Mathematical
Inequalities
& Applications
Volume 16, Number 1 (2013), 175–191 doi:10.7153/mia-16-13

APPROXIMATION BY $(C,1)$ AND ABEL–POISSON MEANS OF FOURIER SERIES ON HEXAGONAL DOMAINS

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(*Communicated by J. Pečarić*)

Abstract. The approximation problems by Cesàro $(C,1)$ means and by Abel-Poisson means of Fourier series on hexagonal domains are studied. The estimates for the rate of convergence of these means are obtained for functions in Lipschitz classes.

1. Introduction

Approximation by trigonometric polynomials, or equivalently by complex exponentials, is at the heart of approximation theory. The most important trigonometric polynomials used in the approximation theory are obtained by linear summation methods of Fourier series of 2π-periodic functions on the real line (Cesàro means, Abel-Poisson means, de la Vallèe-Poussin means, etc.). Much of the advance in the theory of trigonometric approximation is due to the periodicity of the functions. The elegant presentations of results on trigonometric approximation can be found in the monographs [7], [1] and [2].

A straightforward extension to several variables is the tensor product type, where one works with functions that are 2π-periodic in each of their variables. But, in the case of non tensor-product domain one needs another definition of periodicity. For such domains there are other definitions of periodicity, and the most notable one is the periodicity defined by the lattices. A lattice is the discrete subgroup $A\mathbb{Z}^d$ of the $d-$dimensional Euclidean space $\mathbb{R}^d$, where A is a nonsingular matrix, and the periodic function satisfies $f(x+Ak) = f(x)$ for all $k \in \mathbb{Z}^d$. With such periodicity, one works with exponentials of the form $e^{2\pi i\langle \alpha,x\rangle}$, where α and x are in proper sets of $\mathbb{R}^d$, not necessarily the usual trigonometric polynomials.

A theorem of Fuglede ([3]) states that a set tiles $\mathbb{R}^d$ by lattice translation if and only if it has an orthonormal basis of exponentials $e^{2\pi i\langle \alpha,x\rangle}$ with α in the dual lattice. Such a set is called a spectral set. This Theorem suggests that one can study Fourier series and approximation problems on a spectral set. For the simplest spectral sets, cubes in $\mathbb{R}^d$, the Fourier series with respect to the lattice coincides with the classical Fourier series of functions of d variables. Besides the usual rectangular domain in $\mathbb{R}^2$, the simplest spectral set is the regular hexagon.

Mathematics subject classification (2010): 41A25, 42A10, 42B08.
Keywords and phrases: Hexagonal Fourier series, Cesàro means, Abel-Poisson means, Lipschitz class.

Discrete Fourier analysis on lattices was developed in [4]. In the paper [4], the case of hexagon lattice was studied in details; in particular, Lagrange interpolation and cubature formulas by trigonometric functions on a regular hexagon and on an equilateral triangle were studied. In [6], the author studied Cesàro and Abel summability of Fourier series over the regular hexagon, and deduced compact formulas for the Fejèr and Poisson kernels of hexagonal Fourier series. Furthermore, in the same paper, the direct and inverse approximation theorems were established in terms of a modulus of smoothness. In the same paper the author said that “The purpose of this paper is to show, using the hexagonal domain as an example, that Fourier series on a spectral set have a rich structure that permits an extensive theory of Fourier series and approximation. It is our hope that this work may stimulate further studies in this area.” By this motivation, as an introductory work, we try to obtain results about the degree of approximation by Cesàro and Abel-Poisson means of hexagonal Fourier series. We have to point out that, in our proofs we used the methods of paper [6].

2. Hexagonal Fourier series

In this section, we shall give basic definitions and properties of Fourier series with respect to the lattices, and as a special case we shall study the hexagonal Fourier series. More detailed information can be found, as mentioned above, in [4] and [6].

Let A be a $d\times d$ matrix whose columns are linearly independent vectors. The set

$$L_A = A\mathbb{Z}^d := \left\{ Ak : k \in \mathbb{Z}^d \right\}$$

is called the $(d-\text{dimensional})$ lattice generated by A, and A is called the generator matrix of L_A. The lattice generated by the matrix $A^{-tr} := (A^{tr})^{-1}$ is called the dual lattice of L_A and is denoted by $L_A^{\perp}$. It is easy to show that

$$L_A^{\perp} = \left\{ x \in \mathbb{R}^d : \langle x,y\rangle \in \mathbb{Z} \text{ for all } y \in L_A \right\},$$

where $\langle x,y\rangle$ is the Euclidean inner product of x and y, that is $\langle x,y\rangle = x^{tr}y$.

Let $\Omega\subset\mathbb{R}^d$ be a bounded set. Ω is said to tile $\mathbb{R}^d$ with the lattice L_A $\left(\Omega + L_A = \mathbb{R}^d\right)$ if

$$\sum_{\alpha\in L_A} \chi_\Omega(x+\alpha) = 1, \text{ for almost all } x\in\mathbb{R}^d.$$

The space $L^2(\Omega)$ becomes a Hilbert space, where the inner product is defined by

$$\langle f,g\rangle_\Omega := \frac{1}{m(\Omega)} \int_\Omega f(x)\overline{g(x)}dx, \quad f,g\in L^2(\Omega). \tag{1}$$

A theorem of Fuglede ([3]) states that an open bounded set $\Omega\subset\mathbb{R}^d$ tiles $\mathbb{R}^d$ with the lattice L_A if and only if the set

$$\left\{ e^{2\pi i\langle \alpha,x\rangle} : \alpha \in L_A^{\perp} \right\}$$

is an orthonormal basis of $L^2(\Omega)$ with respect to the inner product (1).

It is known that if $\Omega + L_A = \mathbb{R}^d$, then the measure of Ω is $m(\Omega) = |\det A|$.

If $\Omega + L_A = \mathbb{R}^d$ then the set Ω is called a spectral set for the lattice L_A. In this case we write Ω_A in place of Ω.

Since $\alpha \in L_A^{\perp}$ means $\alpha = A^{-tr}k$ for some $k \in \mathbb{Z}^d$, the orthogonality relation becomes

$$\frac{1}{|\det A|}\int_{\Omega_A} e^{2\pi i\langle A^{-tr}k,x\rangle}dx = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}, \quad k \in \mathbb{Z}^d. \tag{2}$$

The Fourier series of a function $f \in L^1(\Omega)$ is

$$f(x) \sim \sum_{k\in\mathbb{Z}^d} c_k e^{2\pi i\langle A^{-tr}k,x\rangle},$$

where

$$c_k = \frac{1}{|\det A|}\int_{\Omega_A} f(x)\, e^{-2\pi i\langle A^{-tr}k,x\rangle}dx, \quad k \in \mathbb{Z}^d.$$

For a given lattice L_A the spectral set is not unique. We fix Ω such that Ω contains 0 in its interior, and that the tiling

$$\Omega + L_A = \mathbb{R}^d$$

holds pointwise and without overlapping:

$$\sum_{k\in\mathbb{Z}^d} \chi_\Omega(x+Ak) = 1, \quad \forall x \in \mathbb{R}^d \tag{3}$$

and

$$(\Omega + Ak)\cap(\Omega + Aj) = \emptyset, \quad k \neq j. \tag{4}$$

For example we can take $\Omega = \left[-\frac{1}{2},\frac{1}{2}\right)^d$ for the standard lattice $\mathbb{Z}^d$.

A function defined on $\mathbb{R}^d$ is called a periodic function with respect to the lattice $A\mathbb{Z}^d$ or A-periodic if

$$f(x+Ak) = f(x), \quad \forall k \in \mathbb{Z}^d.$$

Since the function $x \to e^{2\pi i\langle A^{-tr}k,x\rangle}$ is periodic with respect to the lattice $A\mathbb{Z}^d$, the orthogonality relation (2) is independent of the choice of Ω.

The points $x, y \in \mathbb{R}^d$ are said to be congruent with respect to the lattice $A\mathbb{Z}^d$ if $x - y \in A\mathbb{Z}^d$. In this case we write $x \equiv y \pmod{A}$.

The generator matrix and the spectral set of the hexagonal lattice $L_H = H\mathbb{Z}^2$ are given by

$$H = \begin{bmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{bmatrix}$$

and

$$\Omega_H = \left\{(x_1,x_2) \in \mathbb{R}^2 : -1 \leqslant x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1\right\}.$$

It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfies $t_1+t_2+t_3=0$ ([5], [4]). If we define

$$t_1 := -\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2}, \quad t_2 := x_2, \quad t_3 := -\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{5}$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1,t_2,t_3) \in \mathbb{R}^3 : -1 \leqslant t_1,t_2,-t_3 < 1,\ t_1+t_2+t_3 = 0\right\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H = \left\{\mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3 : t_1+t_2+t_3 = 0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is

$$\mathbb{Z}^3_H = \mathbb{Z}^3 \cap \mathbb{R}^3_H = \left\{\mathbf{k} = (k_1,k_2,k_3) \in \mathbb{Z}^3 : k_1+k_2+k_3 = 0\right\}.$$

The inner product on the hexagon under homogeneous coordinates becomes

$$\begin{aligned}\langle f,g\rangle_H &= \frac{1}{m(\Omega_H)} \int_{\Omega_H} f(x_1,x_2)\overline{g(x_1,x_2)}dx_1dx_2 \\ &= \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t},\end{aligned}$$

where $|\Omega|$ denotes the area of Ω.

If we use the change of variables $x = (x_1,x_2) \to \mathbf{t} = (t_1,t_2,t_3)$, where t_1,t_2,t_3 are defined by (5) we get

$$\left\langle H^{-tr}k,x\right\rangle = \frac{1}{3}\langle \mathbf{k},\mathbf{t}\rangle.$$

Therefore, introducing the notation

$$\phi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3}\langle \mathbf{j},\mathbf{t}\rangle}, \quad \mathbf{j} \in \mathbb{Z}^3_H,$$

the orthogonality relation (2) becomes

$$\left\langle \phi_{\mathbf{k}},\phi_{\mathbf{j}}\right\rangle_{\Omega} = \begin{cases} 1, & \mathbf{k}=\mathbf{j} \\ 0, & \mathbf{k}\neq \mathbf{j}\end{cases}, \quad \mathbf{k},\mathbf{j} \in \mathbb{Z}^3_H,$$

and as a corollary of Fuglede's theorem the set $\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^3_H\right\}$ forms an orthonormal basis for $L^2(\Omega)$.

Under the homogeneous coordinates, $x \equiv y \pmod{H}$ becomes $\mathbf{t} \equiv \mathbf{s} \pmod{3}$, where

$$\mathbf{t} \equiv \mathbf{s} \pmod{3} \Leftrightarrow t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod{3}.$$

Hence a function f is H-periodic if and only if $f(\mathbf{t}) = f(\mathbf{t}+\mathbf{j})$ whenever $\mathbf{j} \equiv \mathbf{0} \pmod 3$. It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are H-periodic. If the function f is H-periodic then

$$\int_{\Omega} f(\mathbf{t}+\mathbf{s})\,d\mathbf{t} = \int_{\Omega} f(\mathbf{t})\,d\mathbf{t}, \quad \mathbf{s} \in \mathbb{R}^3_H.$$

For every natural number n, we define two subsets of $\mathbb{Z}^3_H$ by

$$\mathbb{H}^*_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}^3_H : -n \leqslant j_1, j_2, j_3 \leqslant n \right\}$$

and

$$\mathbb{J}_n := \mathbb{H}^*_n \backslash \mathbb{H}^*_{n-1}.$$

$\mathbb{H}^*_n$ consists of all integer points inside the hexagon $n\overline{\Omega}$ and $\mathbb{J}_n$ is the intersection of $\mathbb{H}^*_n$ with the boundary of $n\Omega$. The elements of the set

$$\mathscr{H}^*_n := \operatorname{span}\left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}^*_n \right\}, \quad n \in \mathbb{N}$$

are called the trigonometric polynomials over Ω. It is clear that the dimension of $\mathscr{H}^*_n$ is $\#\mathbb{H}^*_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an H-periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}^3_H} c_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{6}$$

where

$$c_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t})\, e^{-\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle}\, d\mathbf{t}, \quad \mathbf{j} \in \mathbb{Z}^3_H.$$

In the study of the summability of hexagonal Fourier series it is more convenient to write the series (6) as blocks are groupped according to $\mathbb{J}_n$:

$$f(\mathbf{t}) \sim \sum_{k=0}^{\infty} \sum_{\mathbf{j} \in \mathbb{J}_k} c_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}). \tag{7}$$

The nth partial sums of the series (6) are defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}^*_n} c_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}) = \sum_{k=0}^{n} \sum_{\mathbf{j} \in \mathbb{J}_k} c_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is easy to show that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}-\mathbf{s})\, D_n(\mathbf{s})\, d\mathbf{s},$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}^*_n} \phi_{\mathbf{j}}(\mathbf{t}) = \sum_{k=0}^{n} \sum_{\mathbf{j} \in \mathbb{J}_k} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that ([5], [4]) the Dirichlet kernel has the compact formula

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}),$$

where

$$\Theta_n(\mathbf{t}) = \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}, \quad \mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3_H.$$

We denote by $C_H\left(\overline{\Omega}\right)$ the Banach space of $H-$periodic complex valued continuous functions, whose norm is the uniform norm:

$$\|f\|_\infty = \sup\left\{|f(\mathbf{t})| : \mathbf{t} \in \overline{\Omega}\right\}.$$

The Lipschitz class $Lip_\alpha\left(\overline{\Omega}\right),\ 0 < \alpha \leqslant 1$ is defined by

$$Lip_\alpha\left(\overline{\Omega}\right) = \left\{f \in C_H\left(\overline{\Omega}\right) : |f(\mathbf{t}) - f(\mathbf{s})| \leqslant M\|\mathbf{t}-\mathbf{s}\|_\infty^\alpha\right\},$$

where $\|\mathbf{t}\|_\infty = \max\{|t_1|,|t_2|,|t_3|\}$.

3. Approximation by $(C,1)$ means of hexagonal Fourier series

The Cesàro $(C,\delta),\delta \geqslant 0$ means of the Fourier series (7) are defined by

$$S_n^{(\delta)}(f)(\mathbf{t}) := \frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})K_n^{(\delta)}(\mathbf{s})\,d\mathbf{s},$$

where

$$K_n^{(\delta)}(\mathbf{t}) := \frac{1}{A_n^\delta}\sum_{k=0}^{n}A_{n-k}^\delta\sum_{\mathbf{j}\in\mathbb{J}_k}\phi_{\mathbf{j}}(\mathbf{t}), \quad A_n^\delta = \binom{n+\delta}{n}.$$

It is evident that $K_n^{(0)}(\mathbf{t}) = D_n(\mathbf{t})$, hence $S_n^{(0)}(f)(\mathbf{t}) = S_n(f)(\mathbf{t})$, where

$$K_n^{(1)}(\mathbf{t}) = \frac{1}{n+1}\sum_{k=0}^{n}D_k(\mathbf{t}) = \frac{1}{n+1}\Theta_n(\mathbf{t}).$$

By orthogonality of $\phi_{\mathbf{j}}$'s it follows that

$$\frac{1}{|\Omega|}\int_\Omega K_n^{(1)}(\mathbf{t})\,d\mathbf{t} = 1.$$

The famous theorem of Fejèr states that if the function f is $2\pi-$periodic and continuous, then the sequence of $(C,1)$ means of its Fourier series converges uniformly to f (see, for example [7, p. 89]).

The analogue of Fejèr's theorem for hexagonal Fourier series was proved in [6]:

THEOREM A. *If* $f \in C_H\left(\overline{\Omega}\right)$, *then the sequence* $S_n^{(1)}(f)$ *of* $(C,1)$ *means converges uniformly to* f *on* $\overline{\Omega}$.

In the case of Classical Fourier series, S. N. Bernstein proved the following theorem about the rate of convergence of $(C,1)$ means (see, for example [1, pp. 80–82]):

THEOREM B. *Let* f *be a* $2\pi-$*periodic and continuous function. If* $f \in Lip\alpha$, *then*

$$\left\|f - S_n^{(1)}(f)\right\|_\infty = \begin{cases} O\left(n^{-\alpha}\right), & 0<\alpha<1 \\ O\left(n^{-1}\log n\right), & \alpha = 1 \end{cases}$$

for $n = 1,2,\dots$.

In this work we try to obtain similar estimate for $(C,1)$ means of hexagonal Fourier series. The main theorem of this section is Theorem 1.

THEOREM 1. *If* $f \in Lip_\alpha\left(\overline{\Omega}\right)$ *then*

$$\left\|f - S_n^{(1)}(f)\right\|_\infty = \begin{cases} O\left(n^{-\alpha}\right), & 0<\alpha<1 \\ O\left(n^{-1}(\log n)^2\right), & \alpha = 1 \end{cases}$$

for $n = 1,2,\dots$.

Proof. Let $f \in Lip_\alpha\left(\overline{\Omega}\right)$. Since

$$f(\mathbf{t}) - S_n^{(1)}(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_\Omega \left(f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})\right) K_n^{(1)}(\mathbf{s})\,d\mathbf{s},$$

we have

$$\begin{aligned} \left|f(\mathbf{t}) - S_n^{(1)}(f)(\mathbf{t})\right| &\leqslant \frac{1}{|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})| \left|K_n^{(1)}(\mathbf{s})\right| d\mathbf{s} \\ &\leqslant \frac{M}{|\Omega|}\int_\Omega \|\mathbf{s}\|_\infty^\alpha \left|K_n^{(1)}(\mathbf{s})\right| d\mathbf{s} \\ &= \frac{M}{(n+1)|\Omega|}\int_\Omega \|\mathbf{s}\|_\infty^\alpha |\Theta_n(\mathbf{s})|\,d\mathbf{s}. \end{aligned} \tag{8}$$

So we have to estimate the integral

$$\int_\Omega \|\mathbf{t}\|_\infty^\alpha |\Theta_n(\mathbf{t})|\,d\mathbf{t}.$$

But, since $\|\mathbf{t}\|_\infty^\alpha |\Theta_n(\mathbf{t})|$ is a symmetric function of t_1, t_2, t_3 it is sufficient to consider the integral over the triangle

$$\begin{aligned} \Delta &:= \left\{\mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}_H^3 : 0 \leqslant t_1, t_2, -t_3 \leqslant 1\right\} \\ &= \left\{(t_1,t_2) : t_1 \geqslant 0,\ t_2 \geqslant 0,\ t_1 + t_2 \leqslant 1\right\}, \end{aligned}$$

which is one of the six equilateral triangles in Ω. If we use the change of variables

$$s_1 := \frac{t_1 - t_3}{3} = \frac{2t_1 + t_2}{3}, \quad s_2 := \frac{t_2 - t_3}{3} = \frac{t_1 + 2t_2}{3} \tag{9}$$

as in [6], the integral

$$A_n := \int_{\Delta} \|\mathbf{t}\|_\infty^\alpha |\Theta_n(\mathbf{t})| \, d\mathbf{t}$$

becomes

$$A_n = 3 \int_{\widetilde{\Delta}} (s_1 + s_2)^\alpha \left| \frac{\sin((n+1)\pi s_1)\sin((n+1)\pi s_2)\sin((n+1)\pi(s_1 - s_2))}{\sin(\pi s_1)\sin(\pi s_2)\sin(\pi(s_1 - s_2))} \right| ds_1 ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta} := \{(s_1, s_2) : 0 \leqslant s_1 \leqslant 2s_2, \ 0 \leqslant s_2 \leqslant 2s_1, \ s_1 + s_2 \leqslant 1\}.$$

Since the integrant is a symmetric function of s_1 and s_2 we have

$$A_n = 6 \int_{\Delta^*} (s_1 + s_2)^\alpha \left| \frac{\sin((n+1)\pi s_1)\sin((n+1)\pi s_2)\sin((n+1)\pi(s_1 - s_2))}{\sin(\pi s_1)\sin(\pi s_2)\sin(\pi(s_1 - s_2))} \right| ds_1 ds_2,$$

where Δ^* is the half of $\widetilde{\Delta}$:

$$\Delta^* := \left\{(s_1, s_2) \in \widetilde{\Delta} : s_1 \leqslant s_2\right\} = \{(s_1, s_2) : s_1 \leqslant s_2 \leqslant 2s_1, \ s_1 + s_2 \leqslant 1\}.$$

The change of variables

$$s_1 := \frac{u_1 - u_2}{2}, \quad s_2 := \frac{u_1 + u_2}{2} \tag{10}$$

transforms the triangle Δ^* to another triangle

$$\Gamma := \left\{(u_1, u_2) : 0 \leqslant u_2 \leqslant \frac{u_1}{3}, \ 0 \leqslant u_1 \leqslant 1\right\},$$

and hence we get

$$A_n = 3 \int_{\Gamma} u_1^\alpha |\Theta_n^*(u_1, u_2)| \, du_1 du_2,$$

where

$$\Theta_n^*(u_1, u_2) := \frac{\sin\left(\frac{(n+1)(u_1 - u_2)\pi}{2}\right)\sin\left(\frac{(n+1)(u_1 + u_2)\pi}{2}\right)\sin((n+1)u_2\pi)}{\sin\left(\frac{(u_1 - u_2)\pi}{2}\right)\sin\left(\frac{(u_1 + u_2)\pi}{2}\right)\sin(u_2\pi)}.$$

We need the well known inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \leqslant n \tag{11}$$

and

$$\sin t \geqslant \frac{2}{\pi} t, \quad 0 \leqslant t \leqslant \frac{\pi}{2} \tag{12}$$

to estimate the last integral.

We can write

$$A_n = 3 \left(\int\limits_{\Gamma_1} + \int\limits_{\Gamma_2} + \int\limits_{\Gamma_3} \right) u_1^{\alpha} \left| \Theta_n^* (u_1, u_2) \right| du_1 du_2,$$

where

$$\Gamma_1 := \left\{ (u_1, u_2) \in \Gamma : u_1 \leqslant \frac{3}{n} \right\},$$
$$\Gamma_2 := \left\{ (u_1, u_2) \in \Gamma : \frac{3}{n} \leqslant u_1,\ u_2 \leqslant \frac{1}{n} \right\},$$
$$\Gamma_3 := \left\{ (u_1, u_2) \in \Gamma : \frac{3}{n} \leqslant u_1,\ \frac{1}{n} \leqslant u_2 \right\}.$$

Let $0 < \alpha < 1$.

By (11),

$$\int\limits_{\Gamma_1} u_1^{\alpha} \left| \Theta_n^* (u_1, u_2) \right| du_1 du_2 \leqslant (n+1)^3 \int\limits_{\Gamma_1} u_1^{\alpha} du_1 du_2 = (n+1)^3 \int\limits_0^{1/n} \left(\int\limits_{3u_2}^{3/n} u_1^{\alpha} du_1 \right) du_2$$
$$= \frac{3^{\alpha+1}}{\alpha+1} (n+1)^3 \int\limits_0^{1/n} \left(\frac{1}{n^{\alpha+1}} - u_2^{\alpha+1} \right) du_2$$
$$= \frac{3^{\alpha+1}}{\alpha+1} \left(1 - \frac{1}{\alpha+2} \right) (n+1)^3 \frac{1}{n^{\alpha+2}} \leqslant c_1(\alpha) n^{-\alpha+1}.$$

Since

$$\frac{2u_1}{3} \leqslant u_1 - u_2 \leqslant \frac{\pi}{2}, \quad u_1 \leqslant u_1 + u_2 \leqslant \frac{\pi}{2},$$

by (12) we get

$$\int\limits_{\Gamma_2} u_1^{\alpha} \left| \Theta_n^* (u_1, u_2) \right| du_1 du_2 = \int\limits_0^{1/n} \left(\int\limits_{3/n}^{1} u_1^{\alpha} \left| \Theta_n^* (u_1, u_2) \right| du_1 \right) du_2$$
$$= \int\limits_0^{1/n} \left(\int\limits_{3/n}^{1} u_1^{\alpha} \left| \frac{\sin\left(\frac{(n+1)(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(n+1)(u_1+u_2)\pi}{2}\right)}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(u_1+u_2)\pi}{2}\right)} \right| du_1 \right) \left| \frac{\sin((n+1) u_2 \pi)}{\sin(u_2 \pi)} \right| du_2$$
$$\leqslant \int\limits_0^{1/n} \left(\int\limits_{3/n}^{1} u_1^{\alpha} \frac{du_1}{\left| \sin\left(\frac{(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(u_1+u_2)\pi}{2}\right) \right|} \right) \left| \frac{\sin((n+1) u_2 \pi)}{\sin(u_2 \pi)} \right| du_2$$

$$\leqslant \frac{3}{2}\int_{0}^{1/n}\left(\int_{3/n}^{1}u_1^{\alpha-2}du_1\right)\left|\frac{\sin\left((n+1)u_2\pi\right)}{\sin\left(u_2\pi\right)}\right|du_2$$

$$=\frac{3}{2}\frac{1}{1-\alpha}\left(\frac{3}{n^{\alpha-1}}-1\right)\int_{0}^{1/n}\left|\frac{\sin\left((n+1)u_2\pi\right)}{\sin\left(u_2\pi\right)}\right|du_2.$$

If we use (11) again, we obtain

$$\int_{\Gamma_2}u_1^{\alpha}\left|\Theta_n^*(u_1,u_2)\right|du_1du_2\leqslant c_2(\alpha)\,n^{-\alpha+1}.$$

By the inequality (12) we obtain

$$\int_{\Gamma_3}u_1^{\alpha}\left|\Theta_n^*(u_1,u_2)\right|du_1du_2=\int_{1/n}^{1/3}\left(\int_{3u_2}^{1}u_1^{\alpha}\left|\Theta_n^*(u_1,u_2)\right|du_1\right)du_2$$

$$=\int_{1/n}^{1/3}\left(\int_{3u_2}^{1}u_1^{\alpha}\left|\frac{\sin\left(\frac{(n+1)(u_1-u_2)\pi}{2}\right)\sin\left(\frac{(n+1)(u_1+u_2)\pi}{2}\right)}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right)\sin\left(\frac{(u_1+u_2)\pi}{2}\right)}\right|du_1\right)\left|\frac{\sin\left((n+1)u_2\pi\right)}{\sin\left(u_2\pi\right)}\right|du_2$$

$$\leqslant\int_{1/n}^{1/3}\left(\int_{3u_2}^{1}u_1^{\alpha}\frac{du_1}{\left|\sin\left(\frac{(u_1-u_2)\pi}{2}\right)\sin\left(\frac{(u_1+u_2)\pi}{2}\right)\right|}\right)\frac{du_2}{\left|\sin\left(u_2\pi\right)\right|}$$

$$\leqslant\frac{3}{2}\int_{1/n}^{1/3}\left(\int_{3u_2}^{1}u_1^{\alpha-2}du_1\right)\frac{du_2}{u_2}=\frac{3}{2}\frac{1}{1-\alpha}\int_{1/n}^{1/3}\left((3u_2)^{\alpha-1}-1\right)\frac{du_2}{u_2}$$

$$\leqslant\frac{3^{\alpha}}{2}\frac{1}{1-\alpha}\int_{1/n}^{1/3}u_2^{\alpha-2}du_2\leqslant c_3(\alpha)\,n^{-\alpha+1}.$$

Therefore, combining these three estimates and (8) we get

$$\left|f(\mathbf{t})-S_n^{(1)}(f)(\mathbf{t})\right|\leqslant c_4(\alpha)\,n^{-\alpha},\quad \mathbf{t}\in\overline{\Omega}.$$

Now let $\alpha=1$. We have to estimate

$$A_n=3\left(\int_{\Gamma_1}+\int_{\Gamma_2}+\int_{\Gamma_3}\right)u_1\left|\Theta_n^*(u_1,u_2)\right|du_1du_2.$$

By inequality (11),

$$\int_{\Gamma_1} u_1 |\Theta_n^*(u_1,u_2)| du_1 du_2 \leqslant (n+1)^3 \int_{\Gamma_1} u_1 du_1 du_2 = (n+1)^3 \int_0^{1/n} \left(\int_{3u_2}^{3/n} u_1 du_1 \right) du_2$$
$$= \frac{9}{2}(n+1)^3 \int_0^{1/n} \left(\frac{1}{n^2} - u_2^2 \right) du_2 = 3\frac{(n+1)^3}{n^3} \leqslant 24.$$

By using (12) and (11) we obtain,

$$\int_{\Gamma_2} u_1 |\Theta_n^*(u_1,u_2)| du_1 du_2 = \int_0^{1/n} \left(\int_{3/n}^{1} u_1 |\Theta_n^*(u_1,u_2)| du_1 \right) du_2$$
$$= \int_0^{1/n} \left(\int_{3/n}^{1} u_1 \left| \frac{\sin\left(\frac{(n+1)(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(n+1)(u_1+u_2)\pi}{2}\right)}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(u_1+u_2)\pi}{2}\right)} \right| du_1 \right) \left| \frac{\sin((n+1)u_2\pi)}{\sin(u_2\pi)} \right| du_2$$
$$\leqslant \int_0^{1/n} \left(\int_{3/n}^{1} u_1 \frac{du_1}{\left|\sin\left(\frac{(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(u_1+u_2)\pi}{2}\right)\right|} \right) \left| \frac{\sin((n+1)u_2\pi)}{\sin(u_2\pi)} \right| du_2$$
$$\leqslant \frac{3}{2} \int_0^{1/n} \left(\int_{3/n}^{1} \frac{du_1}{u_1} \right) \left| \frac{\sin\left(\frac{(n+1)u_2\pi}{2}\right)}{\sin\left(\frac{u_2\pi}{2}\right)} \right| du_2 = \frac{3}{2}(\log n - \log 3) \int_0^{1/n} \left| \frac{\sin((n+1)u_2\pi)}{\sin(u_2\pi)} \right| du_2$$
$$\leqslant \frac{3}{2}\frac{n+1}{n} \log n \leqslant 3\log n.$$

By inequality (12),

$$\int_{\Gamma_3} u_1 |\Theta_n^*(u_1,u_2)| du_1 du_2 = \int_{1/n}^{1/3} \left(\int_{3u_2}^{1} u_1 |\Theta_n^*(u_1,u_2)| du_1 \right) du_2$$
$$= \int_{1/n}^{1/3} \left(\int_{3u_2}^{1} u_1 \left| \frac{\sin\left(\frac{(n+1)(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(n+1)(u_1+u_2)\pi}{2}\right)}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right) \sin\left(\frac{(u_1+u_2)\pi}{2}\right)} \right| du_1 \right) \left| \frac{\sin((n+1)u_2\pi)}{\sin(u_2\pi)} \right| du_2$$
$$\leqslant \frac{3}{2} \int_{1/n}^{1/3} \left(\int_{3u_2}^{1} \frac{du_1}{u_1} \right) \left| \frac{\sin\left(\frac{(n+1)u_2\pi}{2}\right)}{\sin\left(\frac{u_2\pi}{2}\right)} \right| du_2 = \frac{3}{2} \int_{1/n}^{1/3} (-\log 3u_2) \left| \frac{\sin((n+1)u_2\pi)}{\sin(u_2\pi)} \right| du_2$$
$$\leqslant \frac{3}{2} \int_{1/n}^{1/3} \frac{(-\log 3u_2)}{u_2} du_2 \leqslant \frac{3}{4}(\log n)^2.$$

Hence we obtain the estimate

$$\left|f(\mathbf{t})-S_n^{(1)}(f)(\mathbf{t})\right|\leqslant c\frac{(\log n)^2}{n},\quad \mathbf{t}\in\overline{\Omega},$$

which completes the proof. □

REMARK. It is interesting that in the case $\alpha=1$, the quantity $\log n$ in Theorem B is replaced by $(\log n)^2$. It is known that the estimate in Theorem B is best possible (see, for example [1, pp. 106–108]. Naturally one can ask that "is the estimate in Theorem 1 best possible, or not?". We think that this estimate is the best possible, but, however we couldn't find a function $f\in Lip_1\left(\overline{\Omega}\right)$ such that

$$\left\|f-S_n^{(1)}(f)\right\|_\infty\geqslant c\,\frac{(\log n)^2}{n}.$$

4. Approximation by Abel-Poisson means of hexagonal Fourier series

The Abel-Poisson means of an H-periodic function $f\in L^1(\Omega)$ are defined by

$$U_r(f)(\mathbf{t}):=\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})P_r(\mathbf{s})\,d\mathbf{s},$$

where

$$P_r(\mathbf{t}):=\sum_{k=0}^{\infty}\sum_{\mathbf{j}\in\mathbb{J}_k}r^k\phi_{\mathbf{j}}(\mathbf{t}),\quad 0\leqslant r<1$$

is the Poisson kernel. It is clear that if the function f has the Fourier series (7) then

$$U_r(f)(\mathbf{t})=\sum_{k=0}^{\infty}\sum_{\mathbf{j}\in\mathbb{J}_k}r^k c_{\mathbf{j}}\phi_{\mathbf{j}}(\mathbf{t}).$$

The Poisson kernel is nonnegative, satisfies

$$\frac{1}{|\Omega|}\int_\Omega P_r(\mathbf{t})\,d\mathbf{t}=1,$$

and has the compact formula

$$\begin{aligned}P_r(\mathbf{t})=&\frac{(1-r)^3\left(1-r^3\right)}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)}+\frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)}\\&+\frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)}+\frac{r(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)},\end{aligned}$$

where $q_r(t) = 1 - 2r\cos t + r^2$ (see [6]). The Poisson kernel satisfies

$$P_r(\mathbf{t}) \leqslant \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_1-t_2)}{3}\right) q_r\left(\frac{2\pi(t_2-t_3)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_2-t_3)}{3}\right) q_r\left(\frac{2\pi(t_3-t_1)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(t_3-t_1)}{3}\right) q_r\left(\frac{2\pi(t_1-t_2)}{3}\right)} \tag{13}$$

for all $\mathbf{t} \in \mathbb{R}_H^3$.

It is clear that

$$\frac{(1-r)^2}{q_r(t)\,q_r(s)} = \frac{1}{(1+r)^2} P_r(t) P_r(s), \tag{14}$$

where

$$P_r(t) = \frac{1-r^2}{q_r(t)}$$

is the classical Poisson kernel.

The Poisson kernel $P_r(t)$ is an even function and satisfies the inequalities ([7, pp. 96–97])

$$P_r(t) \leqslant \frac{2}{1-r}, \quad 0 \leqslant t \leqslant \pi,\ 0 \leqslant r < 1 \tag{15}$$

and

$$P_r(t) \leqslant c\,\frac{1-r}{t^2}, \quad 0 < t \leqslant \pi,\ 0 \leqslant r < 1. \tag{16}$$

It is known that the Abel-Poisson means of a 2π-periodic continuous function converge uniformly to this function ([7, p. 97]). This property is also valid for H-periodic continuous functions:

THEOREM C. *([6]) If $f \in C_H\left(\overline{\Omega}\right)$, then the Abel-Poisson means $U_r(f)$ converge uniformly to f on $\overline{\Omega}$ as $r \to 1-$.*

The rate of convergence of Abel-Poisson means of 2π-periodic functions was given as follows ([1, p. 110]):

THEOREM D. *Let f be a $2\pi-$periodic continuous function. If $f \in Lip\alpha$, then*

$$\|f - U_r(f)\|_\infty = \begin{cases} O\left((1-r)^\alpha\right), & 0 < \alpha < 1 \\ O\left((1-r)\left|\log(1-r)\right|\right), & \alpha = 1 \end{cases}$$

for $r \to 1-$.

For the Abel-Poisson means of H-periodic continuous functions we obtained the following theorem:

THEOREM 2. *If $f \in Lip_\alpha\left(\overline{\Omega}\right)$ then*

$$\|f - U_r(f)\|_\infty = \begin{cases} O\left((1-r)^\alpha\right), & 0 < \alpha < 1 \\ O\left((1-r)\left(\log(1-r)\right)^2\right), & \alpha = 1 \end{cases}$$

for $r \to 1-$.

Proof. Let $f \in Lip_\alpha\left(\overline{\Omega}\right)$.

$$f(\mathbf{t}) - U_r(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_\Omega (f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s}))P_r(\mathbf{s})\,d\mathbf{s},$$

hence

$$\begin{aligned}|f(\mathbf{t}) - U_r(f)(\mathbf{t})| &\leqslant \frac{1}{|\Omega|}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{s})|\,P_r(\mathbf{s})\,d\mathbf{s} \\ &\leqslant \frac{M}{|\Omega|}\int_\Omega \|\mathbf{s}\|_\infty^\alpha P_r(\mathbf{s})\,d\mathbf{s}.\end{aligned}$$

By (13),

$$|f(\mathbf{t}) - U_r(f)(\mathbf{t})| \leqslant \frac{M}{|\Omega|}\int_\Omega \|\mathbf{s}\|_\infty^\alpha Q_r(\mathbf{s})\,d\mathbf{s}, \tag{17}$$

where

$$\begin{aligned}Q_r(\mathbf{s}) := &\frac{2(1-r)^2}{q_r\left(\frac{2\pi(s_1-s_2)}{3}\right)q_r\left(\frac{2\pi(s_2-s_3)}{3}\right)} + \frac{2(1-r)^2}{q_r\left(\frac{2\pi(s_2-s_3)}{3}\right)q_r\left(\frac{2\pi(s_3-s_1)}{3}\right)} \\ &+ \frac{2(1-r)^2}{q_r\left(\frac{2\pi(s_3-s_1)}{3}\right)q_r\left(\frac{2\pi(s_1-s_2)}{3}\right)}.\end{aligned}$$

Hence we shall estimate

$$\int_\Omega \|\mathbf{t}\|_\infty^\alpha Q_r(\mathbf{t})\,d\mathbf{t}.$$

Since $\|\mathbf{t}\|_\infty^\alpha Q_r(\mathbf{t})$ is a symmetric function of t_1, t_2, t_3 it is sufficient to consider the integral over the triangle Δ as in the proof of Theorem 1.

By (14),

$$\begin{aligned}Q_r(\mathbf{t}) \leqslant\ &2P_r\left(\frac{2\pi(t_1-t_2)}{3}\right)P_r\left(\frac{2\pi(t_2-t_3)}{3}\right) + 2P_r\left(\frac{2\pi(t_2-t_3)}{3}\right)P_r\left(\frac{2\pi(t_3-t_1)}{3}\right) \\ &+2P_r\left(\frac{2\pi(t_3-t_1)}{3}\right)P_r\left(\frac{2\pi(t_1-t_2)}{3}\right).\end{aligned}$$

If we use this inequality and the transformations (9) and (10), we obtain

$$\int_\Delta \|\mathbf{t}\|_\infty^\alpha Q_r(\mathbf{t})\,d\mathbf{t} \leqslant 6\int_\Gamma u_1^\alpha Q_r^*(u_1,u_2)\,du_1du_2,$$

where

$$\begin{aligned}Q_r^*(u_1,u_2) := &P_r(\pi(u_1+u_2))P_r(2\pi u_2) + P_r(\pi(u_1-u_2))P_r(\pi(u_1+u_2)) \\ &+P_r(\pi(u_1-u_2))P_r(2\pi u_2)\end{aligned}$$

and $\Gamma = \left\{(u_1,u_2) : 0 \leqslant u_2 \leqslant \frac{u_1}{3}, 0 \leqslant u_1 \leqslant 1\right\}$, as in the proof of Theorem 1.

We can write $\Gamma = \Gamma_1' \cup \Gamma_2' \cup \Gamma_3'$, where

$$\begin{aligned}
\Gamma_1' &:= \{(u_1,u_2) \in \Gamma : u_1 \leqslant 1-r\}, \\
\Gamma_2' &:= \left\{(u_1,u_2) \in \Gamma : 1-r \leqslant u_1,\ u_2 \leqslant \frac{1-r}{3}\right\}, \\
\Gamma_3' &:= \left\{(u_1,u_2) \in \Gamma : 1-r \leqslant u_1,\ \frac{1-r}{3} \leqslant u_2\right\}.
\end{aligned}$$

Let $0 < \alpha < 1$.

By (15),

$$\begin{aligned}
\int\limits_{\Gamma_1'} u_1^{\alpha} Q_r^*(u_1,u_2)\, du_1 du_2 &= \int\limits_0^{\frac{1-r}{3}} \left(\int\limits_{3u_2}^{1-r} u_1^{\alpha} Q_r^*(u_1,u_2)\, du_1\right) du_2 \\
&\leqslant \frac{12}{(1-r)^2} \int\limits_0^{\frac{1-r}{3}} \left(\int\limits_{3u_2}^{1-r} u_1^{\alpha} du_1\right) du_2 \\
&= \frac{12}{(1+\alpha)(1-r)^2} \int\limits_0^{\frac{1-r}{3}} \left((1-r)^{1+\alpha} - (3u_2)^{1+\alpha}\right) du_2 \\
&= c_5(\alpha)(1-r)^{\alpha}.
\end{aligned}$$

By inequalities (15) and (16),

$$\begin{aligned}
\int\limits_{\Gamma_2'} u_1^{\alpha} Q_r^*(u_1,u_2)\, du_1 du_2 &= \int\limits_0^{\frac{1-r}{3}} \left(\int\limits_{1-r}^{1} u_1^{\alpha} Q_r^*(u_1,u_2)\, du_1\right) du_2 \\
&\leqslant c \int\limits_0^{\frac{1-r}{3}} \left\{\int\limits_{1-r}^{1} u_1^{\alpha} \left(\frac{1}{\pi^2 (u_1+u_2)^2} + \frac{1}{\pi^2 (u_1-u_2)^2}\right) du_1\right\} du_2 \\
&\leqslant c \int\limits_0^{\frac{1-r}{3}} \left(\int\limits_{1-r}^{1} u_1^{\alpha-2} du_1\right) du_2 \\
&= c_6(\alpha)(1-r)^{\alpha},
\end{aligned}$$

since $u_1 - u_2 \geqslant \frac{2}{3}u_1$ and $u_1 + u_2 \geqslant u_1$.

By the inequality (16),

$$\int_{\Gamma_3'} u_1^{\alpha} Q_r^*(u_1,u_2)\,du_1du_2 = \int_{\frac{1-r}{3}}^{1/3}\left(\int_{3u_2}^{1} u_1^{\alpha} Q_r^*(u_1,u_2)\,du_1\right)du_2$$
$$\leqslant c(1-r)^2\int_{\frac{1-r}{3}}^{1/3}\left(\int_{3u_2}^{1} u_1^{\alpha}\frac{1}{u_1^2u_2^2}du_1\right)du_2$$
$$= c\frac{(1-r)^2}{1-\alpha}\int_{\frac{1-r}{3}}^{1/3}\left((3u_2)^{\alpha-1}-1\right)\frac{1}{u_2^2}du_2$$
$$\leqslant c_7(\alpha)(1-r)^{\alpha}.$$

Combining these estimates and considering (17) we get

$$|f(\mathbf{t})-U_r(f)(\mathbf{t})|\leqslant c_8(\alpha)(1-r)^{\alpha},\quad \mathbf{t}\in\overline{\Omega}.$$

Now let $\alpha=1$.
By (15),

$$\int_{\Gamma_1'} u_1 Q_r^*(u_1,u_2)\,du_1du_2 = \int_{0}^{\frac{1-r}{3}}\left(\int_{3u_2}^{1-r} u_1 Q_r^*(u_1,u_2)\,du_1\right)du_2$$
$$\leqslant \frac{12}{(1-r)^2}\int_{0}^{\frac{1-r}{3}}\left(\int_{3u_2}^{1-r} u_1\,du_1\right)du_2$$
$$= c(1-r).$$

By (15) and (16),

$$\int_{\Gamma_2'} u_1 Q_r^*(u_1,u_2)\,du_1du_2 = \int_{0}^{\frac{1-r}{3}}\left(\int_{1-r}^{1} u_1 Q_r^*(u_1,u_2)\,du_1\right)du_2$$
$$\leqslant c\int_{0}^{\frac{1-r}{3}}\left\{\int_{1-r}^{1} u_1\left(\frac{1}{\pi^2(u_1+u_2)^2}+\frac{1}{\pi^2(u_1-u_2)^2}\right)du_1\right\}du_2$$
$$\leqslant c\int_{0}^{\frac{1-r}{3}}\left(\int_{1-r}^{1}\frac{1}{u_1}du_1\right)du_2$$
$$= c(1-r)(-\log(1-r)).$$

Using (16) again,

$$\begin{aligned}\int_{\Gamma_3'} u_1 Q_r^*(u_1,u_2)\,du_1du_2 &= \int_{\frac{1-r}{3}}^{1/3}\left(\int_{3u_2}^{1} u_1 Q_r^*(u_1,u_2)\,du_1\right)du_2\\ &\leqslant c(1-r)^2\int_{\frac{1-r}{3}}^{1/3}\left(\int_{3u_2}^{1}\frac{1}{u_1u_2^2}du_1\right)du_2\\ &= c(1-r)^2\int_{\frac{1-r}{3}}^{1/3}(-\log 3u_2)\frac{1}{u_2^2}du_2\\ &\leqslant c(1-r)(\log(1-r))^2.\end{aligned}$$

Hence

$$|f(\mathbf{t})-U_r(f)(\mathbf{t})| \leqslant c(1-r)\left(1-\log(1-r)+(\log(1-r))^2\right)$$

for $\mathbf{t}\in\overline{\Omega}$. □

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(Received July 13, 2011)

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