Mathematical Inequalities & Applications
Volume 16, Number 1 (2013), 21–39 doi:10.7153/mia-16-02

ON FRACTIONAL SMOOTHNESS OF FUNCTIONS RELATED TO p–VARIATION

MARTIN LIND

(*Communicated by Lars Erik Persson*)

Abstract. This paper is concerned with the study of two functionals of variational type - the Riesz type generalized variation $v_{p,\alpha}(f)$ $(1<p<\infty, 0\leqslant\alpha\leqslant 1-1/p)$ and the moduli of p-continuity $\omega_{1-1/p}(f;\delta)$. These functionals generate scales of spaces connecting the class V_p of functions of bounded p-variation and the Sobolev space W_p^1. Some limiting relations in these scales are proved. Sharp estimates of $v_{p,\alpha}(f)$ in terms of $\omega_{1-1/p}(f;\delta)$ are obtained.

1. Introduction

Let f be a periodic function with the period 1 on the real line. A set $\Pi=\{x_0,x_1,\ldots,x_n\}$ of points such that

$$x_0<x_1<\ldots<x_n, \quad \text{where} \quad x_n=x_0+1,$$

will be called *a partition of a period* (or simply *a partition*). Let $1\leqslant p<\infty$. For any partition Π, set

$$v_p(f;\Pi)=\left(\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)|^p\right)^{1/p}.$$

We say that f is *a function of bounded p-variation* and write $f\in V_p$ if

$$v_p(f)=\sup_{\Pi} v_p(f;\Pi)<\infty,$$

where the supremum is taken over all partitions Π. This definition was given by N. Wiener [17]. The following strict inclusions hold

$$V_p\subset V_q \quad \text{for} \quad 1\leqslant p<q<\infty.$$

For $1\leqslant p<\infty$, we denote by W_p^1 the class of all absolutely continuous 1-periodic functions f such that $f'\in L^p$. F. Riesz (see, e.g., [9, Ch. 9]) found a variational type characterization of W_p^1. This result was formulated in the framework of the following

Mathematics subject classification (2010): Primary 26A45, secondary 46E35.
Keywords and phrases: Functions of bounded p-variation, generalized variations, optimal constants, function spaces.

general definition (see [11]). Let $1<p<\infty$, $\alpha\geqslant 0$, and let f be a 1-periodic function. For any partition $\Pi=\{x_0,x_1,\dots,x_n\}$ of a period, we set

$$v_{p,\alpha}(f;\Pi)=\left(\sum_{k=0}^{n-1}\frac{|f(x_{k+1})-f(x_k)|^p}{(x_{k+1}-x_k)^{\alpha p}}\right)^{1/p}. \tag{1.1}$$

We denote by V_p^α the class of all 1-periodic functions f such that

$$v_{p,\alpha}(f)=\sup_\Pi v_{p,\alpha}(f;\Pi)<\infty,$$

where the supremum is taken over all partitions of a period. Let $p'=p/(p-1)$. If $\alpha>1/p'$, then V_p^α contains only constants. For $\alpha=1/p'$, the F. Riesz theorem states that a function f belongs to the class $V_p^{1/p'}$ if and only if $f\in W_p^1$. For $\alpha=0$ we obtain the class V_p. Thus, V_p^α for $0<\alpha<1/p'$ form a scale of spaces of fractional smoothness between V_p and W_p^1.

Another characterization of W_p^1 is given by moduli of p-continuity. For a partiton Π, denote $\|\Pi\|=\max_j(x_{j+1}-x_j)$. Wiener [17] introduced the function

$$\omega_{1-1/p}(f;\delta)=\sup_{\|\Pi\|\leqslant\delta} v_p(f;\Pi)\quad 0\leqslant\delta\leqslant 1, \tag{1.2}$$

where the supremum is taken over all partitions Π with $\|\Pi\|\leqslant\delta$. The function (1.2) is called the *modulus of p-continuity* of the function f. If $1<p<\infty$, then the equality

$$\lim_{\delta\to 0+}\omega_{1-1/p}(f;\delta)=0 \tag{1.3}$$

may hold for non-trivial functions. A function f satisfying (1.3) is called *p-continuous*. We denote by C_p the class of all p-continuous functions.

If $f\in W_p^1$ $(1<p<\infty)$, then

$$\omega_{1-1/p}(f;\delta)\leqslant\|f'\|_p\delta^{1/p'}. \tag{1.4}$$

Conversely, if

$$\omega_{1-1/p}(f;\delta)=O(\delta^{1/p'}),$$

then $f\in W_p^1$ (see [13]). Thus, the space W_p^1 can be also characterized in terms of moduli of p-continuity.

The main objectives of this paper are twofold:

1. to obtain sharp relations between $v_{p,\alpha}(f)$ and moduli of p-continuity;
2. to study limits in the scales generated by $v_{p,\alpha}(f)$ and $\omega_{1-1/p}(f;\delta)$.

Obviously, if $f\in V_p^\alpha$ $(0<\alpha\leqslant 1/p')$, then

$$\omega_{1-1/p}(f;\delta)=O(\delta^\alpha) \tag{1.5}$$

Moreover, as we have seen, for $\alpha = 1/p'$, the converse also holds. However, for $0 < \alpha < 1/p'$, the condition (1.5) does not imply that $f \in V_p^\alpha$. On the other hand, it is in general impossible to improve (1.5).

We prove (see Theorem 4.1 below) that if $f \in V_p \ (1 < p < \infty)$, $0 < \alpha < 1/p'$, and

$$I_{p,\alpha}(f) \equiv \left(\int_0^1 (t^{-\alpha}\omega_{1-1/p}(f;t))^p \frac{dt}{t} \right)^{1/p} < \infty, \tag{1.6}$$

then $f \in V_p^\alpha$ and

$$v_{p,\alpha}(f) \leqslant A[v_p(f) + c_{p,\alpha} I_{p,\alpha}(f)], \tag{1.7}$$

where A is an absolute constant and $c_{p,\alpha} = p'\alpha^{1/p}(1/p' - \alpha)^{1/p}$. Further, we show (see Theorem 4.5) that the condition (1.6) cannot be weakened *whatever be the order of decay of the modulus of* p*-continuity*. That is, if ω is any continuous nondecreasing function on $[0,1]$ such that $\omega(0) = 0$, $t^{-1/p'}\omega(t)$ is nonincreasing, and

$$\int_0^1 (t^{-\alpha}\omega(t))^p \frac{dt}{t} = \infty,$$

then there exists a function f such that $\omega_{1-1/p}(f;\delta) \leqslant \omega(\delta) \ (0 \leqslant \delta \leqslant 1)$ and $f \notin V_p^\alpha$.

It is also important to stress that the constant $c_{p,\alpha}$ in (1.7) has the optimal asymptotic behaviour as $\alpha \to 0+$ and $\alpha \to 1/p'-$. Actually, an inequality of the type (1.7) could be derived from estimates of $v_{p,\alpha}(f)$ via L^p-moduli of continuity obtained in [6] (see Remark 4.4 below). However, the constant obtained in this way is not optimal.

For the functionals $v_{p,\alpha}(f)$ we prove the following limiting relations (see Theorem 3.4 below):

1. for any 1-periodic function f, we have

$$\lim_{\alpha \to 1/p'-} v_{p,\alpha}(f) = v_{p,1/p'}(f); \tag{1.8}$$

2. if $f \in V_p^{\alpha_0}$ for some $\alpha_0 > 0$, then

$$\lim_{\alpha \to 0+} v_{p,\alpha}(f) = v_p(f). \tag{1.9}$$

Further, we study limits of the Besov type norms $I_{p,\alpha}(f)$ as $\alpha \to 1/p'-$. This problem was inspired by the results obtained in [2] (see also [3, 8]) concerning limits of usual Besov norms. We prove that for functions $f \in W_p^1$,

$$\lim_{\alpha \to 1/p'-} (1/p' - \alpha)^{1/p} I_{p,\alpha}(f) = p^{-1/p}\|f'\|_p. \tag{1.10}$$

Conversely, if the limit in the left-hand side of (1.10) is finite, then $f \in W_p^1$.

The limiting relations (1.8)–(1.10) show the sharpness of the constant in (1.7).

An essential role in the proofs of our main results play estimates of approximation by Steklov averages in V_p proved below in Lemma 2.2. We use also these estimates to show that the K-functional $K(f,t;V_p,W_p^1)$ is equivalent to $\omega_{1-1/p}(f;t^{p'})$, $1 < p < \infty$.

2. Auxiliary results

We shall begin with some basic properties of the modulus of p-continuity (1.2). It was proved in [13] that for $1 < p < \infty$ and any $n \in \mathbb{N}$

$$\omega_{1-1/p}(f;n\delta) \leqslant n^{1/p'}\omega_{1-1/p}(f;\delta) \quad (0 \leqslant \delta \leqslant 1/n).$$

It follows that

$$\frac{\omega_{1-1/p}(f;\delta)}{\delta^{1/p'}} \leqslant 2^{1/p'}\frac{\omega_{1-1/p}(f;\mu)}{\mu^{1/p'}} \tag{2.1}$$

for $0 < \mu < \delta \leqslant 1$ (that is, the function $\delta^{-1/p'}\omega_{1-1/p}(f;\delta)$ is "almost decreasing"). As a consequence of (2.1), we see that the best order of decay of the modulus of p-continuity for $f \in C_p$ is $\omega_{1-1/p}(f;\delta) = O(\delta^{1/p'})$. This order is attained only for functions $f \in W_p^1$ (see the Introduction).

Let Ω_γ $(0 < \gamma \leqslant 1)$ be the class of all continuous functions ω defined on $[0,1]$ such that $\omega(0) = 0$, $\omega(t)$ is nondecreasing and $\omega(t)/t^\gamma$ is nonincreasing. For historical remarks and some new information concerning conditions of this type (including the close relation to index numbers), we refer to the paper [12] and the references given there.

For $\gamma = 1$, the class Ω_1 is "almost" the same as the class of moduli of continuity (see, e.g., [4, p.41]), in the sense that for any modulus of continuity η, there is $\omega \in \Omega_1$ such that $\omega(t) \leqslant \eta(t) \leqslant 2\omega(t)$, $t \in [0,1]$.

Similarly, A.P. Terehin [15] proved that for $\gamma = 1/p'$, the class $\Omega_{1/p'}$ "almost coincides" with the class of all moduli of p-continuity for functions in C_p. Indeed, let $f \in C_p$ and set

$$\omega^*(t) = t^{1/p'} \inf_{0<u\leqslant t} \frac{\omega_{1-1/p}(f;u)}{u^{1/p'}}. \tag{2.2}$$

Then clearly $\omega^* \in \Omega_{1/p'}$ and by (2.1)

$$\omega^*(t) \leqslant \omega_{1-1/p}(f;t) \leqslant 2^{1/p'}\omega^*(t), \quad 0 \leqslant t \leqslant 1. \tag{2.3}$$

Conversely, for any $\omega \in \Omega_{1/p'}$, in [15] there was constructed a function $f \in C_p$ such that

$$\omega(t) \leqslant \omega_{1-1/p}(f;t) \leqslant 9\omega(t), \quad 0 \leqslant t \leqslant 1.$$

For this reason, we shall call a function $\omega \in \Omega_{1/p'}$ a *modulus of p-continuity*.

Throughout this paper, for any $\omega \in \Omega_\gamma$ we denote

$$\omega_n = \omega(2^{-n}) \quad \text{and} \quad \overline{\omega}_n = 2^{n\gamma}\omega(2^{-n}) \quad (n \in \mathbb{N}). \tag{2.4}$$

Since $\omega(t)$ is nondecreasing and $\omega(t)/t^\gamma$ is nonincreasing, we have

$$\omega_{n+1} \leqslant \omega_n \leqslant 2^\gamma \omega_{n+1} \tag{2.5}$$

and

$$\overline{\omega}_n \leqslant \overline{\omega}_{n+1} \leqslant 2^\gamma \overline{\omega}_n \quad (n \in \mathbb{N}). \tag{2.6}$$

Let $\omega \in \Omega_\gamma$ and assume that

$$\lim_{t\to 0+} \omega(t)/t^\gamma = \infty. \tag{2.7}$$

Then we define the sequence of natural numbers $n_k \equiv n_k(\omega,\gamma)$ as follows. Set $n_0 = 0$ and

$$n_{k+1} = \min\left(n : \max\left(\frac{\omega_n}{\omega_{n_k}}, \frac{\overline{\omega}_{n_k}}{\overline{\omega}_n}\right) \leqslant \frac{1}{2}\right) \quad (k = 0,1,\dots). \tag{2.8}$$

Thus,

$$2\omega_{n_{k+1}} \leqslant \omega_{n_k}, \quad 2\overline{\omega}_{n_k} \leqslant \overline{\omega}_{n_{k+1}} \quad (k = 0,1,\dots), \tag{2.9}$$

and for each $k = 0,1,\dots$ at least one of the inequalities

$$2\omega_{n_{k+1}-1} > \omega_{n_k} \quad \text{or} \quad \overline{\omega}_{n_{k+1}-1} < 2\overline{\omega}_{n_k}$$

holds. By (2.5) and (2.6), this implies that for each $k = 0,1,\dots$ we have at least one of the inequalities

$$\omega_{n_k} < 4\omega_{n_{k+1}} \tag{2.10}$$

or

$$\overline{\omega}_{n_{k+1}} < 4\overline{\omega}_{n_k}. \tag{2.11}$$

Partitions (2.8) for moduli of continuity have been used for a long time, beginning from the works [1, 10, 16].

The following lemma is a slight generalisation of Lemma 2.1 in [6] and it can be proved in exactly the same way.

LEMMA 2.1. *Let $0 < \gamma \leqslant 1$ and let $\omega \in \Omega_\gamma$ satisfy (2.7). Let $1 \leqslant q < \infty$ and $0 < \beta < q\gamma$ be given numbers. Then*

$$\sum_{k=0}^{\infty} 2^{n_k\beta}\omega_{n_k}^q \leqslant 2\omega_0^q + \frac{2^{q+2}}{q\gamma}\beta(q\gamma-\beta)\int_0^1 t^{-\beta}\omega(t)^q\frac{dt}{t}.$$

Let $f \in L^1$. For any $0 < h \leqslant 1$, let

$$f_h(x) = \frac{1}{h}\int_0^h f(x+t)dt \tag{2.12}$$

be the Steklov average of the function f.

LEMMA 2.2. *Let $1 < p < \infty$ and $f \in V_p$. Then*

$$\omega_{1-1/p}(f_h;t) \leqslant \omega_{1-1/p}(f;t),\ 0 \leqslant t \leqslant 1, \tag{2.13}$$

$$\|f_h'\|_p \leqslant h^{-1/p'}\omega_{1-1/p}(f;h) \tag{2.14}$$

and

$$v_p(f-f_h) \leqslant 6\omega_{1-1/p}(f;h). \tag{2.15}$$

Proof. The inequality (2.13) is immediate and (2.14) follows from (3.1). We shall prove (2.15). Let $\Pi = \{x_0, x_1, \ldots, x_n\}$ be any partition and set

$$K' = \{j : x_{j+1} - x_j \leqslant h\}, \quad K'' = \{0, 1, \ldots, n-1\} \setminus K'$$

Set also $g_h = f - f_h$ and

$$V' = \left(\sum_{j \in K'} |g_h(x_{j+1}) - g_h(x_j)|^p \right)^{1/p}$$

and

$$V'' = \left(\sum_{j \in K''} |g_h(x_{j+1}) - g_h(x_j)|^p \right)^{1/p}.$$

Then $v_p(g_h; \Pi) \leqslant V' + V''$. By Minkowski's inequality

$$\begin{aligned} V' &\leqslant \left(\sum_{j \in K'} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p} + \left(\sum_{j \in K'} |f_h(x_{j+1}) - f_h(x_j)|^p \right)^{1/p} \\ &\leqslant \omega_{1-1/p}(f;h) + \omega_{1-1/p}(f_h;h). \end{aligned}$$

Using (2.13), we get

$$V' \leqslant 2\omega_{1-1/p}(f;h). \tag{2.16}$$

We now estimate V''. We have

$$(V'')^p = h^{-p} \sum_{j \in K''} \left| \int_0^h [f(x_{j+1}) - f(x_{j+1} + t) - f(x_j) + f(x_j + t)] dt \right|^p.$$

Applying the trivial inequality $|a+b|^p \leqslant 2^p(|a|^p + |b|^p)$ and Hölder's inequality, we obtain

$$\begin{aligned} (V'')^p \leqslant 2^p h^{-1} \int_0^h \Bigg[\sum_{j \in K''} |f(x_{j+1} + t) - f(x_{j+1})|^p + \\ + \sum_{j \in K''} |f(x_j + t) - f(x_j)|^p \Bigg] dt. \end{aligned}$$

For $t \in [0,h]$ and $j \in K''$ we have $[x_j, x_j + t] \subset [x_j, x_{j+1})$, and hence $[x_j, x_j + t] \cap [x_i, x_i + t] = \emptyset$ for $i, j \in K''$, $i \neq j$. Moreover, since $j \leqslant n-1$ and $j \in K''$, we have that $x_j + t \leqslant x_{j+1} \leqslant x_n$. Thus,

$$\bigcup_{j \in K''} [x_j, x_j + t] \subset [x_0, x_n],$$

and

$$\sum_{j \in K''} |f(x_j + t) - f(x_j)|^p \leqslant \omega_{1-1/p}(f;h)^p$$

for each $t \in [0,h]$. Furthermore, if $i,j \in K''$ and $i<j$, then $x_{i+1}+t \leqslant x_j+t \leqslant x_{j+1}$. Whence, $[x_{i+1},x_{i+1}+t] \cap [x_{j+1},x_{j+1}+t] = \emptyset$, $i<j$, and

$$\bigcup_{j\in K''} [x_{j+1},x_{j+1}+t] \subset [x_0+t,x_n+t].$$

Thus,

$$\sum_{j\in K''} |f(x_{j+1}+t)-f(x_{j+1})|^p \leqslant \omega_{1-1/p}(f;h)^p$$

for each $t \in [0,h]$. It follows that

$$V'' \leqslant 2^{1+1/p}\omega_{1-1/p}(f;h). \tag{2.17}$$

By (2.16) and (2.17) we obtain

$$v_p(f-f_h) \leqslant 6\omega_{1-1/p}(f;h).$$

This completes the proof. □

REMARK 2.3. Applying Lemma 2.2, we can show that the Peetre K-functional $K(f,t;V_p,W_p^1)$ is equivalent to $\omega_{1-1/p}(f;t^{p'})$.

Set $\|f\|_{V_p} = |f(0)|+v_p(f)$ for $f \in V_p$. It is simple to show that $\|\cdot\|_{V_p}$ is a norm on V_p and that V_p is a Banach space with respect to this norm.

As in [4, p.172], we define the K-functional for the pair (V_p,W_p^1) by the equality

$$K(f,t;V_p,W_p^1) = \inf_{g\in W_p^1} (\|f-g\|_{V_p}+t\|g'\|_p).$$

We emphasize that the second term on the right-hand side is only a seminorm on W_p^1.

We shall prove that

$$\omega_{1-1/p}(f;t^{p'}) \leqslant K(f,t;V_p,W_p^1) \leqslant 8\omega_{1-1/p}(f;t^{p'}). \tag{2.18}$$

Fix an arbitrary $t \in (0,1]$ and set $h=t^{p'}$. Let $g=f_h$ be the Steklov average (2.12), then $g \in W_p^1$. By (2.14) and (2.15), we have that

$$|f(0)-g(0)|+v_p(f-g)+h^{1/p'}\|g'\|_p \leqslant 8\omega_{1-1/p}(f;h).$$

Substituting $h=t^{p'}$ above yields

$$\|f-g\|_{V_p}+t\|g'\|_p \leqslant 8\omega_{1-1/p}(f;t^{p'}),$$

and therefore,

$$K(f,t;V_p,W_p^1) \leqslant 8\omega_{1-1/p}(f;t^{p'}).$$

On the other hand, for any $g \in W_p^1$, we have by (1.4) that

$$\begin{aligned}\omega_{1-1/p}(f;t^{p'}) &\leqslant \omega_{1-1/p}(f-g;t^{p'})+\omega_{1-1/p}(g;t^{p'})\\ &\leqslant v_p(f-g)+t\|g'\|_p.\end{aligned}$$

Taking infimum over all $g \in W_p^1$, we obtain that

$$\omega_{1-1/p}(f;t^{p'}) \leqslant K(f,t;V_p,W_p^1).$$

Thus, (2.18) is proved.

3. Limiting relations

The L^p-modulus of continuity of a function $f \in L^p$ is defined by

$$\omega(f;\delta)_p = \sup_{0\leqslant h\leqslant\delta}\left(\int_0^1 |f(x+h)-f(x)|^p dx\right)^{1/p}, \quad 0 \leqslant \delta \leqslant 1.$$

It was proved in [14] that for $\delta \in [0,1]$

$$\omega(f;\delta)_p \leqslant \delta^{1/p}\omega_{1-1/p}(f;\delta) \quad (1<p<\infty). \tag{3.1}$$

Observe that in the non-periodic case (which is much simpler), (3.1) was proved in [18].

Let $f \in L^p\,(1<p<\infty)$. It was proved in [2] that if

$$\sup_{0<s<1}(1-s)\int_0^1 (t^{-s}\omega(f;t)_p)^p\frac{dt}{t} < \infty,$$

then $f \in W_p^1$ and

$$\lim_{s\to1-}(1-s)^{1/p}\left(\int_0^1 (t^{-s}\omega(f;t)_p)^p\frac{dt}{t}\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p}\|f'\|_p.$$

We shall consider a similar limiting relation involving the modulus of p-continuity instead of L^p-modulus of continuity. We begin with the following proposition.

PROPOSITION 3.1. *Let $f \in W_p^1$ $(1<p<\infty)$. Then*

$$\lim_{h\to0+}\frac{\omega_{1-1/p}(f;h)}{h^{1/p'}} = \|f'\|_p. \tag{3.2}$$

Proof. It is a direct consequence of (1.4) that

$$\overline{\lim_{h\to0+}}\frac{\omega_{1-1/p}(f;h)}{h^{1/p'}} \leqslant \|f'\|_p.$$

For $h\in(0,1]$, denote $\Delta_h f(x) = f(x+h)-f(x)$ and set

$$\mu(h) = \|f' - (\Delta_h f)/h\|_p.$$

Then

$$\|f'\|_p \leqslant \mu(h) + \frac{\|\Delta_h f\|_p}{h} \leqslant \mu(h) + \frac{\omega(f;h)_p}{h}.$$

From here and (3.1), we obtain that

$$\|f'\|_p \leqslant \mu(h) + \frac{\omega_{1-1/p}(f;h)}{h^{1/p'}}, \tag{3.3}$$

for any $0 < h \leqslant 1$. Further,

$$\Delta_h f(x) = \int_0^h f'(x+t)dt.$$

Thus, applying Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}\mu(h) &= \left(\int_0^1 \left|f'(x) - \frac{1}{h}\int_0^h f'(x+t)dt\right|^p dx\right)^{1/p} \\ &\leqslant \left(\frac{1}{h}\int_0^h \left(\int_0^1 |f'(x) - f'(x+t)|^p dx\right) dt\right)^{1/p} \leqslant \omega(f';h)_p.\end{aligned}$$

Since $f' \in L^p$, $\omega(f';h)_p \to 0$ as $h \to 0$. Thus, $\mu(h) \to 0$ as $h \to 0$ and we get from (3.3) that

$$\varliminf_{h\to 0+} \frac{\omega_{1-1/p}(f;h)}{h^{1/p'}} \geqslant \|f'\|_p.$$

This completes the proof. □

THEOREM 3.2. *Let f be an 1-periodic function. Then the following statements hold:*

1. *if $f \in W_p^1$ $(1 < p < \infty)$, then*

$$\lim_{s\to 1/p'-} (1/p'-s)^{1/p} \left(\int_0^1 [t^{-s}\omega_{1-1/p}(f;t)]^p \frac{dt}{t}\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p} \|f'\|_p; \tag{3.4}$$

2. *if $f \in C_p$ $(1 < p < \infty)$ and*

$$\varlimsup_{s\to 1/p'-} (1/p'-s) \int_0^1 [t^{-s}\omega_{1-1/p}(f;t)]^p \frac{dt}{t} < \infty,$$

then $f \in W_p^1$.

Proof. We first prove the statement 1. Let $f \in W_p^1$ and $s \in (0, 1/p')$. Set

$$J(s,h) = p(1/p'-s)\int_0^h [t^{-s}\omega_{1-1/p}(f;t)]^p \frac{dt}{t}, \quad 0 \leqslant h \leqslant 1,$$

then we shall prove that

$$\lim_{s\to 1/p'-} J(s,1) = \|f'\|_p^p.$$

By (3.2) we have that for any $\varepsilon > 0$, there is a number $\delta = \delta(\varepsilon) > 0$ such that for $0 < t < \delta$

$$\|f'\|_p^p - \varepsilon < \frac{\omega_{1-1/p}(f;t)^p}{t^{p-1}} < \|f'\|_p^p + \varepsilon. \tag{3.5}$$

Multiplying (3.5) by t^{p-2-sp}, integrating over $[0,\delta]$ and taking into account that $p-1-sp = p(1/p'-s)$ yield the inequalities

$$\delta^{p-1-sp}(\|f'\|_p^p - \varepsilon) \leqslant J(s,\delta) \leqslant \delta^{p-1-sp}(\|f'\|_p^p + \varepsilon).$$

It follows that

$$(1-\delta^{p-1-sp})\|f'\|_p^p - \varepsilon\delta^{p-1-sp} \leqslant \|f'\|_p^p - J(s,\delta) \leqslant (1-\delta^{p-1-sp})\|f'\|_p^p + \varepsilon\delta^{p-1-sp}. \tag{3.6}$$

Furthermore, since $f \in W_p^1$, we also have $f \in V_p$ and

$$p(1/p'-s)\int_\delta^1 [t^{-s}\omega_{1-1/p}(f;t)]^p \frac{dt}{t} \leqslant p(1/p'-s)\delta^{-sp-1}v_p(f)^p.$$

Therefore,

$$\begin{aligned}\left|J(s,1) - \|f'\|_p^p\right| \leqslant\ & (1-\delta^{p-1-sp})\|f'\|_p + \varepsilon\delta^{p-1-sp} \\ & + p(1/p'-s)\delta^{-sp-1}v_p(f)^p.\end{aligned}$$

As $s \to 1/p'-$, the limit of the right hand side of this inequality is equal to ε. Since $\varepsilon > 0$ is arbitrary, the proof of 1. is complete.

Let now $f \in C_p$. For any $0 < h < 1$, let f_h be the Steklov average of f given by (2.12). Then $f_h \in W_p^1$ and $f_h'(x) = [f(x+h) - f(x)]/h$ a.e. Applying (3.4) to the function f_h and using (2.13), we have that

$$\begin{aligned}\frac{1}{p}\|f_h'\|_p^p &= \lim_{s\to 1/p'-}(1/p'-s)\int_0^1 [t^{-s}\omega_{1-1/p}(f_h;t)]^p \frac{dt}{t} \\ &\leqslant \overline{\lim_{s\to 1/p'-}}(1/p'-s)\int_0^1 [t^{-s}\omega_{1-1/p}(f;t)]^p \frac{dt}{t} = C < \infty.\end{aligned}$$

On the other hand,

$$\|f_h'\|_p^p = h^{-p}\int_0^1 |f(x+h) - f(x)|^p dx.$$

Thus,

$$\left(\int_0^1 |f(x+h) - f(x)|^p dx\right)^{1/p} \leqslant Ch, \quad h \in (0,1].$$

Since f is continuous, a theorem of G.H. Hardy and J.E. Littlewood [5, Thm. 24] implies that $f \in W_p^1$. □

REMARK 3.3. M. Milman [8] studied continuity properties of interpolation scales at the endpoints. In particular, it follows from his results that for any $f \in W_p^1$,

$$\lim_{s\to 1-}(1-s)^{1/p}\left(\int_0^1 (t^{-s}K(f,t;V_p,W_p^1))^p\frac{dt}{t}\right)^{1/p} = \left(\frac{1}{p}\right)^{1/p}\|f'\|_p$$

Together with (2.18), this provides another look on (3.4).

We shall also give some limiting relations for the functionals $v_{p,\alpha}(f)$ defined by (1.1).

THEOREM 3.4. *Let f be an 1-periodic function and let $1<p<\infty$. Then the following relations hold:*

1. *for any f we have*
$$\lim_{\alpha\to 1/p'-} v_{p,\alpha}(f) = v_{p,1/p'}(f); \tag{3.7}$$

2. *if $f\in V_p^{\alpha_0}$ for some $\alpha_0>0$, then*
$$\lim_{\alpha\to 0+} v_{p,\alpha}(f) = v_p(f). \tag{3.8}$$

Proof. To prove 1, we first observe that

$$v_{p,\alpha}(f) \leqslant v_{p,1/p'}(f), \quad 0<\alpha<1/p'.$$

Further, let $\Pi = \{x_0,x_1,\dots,x_n\}$ be any partition. Then, since

$$v_{p,\alpha}(f) \geqslant \left(\sum_{k=0}^{n-1}\frac{|f(x_{k+1})-f(x_k)|^p}{(x_{k+1}-x_k)^{\alpha p}}\right)^{1/p},$$

we get

$$\underline{\lim}_{\alpha\to 1/p'-} v_{p,\alpha}(f) \geqslant \left(\sum_{k=0}^{n-1}\frac{|f(x_{k+1})-f(x_k)|^p}{(x_{k+1}-x_k)^{p-1}}\right)^{1/p}.$$

Taking supremum over all partitions, we obtain

$$\underline{\lim}_{\alpha\to 1/p'-} v_{p,\alpha}(f) \geqslant v_{p,1/p'}(f).$$

Thus, (3.7) holds.

We proceed to prove 2. Since

$$v_p(f) \leqslant v_{p,\alpha}(f)$$

for any $\alpha>0$, it is sufficient to show that

$$\overline{\lim}_{\alpha\to 0+} v_{p,\alpha}(f) \leqslant v_p(f).$$

For any partition $\Pi = \{x_0, x_1, \dots, x_n\}$, we set

$$\sigma_k = \{j : 2^{-k-1} < x_{j+1} - x_j \leqslant 2^{-k}\},$$

and

$$S_k(f) = \left(\sum_{j \in \sigma_k} |f(x_{j+1}) - f(x_j)|^p \right)^{1/p}.$$

Then

$$v_{p,\alpha}(f;\Pi) \leqslant 2^{\alpha} \left(\sum_{k=0}^{\infty} 2^{k\alpha p} S_k(f)^p \right)^{1/p}. \tag{3.9}$$

Furthermore, by applying the Abel transform we have

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{k\alpha p} S_k(f)^p &= \sum_{k=0}^{\infty} 2^{k\alpha p} \left[\sum_{j=k}^{\infty} S_j(f)^p - \sum_{j=k+1}^{\infty} S_j(f)^p \right] \\ &= \sum_{k=0}^{\infty} S_k(f)^p + (1 - 2^{-\alpha p}) \sum_{k=1}^{\infty} 2^{k\alpha p} \sum_{j=k}^{\infty} S_j(f)^p. \end{aligned}$$

It is easy to see that

$$\sum_{j=k}^{\infty} S_j(f)^p \leqslant v_{p,\alpha_0}(f)^p 2^{-k\alpha_0 p}.$$

Whence, for $0 < \alpha < \alpha_0$

$$\sum_{k=0}^{\infty} 2^{k\alpha p} S_k(f)^p \leqslant v_p(f)^p + v_{p,\alpha_0}(f)^p \alpha p \sum_{k=1}^{\infty} 2^{-k(\alpha_0 - \alpha)p}.$$

Thus, by (3.9)

$$v_{p,\alpha}(f) \leqslant 2^{\alpha} \left(v_p(f) + \alpha^{1/p} v_{p,\alpha_0}(f) \left(\frac{p}{2^{(\alpha_0 - \alpha)p} - 1} \right)^{1/p} \right)$$

and it follows that

$$\overline{\lim_{\alpha \to 0+}} v_{p,\alpha}(f) \leqslant v_p(f),$$

which concludes the proof. □

REMARK 3.5. The condition that $f \in V_p^{\alpha_0}$ for some $\alpha_0 > 0$ in (ii) cannot be omitted. Indeed, if $f \in V_p$ has a discontinuity at some point, then $v_{p,\alpha}(f) = \infty$ for all $\alpha > 0$ whence $\lim v_{p,\alpha}(f) = \infty$, while $v_p(f) < \infty$.

4. Estimates of the $v_{p,\alpha}$-variation

In this Section we obtain a sharp estimate of $v_{p,\alpha}(f)$ (see (1.1)) in terms of the modulus of p-continuity $\omega_{1-1/p}(f;\delta)$.

THEOREM 4.1. *Let $1<p<\infty$ and let $0<\alpha<1/p'$. Assume that $f\in V_p$ and that*

$$I_{p,\alpha}(f)=\left(\int_0^1[t^{-\alpha}\omega_{1-1/p}(f;t)]^p\frac{dt}{t}\right)^{1/p}<\infty. \tag{4.1}$$

Then $f\in V_p^\alpha$ and

$$v_{p,\alpha}(f)\leqslant A[v_p(f)+c_{p,\alpha}I_{p,\alpha}(f)], \tag{4.2}$$

where A is an absolute constant and

$$c_{p,\alpha}=p'\alpha^{1/p}(1/p'-\alpha)^{1/p}. \tag{4.3}$$

Proof. The condition (4.1) implies that $f\in C_p$. Let $\omega^*(t)$ be given by (2.2) and take $\omega\in\Omega_{1/p'}$ such that

$$\omega^*(t)\leqslant\omega(t),\quad t\in[0,1] \tag{4.4}$$

and

$$\lim_{t\to 0+}\omega(t)/t^{1/p'}=\infty. \tag{4.5}$$

We specify later how such ω can be obtained. As before, set $\omega_n=\omega(2^{-n})$ and $\overline{\omega}_n=2^{n/p'}\omega_n$. Let the natural numbers $n_k\equiv n_k(\omega,1/p')$, $k=0,1,...,$ be defined by (2.8). Set $\mu(k)=k$ if (2.10) holds and $\mu(k)=k+1$ if (2.11) holds, and define

$$g_k(x)=2^{n_{\mu(k)}}\int_0^{2^{-n_{\mu(k)}}}f(x+t)dt.$$

Fix a partition $\Pi=\{x_0,x_1,...,x_n\}$ and set

$$\sigma_k=\{j:2^{-n_{k+1}}<x_{j+1}-x_j\leqslant 2^{-n_k}\}.$$

For any function φ we define

$$R_k(\varphi)=\left(\sum_{j\in\sigma_k}\frac{|\varphi(x_{j+1})-\varphi(x_j)|^p}{(x_{j+1}-x_j)^{\alpha p}}\right)^{1/p}$$

and

$$S_k(\varphi)=\left(\sum_{j\in\sigma_k}|\varphi(x_{j+1})-\varphi(x_j)|^p\right)^{1/p}.$$

By Hölder's inequality we have for $j \in \sigma_k$

$$\begin{aligned}\frac{|g_k(x_{j+1})-g_k(x_j)|^p}{(x_{j+1}-x_j)^{\alpha p}} &= \frac{1}{(x_{j+1}-x_j)^{\alpha p}}\left|\int_{x_j}^{x_{j+1}} g_k'(t)dt\right|^p \\ &\leqslant (x_{j+1}-x_j)^{p-1-\alpha p}\int_{x_j}^{x_{j+1}} |g_k'(t)|^p dt \\ &\leqslant 2^{-n_k(p-1-\alpha p)}\int_{x_j}^{x_{j+1}} |g_k'(t)|^p dt.\end{aligned}$$

Thus, by (2.14) and (4.4),

$$\begin{aligned}R_k(g_k) &\leqslant 2^{-n_k(1/p'-\alpha)}\|g_k'\|_p \\ &\leqslant 2^{-n_k(1/p'-\alpha)}2^{n_{\mu(k)}/p'}\omega_{1-1/p}(f;2^{-n_{\mu(k)}}) \\ &\leqslant 2^{-n_k(1/p'-\alpha)}2^{n_{\mu(k)}/p'}\omega_{n_{\mu(k)}}.\end{aligned}$$

If $\mu(k)=k$, then $R_k(g_k)\leqslant 2^{n_k\alpha}\omega_{n_k}$. If $\mu(k)=k+1$, then $\overline{\omega}_{n_{k+1}}<4\overline{\omega}_{n_k}$ and

$$R_k(g_k)\leqslant 2^{-n_k(1/p'-\alpha)}\overline{\omega}_{n_{k+1}}\leqslant 2^{n_k\alpha+2}\omega_{n_k}. \tag{4.6}$$

Thus, (4.6) holds for each $k\in\mathbb{N}$. Further,

$$R_k(f-g_k)\leqslant 2^{n_{k+1}\alpha}S_k(f-g_k). \tag{4.7}$$

Applying (4.6) and (4.7), we get

$$\begin{aligned}v_{p,\alpha}(f;\Pi) &\leqslant \left(\sum_{k=0}^{\infty}R_k(g_k)^p\right)^{1/p}+\left(\sum_{k=0}^{\infty}R_k(f-g_k)^p\right)^{1/p} \\ &\leqslant 4\left(\sum_{k=0}^{\infty}2^{n_k\alpha p}\omega_{n_k}^p\right)^{1/p}+\left(\sum_{k=0}^{\infty}2^{n_{k+1}\alpha p}S_k(f-g_k)^p\right)^{1/p}.\end{aligned}$$

We estimate the latter sum. Clearly, $S_k(f-g_k)\leqslant v_p(f-g_k)$. Applying (2.15), we obtain

$$S_k(f-g_k)\leqslant 6\omega_{n_{\mu(k)}}.$$

If $\mu(k)=k$, then $\omega_{n_k}\leqslant 4\omega_{n_{k+1}}$ and

$$2^{n_{k+1}\alpha}S_k(f-g_k)\leqslant 2^{n_{k+1}\alpha+3}\omega_{n_k}\leqslant 2^{n_{k+1}\alpha+5}\omega_{n_{k+1}}.$$

If $\mu(k)=k+1$, then

$$2^{n_{k+1}\alpha}S_k(f-g_k)\leqslant 2^{n_{k+1}\alpha+3}\omega_{n_{k+1}}.$$

Thus

$$\left(\sum_{k=0}^{\infty}2^{n_{k+1}\alpha p}S_k(f-g_k)^p\right)^{1/p}\leqslant 32\left(\sum_{k=0}^{\infty}2^{n_k\alpha p}\omega_{n_k}^p\right)^{1/p}.$$

It follows that

$$v_{p,\alpha}(f;\Pi) \leqslant 36\left(\sum_{k=0}^{\infty} 2^{n_k\alpha p}\omega_{n_k}^p\right)^{1/p}.$$

Applying Lemma 2.1 with $\gamma = 1/p'$, $q = p$ and $\beta = \alpha p$ yields

$$v_{p,\alpha}(f) \leqslant 36\left(2\omega_0^p + 2^{p+2}pp'\alpha(1/p'-\alpha)\int_0^1 t^{-\alpha p}\omega(t)^p\frac{dt}{t}\right)^{1/p}.$$

Set

$$D_{p,\alpha}(\omega) = \left(\int_0^1 t^{-\alpha p}\omega(t)^p\frac{dt}{t}\right)^{1/p}.$$

Since $p^{1/p} \leqslant 2$ and $(p')^{1/p} \leqslant p'$, we obtain

$$v_{p,\alpha}(f) \leqslant 2^{10}\left[\omega_0 + p'\alpha^{1/p}(1/p'-\alpha)^{1/p}D_{p,\alpha}(\omega)\right]. \tag{4.8}$$

If there holds

$$\lim_{t\to 0+}\frac{\omega^*(t)}{t^{1/p'}} = \infty, \tag{4.9}$$

then we take $\omega(t) = \omega^*(t)$. In this case

$$D_{p,\alpha}(\omega) \leqslant I_{p,\alpha}(f)$$

and $\omega_0 \leqslant v_p(f)$, by (2.3). Thus, (4.2) is proved in this case.

If (4.9) does not hold, we take $\omega_\varepsilon(t) = \omega^*(t) + \varepsilon t^\gamma$ where $\alpha < \gamma < 1/p'$. Then $\omega_\varepsilon \in \Omega_{1/p'}$ for each $\varepsilon > 0$ and ω_ε satisfies (4.4) and (4.5). Furthermore, by (2.3) and a simple calculation we have

$$D_{p,\alpha}(\omega_\varepsilon) \leqslant I_{p,\alpha}(f) + \varepsilon(p(\gamma-\alpha))^{1/p}$$

and $\omega_\varepsilon(1) \leqslant v_p(f) + \varepsilon$. Thus, we get from (4.8) that

$$v_{p,\alpha}(f) \leqslant 2^{10}(v_p(f) + \varepsilon + p'\alpha^{1/p}(1/p'-\alpha)^{1/p}[I_{p,\alpha}(f) + \varepsilon(p(\gamma-\alpha))^{1/p}]).$$

Letting $\varepsilon \to 0$ yields (4.2). □

REMARK 4.2. Assume that $f \in W_p^1$ $(1 < p < \infty)$. By Theorem 3.2

$$\lim_{\alpha\to 1/p'-}(1/p'-\alpha)^{1/p}I_{p,\alpha}(f) = p^{-1/p}\|f'\|_p.$$

Further, $v_p(f) \leqslant \|f'\|_p$ for $f \in W_p^1$. Thus, the upper limit as $\alpha \to 1/p'-$ of the right-hand side of (4.2) does not exceed $A\|f'\|_p$ (where A is an absolute constant). On the other hand, by Proposition 3.4, the left-hand side of (4.2) tends to $v_{p,1/p'}(f)$ as $\alpha \to 1/p'-$. Thus,

$$v_{p,1/p'}(f) \leqslant A\|f'\|_p.$$

This agrees with the theorem of F. Riesz (see the Introduction) and shows that the order of the constant (4.3) is optimal as $\alpha \to 1/p'-$.

REMARK 4.3. Assume that $I_{p,\alpha_0}(f) < \infty$ for some $0 < \alpha_0 < 1/p'$. Since $I_{p,\alpha}(f) \leqslant I_{p,\alpha_0}(f)$ for $0 < \alpha \leqslant \alpha_0$, we get that

$$\lim_{\alpha \to 0+} \alpha^{1/p} I_{p,\alpha}(f) \leqslant \lim_{\alpha \to 0+} \alpha^{1/p} I_{p,\alpha_0}(f) = 0.$$

Thus, as $\alpha \to 0+$, the limit of the right-hand side of (4.2) does not exceed $Av_p(f)$ (where A is an absolute constant). On the other hand, if $I_{p,\alpha_0}(f) < \infty$, then $f \in V_p^{\alpha_0}$ by Theorem 4.1. By (3.8), $v_{p,\alpha}(f) \to v_p(f)$ as $\alpha \to 0+$. Thus, the behaviour of the left-hand side of (4.2) agrees with the behaviour of the right-hand side as $\alpha \to 0+$.

REMARK 4.4. Let $1 < p < \infty$ and $0 < \alpha < 1/p'$. In [6], sharp estimates of $v_{p,\alpha}(f)$ in terms of $\omega(f;\delta)_p$ were studied. There it was proved that if $f \in L^p$ and

$$K_{p,\alpha}(f) \equiv \left(\int_0^1 t^{-\alpha p - 1} \omega(f;t)_p^p \frac{dt}{t} \right)^{1/p} < \infty,$$

then there exists $\bar{f} \in V_p^\alpha$ such that $f = \bar{f}$ a.e. and

$$v_{p,\alpha}(\bar{f}) \leqslant A \alpha^{-1/p'} (1/p' - \alpha)^{1/p} K_{p,\alpha}(f), \tag{4.10}$$

where A is an absolute constant. The scheme of the proof of Theorem 4.1 is similar to one used in [6] in the proof of (4.10).

We shall now compare (4.10) and (4.2). For $1 < p < \infty$, $0 < \alpha < 1/p'$ and $f \in V_p$, we have

$$K_{p,\alpha}(f) \leqslant I_{p,\alpha}(f) \leqslant \frac{C}{\alpha} K_{p,\alpha}(f), \tag{4.11}$$

where C is an absolute constant. Indeed, the left inequality is an immediate consequence of (3.1), while the right inequality follows from estimates of $\omega_{1-1/p}(f;\delta)$ in terms of $\omega(f;\delta)_p$ obtained in [14] (see also [6]) combined with Hardy's inequality (see [7, p.7]).

Applying (4.10) and the left inequality of (4.11), we get

$$v_{p,\alpha}(f) \leqslant A c'_{p,\alpha} I_{p,\alpha}(f),$$

where A is an absolute constant and $c'_{p,\alpha} = \alpha^{-1/p'}(1/p' - \alpha)^{1/p}$. Observe that for small $\alpha > 0$, the constant $c'_{p,\alpha}$ is much larger than the constant $c_{p,\alpha}$ given by (4.3). Indeed, $c'_{p,\alpha} \to \infty$ as $\alpha \to 0+$, while $c_{p,\alpha} \to 0$ as $\alpha \to 0+$. Thus, (4.2) with the sharp constant (4.3) cannot be obtained from (4.10). However, note that the order of the constant in (4.10) as $\alpha \to 0+$ is optimal (see [6], Remark 5.3).

Now we show that for $0 < \alpha < 1/p'$ the condition (4.1) is sharp.

THEOREM 4.5. *Let $1 < p < \infty$ and $0 < \alpha < 1/p'$. Assume that $\omega \in \Omega_{1/p'}$ is any modulus of p-continuity such that*

$$\int_0^1 (t^{-\alpha} \omega(t))^p \frac{dt}{t} = \infty. \tag{4.12}$$

Then there is a function $f \in V_p$ such that $\omega_{1-1/p}(f;\delta) \leqslant \omega(\delta)$ but $f \notin V_p^\alpha$.

Proof. Define $\omega_n, \overline{\omega}_n$ by (2.4) with $\gamma = 1/p'$. The condition (4.12) implies that $\omega(\delta) \neq O(\delta^{1/p'})$, thus we may construct $\{n_k\}_{k=0}^{\infty}$ by (2.8).

For $k = 1, 2, \ldots$, set $\xi_k = 2^{-n_k}$, $\delta_k = 2^{-n_k-2}$ and $I_k = [\xi_k - \delta_k, \xi_k + \delta_k]$. Then $I_k \subset (0,1)$. Further, since $n_{k+1} \geqslant n_k + 1$, we have $\xi_{k+1} + \delta_{k+1} < \xi_k - \delta_k$ and thus the intervals $\{I_k\}_{k\in\mathbb{N}}$ are pairwise disjoint and ordered from the right to the left.

For $k \in \mathbb{N}$, define φ_k as a continuous 1-periodic function such that $\varphi_k(x) = 0$ for $x \in [0,1] \setminus I_k$, $\varphi_k(\xi_k) = \omega_{n_k}$, and φ_k is linear on $[\xi_k - \delta_k, \xi_k]$ and $[\xi_k, \xi_k + \delta_k]$. Set

$$f(x) = \sum_{k=1}^{\infty} \varphi_k(x).$$

We shall estimate $\omega_{1-1/p}(f; 2^{-s})$ for $s \in \mathbb{N}$. Assume that $n_m \leqslant s < n_{m+1}$ for some $m \geqslant 1$. Clearly, there holds

$$\omega_{1-1/p}(f; 2^{-s}) \leqslant \sum_{k=1}^{\infty} \omega_{1-1/p}(\varphi_k; 2^{-s}).$$

For each $k \geqslant m+1$ we have the trivial estimate

$$\omega_{1-1/p}(\varphi_k; 2^{-s}) \leqslant v_1(\varphi_k) = 2\omega_{n_k}.$$

Fix $1 \leqslant k \leqslant m$. Observe that

$$|\varphi_k'(x)| = 2^{n_k+2}\omega_{n_k}, \quad x \in (\xi_k - \delta_k, \xi_k) \cup (\xi_k, \xi_k + \delta_k),$$

and

$$\varphi_k'(x) = 0, \quad x \in [0,1] \setminus I_k.$$

By (1.4), we have

$$\begin{aligned}\omega_{1-1/p}(\varphi_k; 2^{-s}) &\leqslant 2^{-s/p'} \|\varphi_k'\|_p = 2^{-s/p'} \left(\int_{I_k} 2^{(n_k+2)p} \omega_{n_k}^p \, dx \right)^{1/p} \\ &= 2^{-s/p'+2-1/p} \overline{\omega}_{n_k}.\end{aligned}$$

By (2.9),

$$\begin{aligned}\omega_{1-1/p}(f; 2^{-s}) &\leqslant 4 \left[2^{-s/p'} \sum_{k=1}^{m} \overline{\omega}_{n_k} + \sum_{k=m+1}^{\infty} \omega_{n_k} \right] \\ &\leqslant 8(2^{-s/p'} \overline{\omega}_{n_m} + \omega_{n_{m+1}}).\end{aligned}$$

Further, since $n_m \leqslant s < n_{m+1}$, we have $\overline{\omega}_{n_m} \leqslant \overline{\omega}_s = 2^{s/p'}\omega_s$, and $\omega_{n_{m+1}} \leqslant \omega_s$. Thus, $\omega_{1-1/p}(f; 2^{-s}) \leqslant 16\omega_s$. This implies that

$$\omega_{1-1/p}(f; \delta) \leqslant 32\omega(\delta) \quad \text{for } 0 \leqslant \delta \leqslant 1.$$

We shall prove that $f \notin V_p^{\alpha}$. For any $N \in \mathbb{N}$, consider the points

$$0 < \xi_N - \delta_N < \xi_N < \xi_{N-1} - \delta_{N-1} < \ldots. < \xi_1 - \delta_1 < \xi_1 < 1.$$

Clearly

$$v_{p,\alpha}(f) \geqslant \left(\sum_{k=1}^{N} \frac{|f(\xi_k) - f(\xi_k - \delta_k)|^p}{\delta_k^{\alpha p}} \right)^{1/p} = 4^\alpha \left(\sum_{k=1}^{N} 2^{n_k \alpha p} \omega_{n_k}^p \right)^{1/p}.$$

Thus,

$$v_{p,\alpha}(f) \geqslant 4^\alpha \left(\sum_{k=1}^{\infty} 2^{n_k \alpha p} \omega_{n_k}^p \right)^{1/p}.$$

It remains to show that the series at the right-hand side diverges.

If (2.10) holds, then

$$\int_{2^{-n_{k+1}}}^{2^{-n_k}} (t^{-\alpha} \omega(t))^p \frac{dt}{t} \leqslant \frac{4^p}{\alpha p} 2^{n_{k+1} \alpha p} \omega_{n_{k+1}}^p.$$

If (2.11) holds, then

$$\int_{2^{-n_{k+1}}}^{2^{-n_k}} (t^{-\alpha} \omega(t))^p \frac{dt}{t} \leqslant \frac{4^p}{p - 1 - \alpha p} 2^{n_k \alpha p} \omega_{n_k}^p.$$

These estimates and (4.12) yield that

$$\sum_{k=1}^{\infty} 2^{n_k \alpha p} \omega_{n_k}^p = \infty. \qquad \square$$

Acknowledgements

The author is very grateful to Professor V.I. Kolyada, under whose guidance this work was completed, and to the referee for his/her useful remarks.

REFERENCES

[1] V. A. ANDRIENKO, *Necessary conditions for imbedding the function classes H_p^ω*, Mat. Sb. (N.S.) **78 (120)** (1969), 280–300; English transl.: Math. USSR Sb. **7** (1969) No. 2, 273–292.

[2] J. BOURGAIN, H. BRÉZIS AND P. MIRONESCU, *Another look at Sobolev spaces*, Optimal Control and Partial Differential Equations. In honour of Professor Alain Bensoussan's 60th Birthday. J. L. Menaldi, E. Rofman, A. Sulem (eds), IOS Press, Amsterdam, 2001, 439–455.

[3] H. BRÉZIS, *How to recognize constant functions. Connections with Sobolev spaces*, (Russian) Uspekhi Mat. Nauk **57** (2002), no. 4(346), 59–74; English transl. in Russian Math. Surveys **57** (2002), no. 4, 693–708.

[4] R. A. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer-Verlag, Berlin Heidelberg, 1993.

[5] G. H. HARDY AND J. E. LITTLEWOOD, *Some properties of fractional integrals. I*, Math. Z. **27** (1928), 565–606.

[6] V. I. KOLYADA AND M. LIND, *On functions of bounded p-variation*, J. Math. Anal. Appl. **356** (2009), no. 2, 582–604.

[7] A. KUFNER AND L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific, 2003.

[8] M. MILMAN, *Notes on limits of Sobolev spaces and the continuity of interpolation scales*, Trans. Amer. Math. Soc. **357** (2005), no. 9, 3425–3442.

[9] I. P. NATANSON, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955.
[10] K. I. OSKOLKOV, *Approximation properties of integrable functions on sets of full measure*, Mat. Sb. **103** (1977), 563–589; English transl. in Math. USSR Sb. **32** (1977), 489–514.
[11] J. PEETRE, *New thoughts on Besov spaces*, Duke Univ. Math. Ser., Durham University Press, Durham, 1976.
[12] L. E. PERSSON, N. SAMKO AND P. WALL, *Quasi-monotone weight functions and their characteristics and applications*, Math. Inequal. Appl., **15**, 3 (2012), 685–705.
[13] A. P. TEREHIN, *Approximation of functions of bounded $p-$variation*, Izv. Vyssh. Uchebn. Zaved. Mat. (1965), no. 2, 171–187 (Russian).
[14] A. P. TEREHIN, *Integral smoothness properties of periodic functions of bounded p-variation*, Mat. Zametki **2** (1967), 289-300 (Russian).
[15] A. P. TEREHIN, *Functions of bounded p-variation with a given modulus of continuity*, Mat. Zametki **53** (1972), no. 6, 523–530; English transl. in Math. Notes, **12** (1972), 751–755
[16] P. L. UL'YANOV, *Imbeddings of certain function classes H_p^ω*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 649–686; English transl. in Math. USSR Izv. **2** (1968).
[17] N. WIENER, *The quadratic variation of a function and its Fourier coefficients*, J. Math. Phys. **3** (1924), 72–94.
[18] L. C. YOUNG, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), 251–282.

(Received February 9, 2011)

Martin Lind
Department of Mathematics
Karlstad University
Universitetsgatan 2
651 88 Karlstad
Sweden
e-mail: martin.lind@kau.se