - \|x_{i+1}^*\|) \right\}$$
$$= \sum_{i=1}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|).$$

Applying Theorem 3.2, we have

$$0 \leqslant \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_j^*}{\|x_j^*\|} \right\| \right) (\|x_i^*\| - \|x_{i+1}^*\|) \leqslant \sum_{j=1}^{n} \|x_j^*\| - \left\| \sum_{j=1}^{n} x_j^* \right\|.$$

We next prove a general case by using technique in the proof of [13, Theorem 2]. Let $\|x_1^*\| \geqslant \|x_2^*\| \geqslant \cdots \geqslant \|x_n^*\|$. For each fixed integer m with $m > n$, we set

$$x_{i,m}^* = \left(1 - \frac{i}{m}\right) x_i^* \quad (1 \leqslant i \leqslant n).$$

Then we see that $\|x_{1,m}^*\| > \|x_{2,m}^*\| > \cdots > \|x_{n,m}^*\| > 0$, and so we have

$$0 \leqslant \sum_{i=2}^{n} \left(i - \left\| \sum_{j=1}^{i} \frac{x_{j,m}^*}{\|x_{j,m}^*\|} \right\| \right) (\|x_{i,m}^*\| - \|x_{i+1,m}^*\|) \leqslant \sum_{j=1}^{n} \|x_{j,m}^*\| - \left\| \sum_{j=1}^{n} x_{j,m}^* \right\|,$$

where $x_{n+1,m}^* = 0$. Since $x_{i,m}^* \to x_i^*$ $(m \to \infty)$, we have Theorem 1.5. This completes the proof. □

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