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COMPLETE INTERPOLATION OF MATRIX VERSIONS OF HERON AND HEINZ MEANS

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Abstract. The interpolation and comparison of a matrix version of Heron mean, $F_\alpha(a,b) = (1-\alpha)\sqrt{ab}+\alpha\frac{a+b}{2}$, $0 \leqslant \alpha \leqslant 1$, $a,b \in \mathbb{R}^+$ is considered by R. Bhatia in [1]. We shall discuss the complete interpolation and comparison of matrix version of such means by extending the range of α from $[0,1]$ to $\mathbb{R}^+$. We shall also discuss some more results involving Heinz means.

1. Introduction

In what follows, the capital letters $A,B,C,\dots$ denote the $n\times n$ (n arbitrary but fixed) matrices over the algebra of complex numbers, i.e. elements of M_n. By P_n and S_n, we denote the set of positive definite and the set of positive semidefinite matrices respectively. The Schur product of two matrices $A=(a_{ij})_{i,j}$ and $B=(b_{ij})_{i,j}$ in M_n is defined to be the matrix $A\circ B$ whose $i,j-$entry is $a_{ij}b_{ij}$. For any matrix $A\in M_n$, $\sigma(A)$ denotes the set of singular values of A i.e. eigenvalues of $(A^*A)^{1/2}$. The symbol $|||.|||$ denotes unitarily invariant norms throughout this paper.

Heron and Heinz means are two families of means defined respectively as,

$$F_\alpha(a,b) = (1-\alpha)\sqrt{ab}+\alpha\frac{a+b}{2} \tag{1.1}$$

$$H_\alpha(a,b) = \frac{a^{1-\alpha}b^\alpha + a^\alpha b^{1-\alpha}}{2} \tag{1.2}$$

for $0\leqslant\alpha\leqslant 1$ and $a,\ b\in\mathbb{R}^+$. The first family is clearly the linear interpolant between arithmetic and geometric mean and satisfies $F_\alpha \leqslant F_\beta$ whenever $\alpha\leqslant\beta$ and $\alpha,\beta\in\mathbb{R}^+$. Using simple arguments it is proved in [1] that

$$H_\nu(a,b) \leqslant F_{\alpha(\nu)}(a,b) \tag{1.3}$$

for $\alpha(\nu) = (2\nu-1)^2$ and $0\leqslant\nu\leqslant 1$.

There is yet, another mean of interest in several branches of science like geometry, statistics and thermodynamics, the logarithmic mean, defined as

$$L(a,b) = \frac{a-b}{\log a - \log b} = \int_0^1 a^t b^{1-t}\,dt. \tag{1.4}$$

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In [1], R. Bhatia proved that $f(\alpha) \leqslant f(1/2)$ for $\alpha \in [0,1/2]$, while $f(\alpha)$ is an increasing function of α on $[1/2,1]$ with $f(\alpha)$ as one of the possible matrix version of (1.1) and is defined as follows

$$f(\alpha) = \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|, \tag{1.5}$$

for $A,\ B \in P_n$ and $X \in M_n$.

For the matrix version of (1.2) and (1.4), and more about these, the reader may refer to [2],[4] and [5]. In this note, we shall prove $f(\alpha) \leqslant f(1/2)$ for $0 \leqslant \alpha \leqslant 1/2$ and $f(\alpha)$ is an increasing function for $\alpha \in [1/2,\infty)$. This is in fact the generalization of monotonic property of matrix version of (1.1) as in the case of $\alpha \in \mathbb{R}^+$ of $F_\alpha(a,b)$ for $a,b \in \mathbb{R}^+$. As an outcome of above result we shall also present a possible generalized matrix analogue of (1.3) i.e.

$$\frac{1}{2}|||A^\nu XB^{1-\nu} + A^{1-\nu}XB^\nu||| \leqslant \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right| \tag{1.6}$$

for $1/4 \leqslant \nu \leqslant 3/4$ and $\alpha \in [1/2,\infty)$.

A comparison of possible matrix version of (1.4) i.e.

$$\left|\left|\left|\int_0^1 A^t XB^{1-t}\,dt\right|\right|\right|$$

for $A,\ B \in P_n,\ \ X \in M_n$ and (1.5) for $\alpha \in [1/2,\infty)$ is established. Some more general results are indicated. We shall further conclude similar result for another linear interpolant matrix version of the Heinz and the arithmetic mean, i.e.

$$g(\alpha) = \left|\left|\left|(1-\frac{\alpha}{2})(A^{2/3}XB^{1/3} + A^{1/3}XB^{2/3}) + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|.$$

2. Main results

We shall use the following Theorem (2.1) and Lemma (2.2) in the sequel.

THEOREM 2.1. *For $A,B \in M_n$ with A as positive semidefinite, we have*

$$|||A \circ B||| \leqslant \max a_{ii}\ |||B|||$$

where a_{ii}'s for $i = 1,2,\cdots,n$ are the diagonal entries of matrix A.

LEMMA 2.2. *Let $\sigma_1,\sigma_2,\cdots,\sigma_n$ be positive numbers, $-1 \leqslant r \leqslant 1$, and $-2 < t \leqslant 2$. Then the $n \times n$ matrix*

$$W = \left(\frac{\sigma_i^r + \sigma_j^r}{\sigma_i^2 + t\sigma_i\sigma_j + \sigma_j^2}\right)$$

is positive semidefinite.

For a proof of Theorem (2.1) reader may refer to Horn and Johnson ([6], p. 343) and for Lemma (2.2), Bhatia and Parthasarthy [3] and Zhan ([8], p. 75–76).

THEOREM 2.3. *For $A,B \in P_n$ and $X \in M_n$ and $|||.|||$ any unitarily invariant norm, the function*

$$f(\alpha) = \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|$$

is increasing for $\alpha \in [1/2,\infty)$ and $f(\alpha) \leqslant f(1/2)$ for $\alpha \in [0,1/2]$.

Proof. We first prove the result for $\alpha > 0$ and $A = B$, i.e.,

$$f(\alpha) = \left|\left|\left|(1-\alpha)A^{1/2}XA^{1/2} + \alpha\left(\frac{AX+XA}{2}\right)\right|\right|\right| = \frac{\alpha}{2}p(\alpha),$$

where, $p(\alpha) = |||\left(h(\alpha)A^{1/2}XA^{1/2} + AX + XA\right)|||$ and $h(\alpha) = 2(\alpha^{-1} - 1)$. We may assume without loss of generality, $A = \mathrm{diag}(\lambda_1, \lambda_2 \cdots, \lambda_n)$, $\lambda_i > 0$, (due to unitarily invariant property of $|||.|||$). Then

$$\begin{aligned}
&h(\alpha)A^{1/2}XA^{1/2} + AX + XA \\
&= \left(\left(h(\alpha)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j\right)x_{ij}\right)_{i,j} \\
&= \left(\frac{h(\alpha)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j}{h(\beta)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j}\right)_{i,j} \circ \left(h(\beta)A^{1/2}XA^{1/2} + AX + XA\right) \\
&= Z \circ \left(h(\beta)A^{1/2}XA^{1/2} + AX + XA\right),
\end{aligned}$$

where

$$Z = \left(\frac{h(\alpha)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j}{h(\beta)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j}\right)_{i,j}.$$

Now the matrix Z can be written as

$$\left(1 + \frac{(h(\alpha)-h(\beta))\lambda_i^{1/2}\lambda_j^{1/2}}{h(\beta)\lambda_i^{1/2}\lambda_j^{1/2}+\lambda_i+\lambda_j}\right)_{i,j} = (1)_{i,j} + \left(\lambda_i^{1/2}\left(\frac{h(\alpha)-h(\beta)}{h(\beta)\lambda_i^{1/2}\lambda_j^{1/2}+\lambda_i+\lambda_j}\right)\lambda_j^{1/2}\right)_{i,j},$$

which will be positive semidefinite if the matrix,

$$\left(\frac{h(\alpha)-h(\beta)}{h(\beta)\lambda_i^{1/2}\lambda_j^{1/2} + \lambda_i + \lambda_j}\right)_{i,j},$$

is positive semidefinite. By Lemma (2.2), the later matrix is positive semidefinite if and only if $h(\alpha) \geqslant h(\beta)$ and $2 \geqslant h(\beta) > -2$. Since $h(\alpha) = 2(\alpha^{-1} - 1)$, so continuously decreasing function on positive half line from $[1/2,\infty) \to (-2,2]$. Hence $h(\alpha) \geqslant h(\beta)$ for all $\beta \geqslant \alpha$. Using Theorem (2.1), we get $p(\alpha) \leqslant \left(\frac{h(\alpha)+2}{h(\beta)+2}\right) p(\beta)$. This proves the result for $A = B$ and $\alpha \in [1/2,\infty)$.

For $\alpha \in (0,1/2]$, we have $2 \leqslant h(\alpha) < \infty$ and $h(\alpha) \geqslant h(1/2) = 2$ and so the matrix Z with $\beta = 1/2$ is positive semidefinite using Lemma (2.2). The case $\alpha = 0$ trivially holds, since by Lemma (2.2) the matrix

$$\left(\frac{\lambda_i^{1/2}\lambda_j^{1/2}}{\lambda_i^{1/2}\lambda_j^{1/2}+\lambda_i+\lambda_j}\right)_{i,j} = \left(\lambda_i^{1/2}\left(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}+\lambda_i+\lambda_j}\right)\lambda_j^{1/2}\right)_{i,j}$$

is positive semidefinite. This gives us the desired result for this case, i.e. $\alpha p(\alpha) \leqslant \frac{1}{2}p(1/2)$. Equivalently saying that $f(\alpha) \leqslant f(1/2)$ for all $\alpha \in [0,1/2]$.

The general case follows on replacing A by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and X by $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. □

COROLLARY 2.4. *Let $A,B,X \in M_n$ with A,B positive definite. Then for any unitarily invariant norm $|||.|||$ and a matrix monotone increasing function $f:(0,\infty) \to (0,\infty)$ with $f^{\perp}(x) = x(f(x))^{-1}$,*

$$\begin{aligned}&\frac{1}{2}|||A^{1/4}(f(A^{1/2})Xf^{\perp}(B^{1/2}) + f^{\perp}(A^{1/2})Xf(B^{1/2}))B^{1/4}||| \\ \leqslant &\left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right| \qquad (2.1)\end{aligned}$$

holds for $\alpha \in [1/2,\infty)$.

Proof. For $\alpha = 1/2$, (2.1) has already been proved in Singh and Vasudeva ([7], Theorem 1.1, p. 622). Now, using Theorem (2.3), we get the desired result. □

REMARK 2.5. We remark here that above corollary (2.4) is one of the possible generalizations of an inequality by Zhan ([8] Th.4.24, p. 76).

Now we shall settle the claim to prove (1.6) as assured earlier.

COROLLARY 2.6. *Let $A,B,X \in M_n$ with A,B positive definite. Then for any unitarily invariant norm $|||.|||$, $1/4 \leqslant \nu \leqslant 3/4$ and $\alpha \in [1/2,\infty)$,*

$$\frac{1}{2}|||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leqslant \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2} + \alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|.$$

Proof. Choosing $f(x) = x^{2\nu-1/2}$ in corollary (2.4), we get the desired result. □

COROLLARY 2.7. *Let $A,B,X \in M_n$ with A,B positive definite. Then for any unitarily invariant norm $|||.|||$ and a matrix monotone increasing function $f:(0,\infty)\to(0,\infty)$*

$$\frac{\lambda}{2f(\lambda)}|||A^{1/4}(f(A^{1/2})X+Xf(B^{1/2}))B^{1/4}||| \leqslant \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right| \tag{2.2}$$

holds, where $\lambda=\min\{\sigma(A),\sigma(B)\}$ and $\alpha\in[1/2,\infty)$.

Proof. For $\alpha=1/2$, (2.2) has already been proved in Singh and Vasudeva ([7], Corollary 2.5, p. 622). Again, using Theorem (2.3), we get the desired result. □

COROLLARY 2.8. *Let $A,B,X \in M_n$ with A,B positive definite. Then for any unitarily invariant norm $|||.|||$ and $\lambda=\min\{\sigma(A),\sigma(B)\}$,*

$$\frac{\lambda}{2\log(1+\lambda)}|||A^{1/4}(\log(I+A^{1/2})X+X\log(I+B^{1/2}))B^{1/4}||| \leqslant \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right| \tag{2.3}$$

holds for $\alpha\in[1/2,\infty)$.

Proof. Taking $f(x)=\log(1+x)$ in corollary (2.7), we get the desired result. □

COROLLARY 2.9. *Let $A,B,X \in M_n$ with A,B positive definite. Then for any unitarily invariant norm $|||.|||$,*

$$|||A^{1/2}XB^{1/2}||| \leqslant |||\int_0^1 A^tXB^{1-t}dt||| \leqslant \left|\left|\left|(1-\alpha)A^{1/2}XB^{1/2}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right| \tag{2.4}$$

holds for $\alpha\in[1/2,\infty)$ (c.f. [2], Th. 5.4.7, p. 163).

Proof. For $\alpha=1/2$, (2.4) has already been proved in Hiai and Kosaki ([5], p. 924). Further, using Theorem (2.3), we get the desired result. □

THEOREM 2.10. *For $A,B\in P_n$ and $X\in M_n$ and $|||.|||$ any unitarily invariant norm, the function*

$$g(\alpha)=\left|\left|\left|(1-\frac{\alpha}{2})(A^{2/3}XB^{1/3}+A^{1/3}XB^{2/3})+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|\right|$$

is increasing for $\alpha\in[1/2,\infty)$ and $g(\alpha)\leqslant g(1/2)$ for $\alpha\in[0,1/2]$.

Proof. Once again following the same lines of the proof of Theorem (2.3), we shall prove the result for $\alpha > 0,\ A = B$ and $A = \operatorname{diag}(\lambda_1, \lambda_2 \cdots, \lambda_n)$.

Suppose

$$g(\alpha) = \left|\left|\left|(1-\frac{\alpha}{2})(A^{2/3}XA^{1/3}+A^{1/3}XA^{2/3})+\alpha\left(\frac{AX+XA}{2}\right)\right|\right|\right| = \frac{\alpha}{2}q(\alpha),$$

where $q(\alpha) = |||h_1(\alpha)(A^{2/3}XA^{1/3}+A^{1/3}XA^{2/3})+AX+XA|||$ and $h_1(\alpha) = 2\alpha^{-1}-1$.

$$\begin{aligned} & h_1(\alpha)(A^{2/3}XA^{1/3}+A^{1/3}XA^{2/3})+AX+XA \\ &= \left(\left(h_1(\alpha)(\lambda_i^{2/3}\lambda_j^{1/3}+\lambda_i^{1/3}\lambda_j^{2/3})+\lambda_i+\lambda_j\right)x_{ij}\right)_{i,j} \\ &= Y \circ \left(h_1(\beta)(A^{2/3}XA^{1/3}+A^{1/3}XA^{2/3})+AX+XA\right). \end{aligned}$$

Now the matrix Y can be written as

$$\begin{aligned} & \left(\frac{h_1(\alpha)(\lambda_i^{2/3}\lambda_j^{1/3}+\lambda_i^{1/3}\lambda_j^{2/3})+\lambda_i+\lambda_j}{h_1(\beta)(\lambda_i^{2/3}\lambda_j^{1/3}+\lambda_i^{1/3}\lambda_j^{2/3})+\lambda_i+\lambda_j}\right)_{i,j} \\ &= \left(1+\frac{(h_1(\alpha)-h_1(\beta))\lambda_i^{1/3}\lambda_j^{1/3}}{(h_1(\beta)-1)\lambda_i^{1/3}\lambda_j^{1/3}+\lambda_i^{2/3}+\lambda_j^{2/3}}\right)_{i,j} \\ &= (1)_{i,j} + \left(\lambda_i^{1/3}\left(\frac{h_1(\alpha)-h_1(\beta)}{(h_1(\beta)-1)\lambda_i^{1/3}\lambda_j^{1/3}+\lambda_i^{2/3}+\lambda_j^{2/3}}\right)\lambda_j^{1/3}\right)_{i,j}. \end{aligned}$$

Again by Lemma (2.2), the later matrix is positive semidefinite if and only if $h_1(\alpha) \geqslant h_1(\beta)$ and $2 \geqslant h_1(\beta)-1 > -2$. Since $h(\alpha) = h_1(\alpha)-1 = (2\alpha^{-1}-2)$, so continuously decreasing function on positive half line and from $[1/2,\infty) \to (-2,2]$. Hence as in Theorem (2.3) we have $h(\alpha) \geqslant h(\beta)$ and so $h_1(\alpha) \geqslant h_1(\beta)$ for all $\beta \geqslant \alpha$.

Using Theorem (2.1), we get $q(\alpha) \leqslant \left(\frac{h_1(\alpha)+1}{h_1(\beta)+1}\right) q(\beta)$. This proves the result for $A = B$ and $\alpha \in [1/2,\infty)$.

For $\alpha \in (0,1/2]$, we have $3 = h_1(1/2) \leqslant h_1(\alpha) < \infty$ and so the matrix Y with $\beta = 1/2$ is positive semidefinite using Lemma (2.2). The case $\alpha = 0$ can easily be seen by the positive semidefiniteness of the matrix $\left(\lambda_i^{1/3}\left(\frac{1}{2\lambda_i^{1/3}\lambda_j^{1/3}+\lambda_i^{2/3}+\lambda_j^{2/3}}\right)\lambda_j^{1/3}\right)_{i,j}$ which is so, using Lemma (2.2). This gives us the desired result for this case, i.e. $\alpha q(\alpha) \leqslant \frac{1}{2}q(1/2)$. Equivalently saying that $g(\alpha) \leqslant g(1/2)$ for all $\alpha \in [0,1/2]$.

The general case follows on replacing A by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and X by $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$. □

COROLLARY 2.11. *Let $A,B,X \in M_n$ with A,B positive definite and $f(\alpha)$ and $g(\alpha)$ are same as taken in Theorem* (2.3) *and* (2.10) *respectively. Then*

$$f(0) \leqslant \frac{1}{2}g(0) \leqslant f(\alpha) \tag{2.5}$$

for $\alpha \in [1/2, \infty)$, *or equivalently, for any unitarily invariant norm* $|||.|||$ *and* $-2 < t \leqslant 2$,

$$|||A^{1/2}XB^{1/2}||| \leqslant \frac{1}{2}|||A^{2/3}XB^{1/3} + A^{1/3}XB^{2/3}||| \leqslant \frac{1}{2+t}|||AX + XB + tA^{1/2}XB^{1/2}|||.$$

Proof. For the first inequality in (2.5) see [1] and for the second inequality take $\nu = 2/3$ in corollary (2.6) and $\alpha = \frac{2}{2+t}$. □

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