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ANOTHER APPROACH FOR SANO'S CHARACTERIZATION OF THE $J-$CHAOTIC ORDER

G. SOARES

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Abstract. A selfadjoint involutive matrix J endows $\mathbb{C}^n$ with the indefinite Krein space structure endowed by the inner product $[\cdot,\cdot]$ given by $[x,y] := \langle Jx,y\rangle$, $x,y \in \mathbb{C}^n$. For a pair of $J-$selfadjoint matrices A,B with positive eigenvalues, $\operatorname{Log}A \geqslant^J \operatorname{Log}B$ is called the $J-$*chaotic order* or the *indefinite chaotic order*. Sano [8], proved as an application of Furuta inequality of indefinite type that $\operatorname{Log}A \geqslant^J \operatorname{Log}B$ if and only if $A^r \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ for all $p>0$ and $r>0$. In this paper, we prove Sano's result using a different approach. In the process, some other results due to Bebiano, Lemos, Providência and Soares [4, 9] are reobtained. The techniques in this paper are inspired by [5].

1. Introduction

Let M_n denote the algebra of $n\times n$ complex matrices. For a selfadjoint involution $J\in M_n$: $J=J^*$ and $J^2=I_n$, we consider $\mathbb{C}^n$ with a Krein space structure induced by the indefinite inner product $[x,y]=\langle Jx,y\rangle$, $x,y\in\mathbb{C}^n$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbb{C}^n$. A matrix $A\in M_n$ is said to be J-*selfadjoint* if $A=A^{[*]}$, where $A^{[*]}=JA^*J$ denotes the J-*adjoint* matrix of A. For J-selfadjoint matrices $A,B\in M_n$, we define the $J-$*order relation* $A\geqslant^J B$ by $[Ax,x]\geqslant[Bx,x]$, $x\in\mathbb{C}^n$, which means that the selfadjoint matrix $J(A-B)$ is positive semidefinite. If $A\in M_n$ is J-selfadjoint and $I_n\geqslant^J A$, then all the eigenvalues of A are real, because I_n-A is the product of the selfadjoint involution J and a positive semidefinite matrix. A matrix $A\in M_n$ is called a J-*contraction* if $I_n\geqslant^J A^{[*]}A$ and a J-*expansion* if $A^{[*]}A\geqslant^J I_n$. When A is a J-contraction (or J-expansion), then all the eigenvalues of the product $A^{[*]}A$ are nonnegative [2].

Let $A,B\in M_n$ be J-selfadjoint matrices with nonnegative eigenvalues such that $I_n\geqslant^J A\geqslant^J B$, then $I_n\geqslant^J A^\alpha\geqslant^J B^\alpha$ holds [7], $0\leqslant\alpha\leqslant 1$. This result is known as the Löwner inequality of indefinite type, obtained by Ando [1] for the particular case $\alpha=\frac{1}{2}$. Motivated by these results, the Furuta inequality of indefinite type [3, 7] was established in the following way:

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THEOREM 1.1. *Let $A,B \in M_n$ be J-selfadjoint with nonnegative eigenvalues and $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$. For each $r \geqslant 0$, the following inequalities hold*

$$\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}, \tag{1}$$

$$\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geqslant^J \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}, \tag{2}$$

for all $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

If we consider $r=0$ in Theorem 1.1, we obtain Löwner inequality of indefinite type.For J-selfadjoint matrices $A,B \in M_n$ with positive eigenvalues, the *J-chaotic order* is defined by $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$, where $\mathrm{Log}(t)$ denotes the principal branch of the logarithm function and is weaker than the usual J-order relation $A \geqslant^J B$. Some characterizations of the J-chaotic order were obtained [3, 4, 8], using Furuta inequality of indefinite type.

Let $A,B \in M_n$ be J-selfadjoint matrices with nonnegative eigenvalues such that $A \geqslant^J B$ (or $B \geqslant^J A$) and let $0 \leqslant \alpha \leqslant 1$. If A is an invertible matrix, then $I_n \geqslant^J A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ (or $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \geqslant^J I_n$) and the J-selfadjoint power $\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha}$ is well defined. Under these assumptions, the *α-power mean* of A and B is defined by

$$A\sharp_\alpha B \;=\; A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha}A^{\frac{1}{2}}.$$

The following theorem on the $\alpha-$power mean is the corresponding result to one due to Fujii, Kamei and Nakamoto [5] in the case of bounded linear operators on a Hilbert space.

THEOREM 1.2. *Let A,B be $J-$selfadjoint matrices with positive eigenvalues such that $\mu I_n \geqslant^J A \geqslant^J B$, for some $\mu > 0$. Then*

$$I_n \geqslant^J A^{-r}\sharp_{\frac{r}{p+r}}B^p \quad \text{and} \quad B^{-r}\sharp_{\frac{r}{p+r}}A^p \geqslant^J I_n$$

for $r \geqslant 0$ and $p \geqslant 0$.

This note is organized as follows. In section 2, we present a proof of Theorem 1.2 using an analogous approach as the the one used by Fujii, Kamei and Nakamoto [5] for Hilbert spaces. In section 2, we reobtain some results already proved in [4, 8, 9].

2. Proof of Theorem 1.2

To prove Theorem 1.2, we recall the following useful lemma.

LEMMA 2.1. *For $J-$selfadjoint matrices A,B with positive eigenvalues, $A \geqslant^J B$ (or $A \leqslant^J B$) and $\alpha,\beta \in [0,1]$. Then*

(i) $(A\sharp_\alpha B)^{-1} = A^{-1}\sharp_\alpha B^{-1}$;

(ii) $A\natural_{\alpha\beta}B = A\natural_{\alpha}\left(A\natural_{\beta}B\right)$;

(iii) $A\natural_{\alpha}B = A\left(A^{-1}\natural_{-\alpha}B^{-1}\right)A$.

The proof of the previous lemma follows directly from the definition of $\alpha-$power mean. In the next proposition, we consider some additional properties of the α-power mean, namely the effect of interchanging A and B and the J*-monotonicity* of $\natural_{\alpha}$ in the second variable (see [4] for the proof).

LEMMA 2.2. *Let $A,B,C \in M_n$ be J-selfadjoint matrices with positive eigenvalues and $0 \leqslant \alpha \leqslant 1$.*

(i) *If $A \geqslant^J \mu I_n \geqslant^J B$ or ($B \geqslant^J \mu I_n \geqslant^J A$) for some $\mu > 0$, then $A\#_{\alpha}B = B\#_{1-\alpha}A$.*

(ii) *If $A \geqslant^J B$ (or $B \geqslant^J A$) and $A \geqslant^J C$ (or $C \geqslant^J A$), then $B \geqslant^J C$ implies $A\natural_{\alpha}B \geqslant^J A\natural_{\alpha}C$.*

The following lemmas will be used throughout this section.

LEMMA 2.3. [7] *If A,B are $J-$selfadjoint matrices with positive eigenvalues and $A \geqslant^J B$, then $B^{-1} \geqslant^J A^{-1}$.*

LEMMA 2.4. [3] *Let $A,B \in M_n$ be $J-$selfadjoint matrices. Then $X^{[*]}AX \geqslant^J X^{[*]}BX$ for all $X \in M_n$ if and only if $A \geqslant^J B$.*

LEMMA 2.5. [7] *Let A be a $J-$selfadjoint matrix with nonnegative eigenvalues and $I_n \geqslant^J A$. Then $I_n \geqslant^J A^{\lambda}$ for all $\lambda > 0$.*

LEMMA 2.6. *Let A,B be $J-$selfadjoint matrices with positive eigenvalues such that $\mu I_n \geqslant^J A \geqslant^J B$, for some $\mu > 0$. Then*

$$I_n \geqslant^J A^{-m}\natural_{\frac{m}{p+m}}B^p \quad \text{and} \quad B^{-m}\natural_{\frac{m}{p+m}}A^p \geqslant^J I_n$$

for $m = 1,2,\ldots,$ and $p \geqslant 0$.

Proof. Without loss of generality we may consider $\mu = 1$. Otherwise, replace A and B by $A_{\mu} = \frac{1}{\mu}A$, $B_{\mu} = \frac{1}{\mu}B$, respectively. The proof of $I_n \geqslant^J A^{-m}\natural_{\frac{m}{p+m}}B^p$ is made by induction on m.

Since $A^{-1} \geqslant^J I_n \geqslant^J B^p$ hold by Lemmas 2.3 and 2.5 and $0 \leqslant \frac{1}{p+1} \leqslant 1$, then by Lemma 2.2 (i),

$$A^{-1}\natural_{\frac{1}{p+1}}B^p = B^p\natural_{\frac{p}{p+1}}A^{-1}. \tag{3}$$

From Lemmas 2.3 and 2.5, we conclude that $B^{-1} \geqslant^J I_n \geqslant^J B^p$ and $A^{-1} \geqslant^J I_n \geqslant^J B^p$. By hypothesis $A \geqslant^J B$, so applying Lemma 2.3, we have $B^{-1} \geqslant^J A^{-1}$, which implies by Lemma 2.2 (ii),

$$I_n = B^p\natural_{\frac{p}{p+1}}B^{-1} \geqslant^J B^p\natural_{\frac{p}{p+1}}A^{-1}. \tag{4}$$

Hence, from (3) and (4), we have the case $m=1$.

Suppose now, that $I_n \geqslant^J A^{-m}\sharp_{\frac{m}{p+m}} B^p$, holds for a given m. By Lemmas 2.4 and 2.5, this is equivalent to $I_n \geqslant^J A^m \geqslant^J (A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{\frac{m}{p+m}}$. Since $0<\frac{1}{m}\leqslant 1$, applying Löwner inequality of indefinite type, we get

$$I_n \geqslant^J A \geqslant^J (A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{\frac{1}{p+m}}. \tag{5}$$

Consider, now, the case $m+1$. From Lemmas 2.3 and 2.5, we may conclude that $A^{-(m+1)} \geqslant^J I_n \geqslant^J B^p$. By other hand, since $I_n \geqslant^J B^p$ (Lemma 2.5), this implies by Lemmas 2.4, 2.3 and 2.5, $A^{-1} \geqslant^J I_n \geqslant^J A^m \geqslant^J A^{\frac{m}{2}}B^pA^{\frac{m}{2}}$. Henceforth, applying Lemma 2.2, (i), we get

$$A^{-m-1}\sharp_{\frac{m+1}{p+m+1}} B^p = A^{-\frac{m}{2}}\left(A^{-1}\sharp_{\frac{m+1}{p+m+1}} A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\right)A^{-\frac{m}{2}} = A^{-\frac{m}{2}}\left(A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}} A^{-1}\right)A^{-\frac{m}{2}}. \tag{6}$$

We proved that $I_n \geqslant^J A^{\frac{m}{2}}B^pA^{\frac{m}{2}}$, so by Lemmas 2.5 and 2.3, we may conclude $(A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{-\frac{1}{p+m}} \geqslant^J I_n \geqslant^J A^{\frac{m}{2}}B^pA^{\frac{m}{2}}$. Applying Lemma 2.3 to (5), we obtain $(A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{\frac{-1}{p+m}} \geqslant^J A^{-1}$ which implies by Lemma 2.2 (ii),

$$A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}}\left(A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\right)^{\frac{-1}{p+m}} \geqslant^J A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}} A^{-1}$$

which is equivalent by Lemma 2.4

$$A^{-\frac{m}{2}}\left(A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}}(A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{\frac{-1}{p+m}}\right)A^{-\frac{m}{2}} \geqslant^J A^{-\frac{m}{2}}\left(A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}} A^{-1}\right)A^{-\frac{m}{2}} \tag{7}$$

By easy computations, we can show that

$$A^{-m}\sharp_{\frac{m}{p+m}} B^p = A^{-\frac{m}{2}}\left(A^{\frac{m}{2}}B^pA^{\frac{m}{2}}\sharp_{\frac{p}{p+m+1}}(A^{\frac{m}{2}}B^pA^{\frac{m}{2}})^{\frac{-1}{p+m}}\right)A^{-\frac{m}{2}}, \tag{8}$$

Hence, by (6), (7), (8) and the inductive hypothesis, we get

$$I_n \geqslant^J A^{-m}\sharp_{\frac{m}{p+m}} B^p \geqslant^J A^{-m-1}\sharp_{\frac{m+1}{p+m+1}} B^p.$$

To prove $B^{-m}\sharp_{\frac{m}{p+m}} A^p \geqslant^J I_n$, we proceed in an analogous way. □

Proof of Theorem 1.2. Without loss of generality we may consider $\mu=1$. Otherwise, replace A and B by $A_\mu = \frac{1}{\mu}A$, $B_\mu = \frac{1}{\mu}B$, respectively.

We first prove $I_n \geqslant^J A^{-r}\sharp_{\frac{r}{p+r}} B^p$. Let us consider the case $0\leqslant r\leqslant 1$. From Lemmas 2.3 and 2.5, we conclude $A^{-r} \geqslant^J I_n \geqslant^J B^p$, and so by Lemma 2.2 (i),

$$A^{-r}\sharp_{\frac{r}{p+r}} B^p = B^p\sharp_{\frac{p}{p+r}} A^{-r}. \tag{9}$$

From Lemmas 2.3 and 2.5, $B^{-r} \geqslant^J I_n \geqslant^J B^p$ and $A^{-r} \geqslant^J I_n \geqslant^J B^p$. On the other hand, by hypothesis, $I_n \geqslant^J A \geqslant^J B$. Using Löwner inequality of indefinite type, $I_n \geqslant^J A^r \geqslant^J B^r$ and by Lemma 2.3, $B^{-r} \geqslant^J A^{-r}$, which implies by Lemma 2.2 (ii),

$$I_n = B^p\sharp_{\frac{p}{p+r}} B^{-r} \geqslant^J B^p\sharp_{\frac{p}{p+r}} A^{-r}. \tag{10}$$

We proved the theorem in the case $0 \leqslant r \leqslant 1$, having in mind (9) and (10).

From Lemmas 2.4, 2.5 and 2.6, we get

$$I_n \geqslant^J A^m \geqslant^J (A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{m}{p+m}}. \tag{11}$$

Let $r = m + \varepsilon$, for positive integer m and $0 \leqslant \varepsilon < 1$. Since $0 \leqslant \frac{\varepsilon}{m} < 1$, applying Löwner inequality of indefinite type to (11), we get

$$I_n \geqslant^J A^{\varepsilon} \geqslant^J (A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{\varepsilon}{p+m}}. \tag{12}$$

By other hand, since $I_n \geqslant^J B^p$ (Lemma 2.5), this implies by Lemmas 2.3, 2.4, and 2.5, $A^{-\varepsilon} \geqslant^J I_n \geqslant^J A^m \geqslant^J A^{\frac{m}{2}} B^p A^{\frac{m}{2}}$. We have from Lemma 2.2, (i), that

$$\begin{aligned} A^{-r} \natural_{\frac{r}{p+r}} B^p &= A^{-(m+\varepsilon)} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} B^p = A^{-\frac{m}{2}} \left(A^{-\varepsilon} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \right) A^{-\frac{m}{2}} \\ &= A^{-\frac{m}{2}} \left(A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} A^{-\varepsilon} \right) A^{-\frac{m}{2}}. \end{aligned} \tag{13}$$

We proved that $I_n \geqslant^J A^{\frac{m}{2}} B^p A^{\frac{m}{2}}$, so by Lemmas 2.5 and 2.3, we may conclude $(A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{-\frac{\varepsilon}{p+m}} \geqslant^J I_n \geqslant^J A^{\frac{m}{2}} B^p A^{\frac{m}{2}}$. Applying Lemma 2.3 to (12), we obtain $(A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{-\varepsilon}{p+m}} \geqslant^J A^{-\varepsilon}$ which implies by Lemma 2.2 (ii)

$$A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} (A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{-\varepsilon}{p+m}} \geqslant^J A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} A^{-\varepsilon} \tag{14}$$

which is equivalent by Lemma 2.4

$$A^{-\frac{m}{2}} \left(A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} (A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{-\varepsilon}{p+m}} \right) A^{-\frac{m}{2}} \geqslant^J A^{-\frac{m}{2}} \left(A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} A^{-\varepsilon} \right) A^{-\frac{m}{2}} \tag{15}$$

By easy computations, we can show that

$$A^{-\frac{m}{2}} \left(A^{\frac{m}{2}} B^p A^{\frac{m}{2}} \natural_{\frac{m+\varepsilon}{p+m+\varepsilon}} (A^{\frac{m}{2}} B^p A^{\frac{m}{2}})^{\frac{-\varepsilon}{p+m}} \right) A^{-\frac{m}{2}} = A^{-m} \natural_{\frac{m}{p+m}} B^p$$

Hence, by Lemma 2.6 and by (13), (14) and (15) we get

$$I_n \geqslant^J A^{-m} \natural_{\frac{m}{p+m}} B^p \geqslant^J A^{-r} \natural_{\frac{r}{p+r}} B^p.$$

The proof of $B^{-r} \natural_{\frac{r}{p+r}} A^p \geqslant^J I_n$ is made in an analogous way. □

3. Some consequences of Theorem 1.2

In this section, we reobtain some known results, but first we need the following lemma.

LEMMA 3.1. *If $A, B \in M_n$ are J-selfadjoint matrices with positive eigenvalues such that $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$, then $A^{-r} \natural_{\frac{r}{p+r}} B^p \geqslant^J B^p$ and $A^{-r} \geqslant^J A^{-r} \natural_{\frac{r}{p+r}} B^p$ for $r \geqslant 0$ and $p \geqslant 0$.*

Proof. Since $I_n \geqslant^J A^r$, from Lemmas 2.4 and 2.5, we may conclude $I_n \geqslant^J B^p \geqslant^J B^{\frac{p}{2}} A^r B^{\frac{p}{2}}$, which is equivalent by Lemmas 2.5 and 2.3 to $\left(B^{\frac{p}{2}} A^r B^{\frac{p}{2}}\right)^{-\frac{p}{p+r}} \geqslant^J I_n$. Applying Lemma 2.4, we have $B^p \sharp_{\frac{p}{p+r}} A^{-r} \geqslant^J B^p$ and the lemma is proved using Lemma 2.2 (i) and nothing that $A^{-r} \geqslant^J I_n \geqslant^J B^p$.

The proof of $A^{-r} \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p$ follows analogous steps. □

The next result is obtained in a quite easy way from Theorem 1.2 and is known as indefinite version of Kamei's satellite to Furuta inequality (see [4] for another proof).

THEOREM 3.1. *If $A, B \in M_n$ are J-selfadjoint matrices with positive eigenvalues such that $\mu I_n \geqslant^J A \geqslant^J B$ for some $\mu > 0$, then*

$$B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \geqslant^J A \geqslant^J B \geqslant^J A^{-r} \sharp_{\frac{1+r}{p+r}} B^p$$

for all $p \geqslant 1$ and $r \geqslant 0$.

Proof. From Lemmas 2.3 and 2.5, we get $A^{-r} \geqslant^J I_n \geqslant^J B^p$, so by Lemma 2.2 (i) and Lemma 2.1 (ii), we conclude

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p = B^p \sharp_{\frac{p-1}{p+r}} A^{-r} = B^p \sharp_{\frac{p-1}{p}} \left(B^p \sharp_{\frac{p}{p+r}} A^{-r}\right) = B^p \sharp_{\frac{p-1}{p}} \left(A^{-r} \sharp_{\frac{r}{p+r}} B^p\right). \tag{16}$$

By Lemmas 3.1 and 2.5, $A^{-r} \sharp_{\frac{r}{p+r}} B^p \geqslant^J B^p$ and $I_n \geqslant^J B^p$, respectively. We can conclude by Theorem 1.2, that $I_n \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p$, which implies by Lemma 2.2 (ii) and (16)

$$A \geqslant^J B = B^p \sharp_{\frac{p-1}{p}} I_n \geqslant^J B^p \sharp_{\frac{p-1}{p}} \left(A^{-r} \sharp_{\frac{r}{p+r}} B^p\right).$$

The proof of $B^{-r} \sharp_{\frac{1+r}{p+r}} A^p \geqslant^J A$ is made in an analogous way. □

Some characterizations of the J-chaotic order were obtained in [3, 8], using Furuta inequality of indefinite type. For example, Sano [8] obtained the following useful result.

THEOREM 3.2. *Let $A, B \in M_n$ be J-selfadjoint matrices with positive eigenvalues such that $I_n \geqslant^J A$ and $I_n \geqslant^J B$. Then $\mathrm{Log}(A) \geqslant^J \mathrm{Log}(B)$ if and only if $A^r \geqslant^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$ for all $p > 0$ and $r > 0$.*

Next we will obtain Sano result's without the use of Furuta inequality of indefinite type. First, we recall that $e^{\mathrm{Log} X} = X$ for any matrix X with all eigenvalues positive and we recall the following essential formula:

$$\lim_{m \to \infty} \left(I_n + \frac{1}{m} A\right)^m = e^A,$$

which holds for any matrix A.

THEOREM 3.3. *Let A,B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I \geqslant^J A,\ \mu I \geqslant^J B$, for some $\mu > 0$. Then* $\operatorname{Log} A \geqslant^J \operatorname{Log} B$ *if and only if*

$$I_n \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p \quad \text{and} \quad B^{-r} \sharp_{\frac{r}{p+r}} A^p \geqslant^J I_n$$

for $r \geqslant 0$ and $p \geqslant 0$.

Proof. Without loss of generality we may consider $\mu = 1$. Otherwise, replace A, B by $A_\mu = \frac{1}{\mu} A$, $B_\mu = \frac{1}{\mu} B$, respectively.

We prove the equivalence between $\operatorname{Log} A \geqslant^J \operatorname{Log} B$ and $I_n \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p$. The other equivalence is obtained in an analogous way.

$(\Rightarrow)$ Since $I_n \geqslant^J A, B$, then $0 = \operatorname{Log} I_n \geqslant^J \operatorname{Log} A, \operatorname{Log} B$, because $\operatorname{Log} x$ is operator monotone. Suppose $\operatorname{Log} A \geqslant^J \operatorname{Log} B$, then

$$A_1 = I_n + \frac{1}{m} \operatorname{Log} A \geqslant^J I_n + \frac{1}{m} \operatorname{Log} B = B_1,$$

holds for sufficiently large natural number m.

By Lemmas 2.3 and 2.5, $A_1^{-mr} \geqslant^J I_n \geqslant^J B_1^{mp}$, applying Theorem 1.2 to A_1 and B_1, we have

$$I_n \geqslant^J A_1^{-mr} \sharp_{\frac{mr}{mp+mr}} B_1^{mp} = \left(I_n + \frac{1}{m} \operatorname{Log} A\right)^{-mr} \sharp_{\frac{r}{p+r}} \left(I_n + \frac{1}{m} \operatorname{Log} B\right)^{mp}. \tag{17}$$

Since, A and B are invertible, we obtain

$$\lim_{m\to\infty} \left(I_n + \frac{1}{m} \operatorname{Log} A\right)^{-mr} = A^{-r} \quad \text{and} \quad \lim_{m\to\infty} \left(I_n + \frac{1}{m} \operatorname{Log} B\right)^{mp} = B^p.$$

Henceforth from (17), we get

$$I_n \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p.$$

$(\Leftarrow)$ Suppose that $I_n \geqslant^J A^{-r} \sharp_{\frac{r}{p+r}} B^p$. This implies, $A^r \geqslant^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$. Since $\operatorname{Log} x$ is an increasing function of x, we get $\operatorname{Log} A \geqslant^J \operatorname{Log} (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{p+r}}$, for all $r \geqslant 0$ and $p \geqslant 0$. Considering $r = 0$, we get $\operatorname{Log} A \geqslant^J \operatorname{Log} B$. $\square$

The following result is understood as a corollary of the previous theorem and was obtained in [9, Theorem 3.5] using Löwner inequality of indefinite type. Next we present another simple proof.

COROLLARY 3.1. *Let A,B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I_n \geqslant^J A,\ \mu I_n \geqslant^J B$, for some $\mu > 0$. Then* $\operatorname{Log} A \geqslant^J \operatorname{Log} B$, *if and only if*

(i) $B^\delta \geqslant^J A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$, *for $0 \leqslant \delta \leqslant p$ and $r \geqslant 0$.*

(ii) $A^\delta \geqslant^J A^{-r} \sharp_{\frac{\delta+r}{p+r}} B^p$, *for $-r \leqslant \delta \leqslant 0$ and $p \geqslant 0$.*

Proof. Without loss of generality we may consider $\mu = 1$. Otherwise, replace A, B by $A_\mu = \frac{1}{\mu}A$, $B_\mu = \frac{1}{\mu}B$, respectively.

(i) Since $A^{-r} \geqslant^J I_n \geqslant^J B^p$, by Lemma 2.2 (i) and Lemma 2.1 (ii), we have

$$A^{-r}\sharp_{\frac{\delta+r}{p+r}}B^p = B^p\sharp_{\frac{p-\delta}{p+r}}A^{-r} = B^p\sharp_{\frac{p-\delta}{p}}\left(B^p\sharp_{\frac{p}{p+r}}A^{-r}\right) = B^p\sharp_{\frac{p-\delta}{p}}\left(A^{-r}\sharp_{\frac{r}{p+r}}B^p\right). \tag{18}$$

By Lemma 3.1 and Lemma 2.5, $A^{-r}\sharp_{\frac{r}{p+r}}B^p \geqslant^J B^p$ and $I_n \geqslant^J B^p$, respectively. Applying Theorem 3.3, we have that $\operatorname{Log} A \geqslant^J \operatorname{Log} B$, is equivalent to $I_n \geqslant^J A^{-r}\sharp_{\frac{r}{p+r}}B^p$, which implies by Lemma 2.2 (ii)

$$B^\delta = B^p\sharp_{\frac{p-\delta}{p}}I_n \geqslant^J B^p\sharp_{\frac{p-\delta}{p}}\left(A^{-r}\sharp_{\frac{r}{p+r}}B^p\right). \tag{19}$$

The result follows from (18).

(ii) Since $A^{-r} \geqslant^J I_n \geqslant^J B^p$, by Lemma 2.1 (ii), we have

$$A^{-r}\sharp_{\frac{\delta+r}{p+r}}B^p = A^{-r}\sharp_{\frac{\delta+r}{r}}\left(A^{-r}\sharp_{\frac{r}{p+r}}B^p\right). \tag{20}$$

By Lemma 3.1, Lemma 2.5 and Lemma 2.3, $A^{-r} \geqslant^J A^{-r}\sharp_{\frac{r}{p+r}}B^p$ and $A^{-r} \geqslant^J I_n$. Applying Theorem 3.3, we have that $\operatorname{Log} A \geqslant^J \operatorname{Log} B$, is equivalent to $I_n \geqslant^J A^{-r}\sharp_{\frac{r}{p+r}}B^p$, which implies by Lemma 2.2 (ii)

$$A^\delta = A^{-r}\sharp_{\frac{\delta+r}{p}}I_n \geqslant^J A^{-r}\sharp_{\frac{\delta+r}{r}}\left(A^{-r}\sharp_{\frac{r}{p+r}}B^p\right).$$

The result follows from (20). □

A real valued continuous function f defined on a real interval $\mathscr{I}$ is said to be J-*increasing* if $f(r) \geqslant^J f(s)$ whenever $r \geqslant s$. Analogously, f is said to be J-*decreasing* if $f(r) \geqslant^J f(s)$ whenever $r \leqslant s$. Concluding this section, we point out the monotonicity of an operator function $A^{-r}\sharp_{\frac{r}{p+r}}B^p$ for $p \geqslant 0$ and $r \geqslant 0$, that was previously obtained in [4].

THEOREM 3.4. *Let A,B be $J-$selfadjoint matrices with positive eigenvalues and $\mu I_n \geqslant^J A$, $\mu I_n \geqslant^J B$, for some $\mu > 0$. Then* $\operatorname{Log} A \geqslant^J \operatorname{Log} B$ *if and only if $A^{-r}\sharp_{\frac{r}{p+r}}B^p$, is $J-$decreasing for $r \geqslant 0$ and $p \geqslant 0$.*

Proof. Without loss of generality we may consider $\mu = 1$. Otherwise, replace A, B by $A_\mu = \frac{1}{\mu}A$, $B_\mu = \frac{1}{\mu}B$, respectively.

Let $p_1 \geqslant p$ and $r_1 \geqslant r$. Since $A^{-r_1} \geqslant^J I_n \geqslant^J B^{p_1}$, by Lemma 2.1 (ii), we have

$$A^{-r_1}\sharp_{\frac{r_1}{p_1+r_1}}B^{p_1} = A^{-r_1}\sharp_{\frac{r_1}{p+r_1}}\left(A^{-r_1}\sharp_{\frac{p+r_1}{p_1+r_1}}B^{p_1}\right). \tag{21}$$

In an analogous way as the proof of Lemma 3.1, we have $A^{-r_1} \geqslant^J A^{-r_1} \sharp_{\frac{p+r_1}{p_1+r_1}} B^{p_1}$. By Lemma 2.5 and Lemma 2.3, and $A^{-r_1} \geqslant^J I_n \geqslant^J B^p$. Applying Corollary 3.1 (i), Lemma 2.2 (ii), we get

$$B^p \sharp_{\frac{p}{p+r_1}} A^{-r_1} = A^{-r_1} \sharp_{\frac{r_1}{p+r_1}} B^p \geqslant^J A^{-r_1} \sharp_{\frac{r_1}{p+r_1}} \left(A^{-r_1} \sharp_{\frac{p+r_1}{p_1+r_1}} B^{p_1} \right). \tag{22}$$

From (21), we obtain $A^{-r_1} \sharp_{\frac{r_1}{p+r_1}} B^p \geqslant^J A^{-r_1} \sharp_{\frac{r_1}{p_1+r_1}} B^{p_1}$. Applying Lemma 2.1 (ii), to (22), we get

$$B^p \sharp_{\frac{p}{p+r_1}} A^{-r_1} = B^p \sharp_{\frac{p}{p+r}} \left(B^p \sharp_{\frac{p+r}{p+r_1}} A^{-r_1} \right) = B^p \sharp_{\frac{p}{p+r}} \left(A^{-r_1} \sharp_{\frac{r_1-r}{p+r_1}} B^p \right).$$

In an analogous way as the proof of Lemma 3.1, we have $A^{-r_1} \sharp_{\frac{r_1-r}{p+r_1}} B^p \geqslant^J B^p$. On the other hand, applying Corollary 3.1 (ii), Lemma 2.2 (ii), we get

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p = B^p \sharp_{\frac{p}{p+r}} A^{-r} \geqslant^J B^p \sharp_{\frac{p}{p+r}} \left(A^{-r_1} \sharp_{\frac{r_1-r}{p+r_1}} B^p \right).$$

Hence, we get

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \geqslant^J B^p \sharp_{\frac{p}{p+r_1}} A^{-r_1} \geqslant^J A^{-r_1} \sharp_{\frac{r_1}{p_1+r_1}} B^{p_1},$$

so we proved the desired result. □

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G. Soares
CM-UTAD, Department of Mathematics
University of Trás-os-Montes and Alto Douro