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INTERPOLATIONS OF SCHWAB–BORCHARDT MEAN

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Abstract. For two means M,N satisfying $M(x,y) \leqslant N(x,y)$ we apply the 'borchardtisation' process to obtain a new mean

$$SB_{M,N} = \frac{\sqrt{N^2 - M^2}}{\arccos(M/N)}.$$

We use some geometric ideas to prove inequalities between the three means. In particular some new inequalities for Seiffert means are established.

1. Introduction

For positive numbers x, y the pair of sequences

$$x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{y_n \frac{x_n + y_n}{2}}, \quad x_0 = x, \quad y_0 = y, \tag{1}$$

converges to a common limit called the Schwab-Borchardt mean

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2 - x^2}}{\arccos(x/y)}, & x < y, \\ \dfrac{\sqrt{x^2 - y^2}}{\operatorname{arccosh}(x/y)}, & y < x, \\ x & x = y. \end{cases}$$

The algorithm (1) was known to Gauss but has been rediscovered by Borchradt and named after him ([1, 6]).

Two means introduced by Seiffert in [13]

$$P(x,y) = \begin{cases} \dfrac{x-y}{2\arcsin \frac{x-y}{x+y}} & x \neq y, \\ x & x = y, \end{cases}$$

and in [15]

$$T(x,y) = \begin{cases} \dfrac{x-y}{2\arctan \frac{x-y}{x+y}} & x \neq y, \\ x & x = y, \end{cases}$$

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are of great interest for many mathematicians. In [6] Neuman and Sándor proved that both are particular cases of the Schwab-Borchardt means, namely

$$P(x,y) = SB\left(\sqrt{xy}, \frac{x+y}{2}\right) \quad \text{and} \quad T(x,y) = SB\left(\frac{x+y}{2}, \sqrt{\frac{x^2+y^2}{2}}\right).$$

Interesting inequalities between P, T, arithmetic, geometric, logarithmic, identric and power means were obtained by many authors (see the bibliography) using analytic approach or properties of the Schwab-Borchardt algorithm. Especially the sequential method developed by Sándor ([8, 9, 10, 6]) provides a way for further refinements.

In this paper we use geometric properties of the 'upper' part of SB to generalise those results and to obtain some new, optimal estimates. The word "optimal" does not mean here that the estimates cannot be improved, rather that the results presented are the best possible in the class (obvious from the statement of theorems) of inequalities.

2. Notation and definitions

We shall be using the following notation: x,y are always positive. For real t we denote by $A_t(x,y)$ the power mean of order t

$$A_t = A_t(x,y) = \begin{cases} \left(\frac{x^t+y^t}{2}\right)^{1/t} & t \neq 0, \\ \sqrt{xy} & t = 0. \end{cases}$$

We also use standard notation $G = A_0$, $A = A_1$ and $Q = A_2$ for geometric, arithmetic and the root mean square means.

We write $a \cong b$ to indicate that a and b are of the same sign.

DEFINITION 2.1. If two means $N(x,y)$ and $M(x,y)$ satisfy $M(x,y) \leqslant N(x,y)$ for all x,y, then we define their *borchardtisation* by

$$SB_{M,N}(x,y) = SB(M(x,y), N(x,y)).$$

With this terminology we have

$$SB = SB_{\min,\max}, \quad P = SB_{G,A}, \quad T = SB_{A,Q}.$$

Consider a right triangle ΔABC (see Fig. 1) with sides

$$|AB| = N, \quad |BC| = M, \quad |AC| = \sqrt{N^2 - M^2},$$

and let P be the intersection point of AB and the circle of radius $|BC|$ centered at B. In the middle of $\overset{\frown}{PC}$ draw a tangent line that meets BA at F and BC at E.

We shall denote by β the radial measure of $\angle B$

$$\beta = \arccos \frac{M}{N}.$$

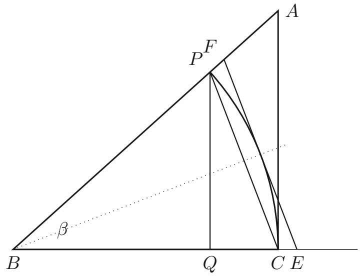

Figure 1.

Thus

$$SB_{M,N} = \frac{|AC|}{\beta} = \frac{|AC||BC|}{|\overset{\frown}{PC}|}. \tag{2}$$

DEFINITION 2.2. As x, y vary, the angle β varies as well between 0 and α, where

$$\alpha = \alpha_{M,N} = \sup_{x,y} \arccos \frac{M(x,y)}{N(x,y)}.$$

We call $\alpha_{M,N}$ the *angle between* M *and* N.

Note that $\alpha_{\min,\max} = \alpha_{G,A} = \frac{\pi}{2}$ (fix y and make x small), while $\alpha_{A,Q} = \frac{\pi}{4}$, because $|AC| = \frac{|x-y|}{2} < \frac{x+y}{2} = |AB|$. In general, the reader can easily show that for $0 < r < s$ the angle between power means equals $\alpha_{A_r,A_s} = \arccos 2^{1/s-1/r}$.

Observe that

$$\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{|AB|-|BC|}{2|AB|}} = \frac{|AC|}{2\sqrt{|AB|\frac{|AB|+|BC|}{2}}}, \tag{3}$$

$$\tan\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{1+\cos\beta}} = \sqrt{\frac{|AB|-|BC|}{|AB|+|BC|}} = \frac{|AC|}{|AB|+|BC|}, \tag{4}$$

and therefore (2) enables us to write the identities

$$|\overset{\frown}{PC}| = |BC|\beta = \frac{SB_{M,N}|\overset{\frown}{PC}|}{SB_{M,N}}, \tag{5}$$

$$|PQ| = |BC|\sin\beta = \frac{SB_{M,N}|\overset{\frown}{PC}|}{N}, \tag{6}$$

$$|AC| = |BC|\tan\beta = \frac{SB_{M,N}|\overset{\frown}{PC}|}{M}, \tag{7}$$

$$|PC| = 2|BC|\sin\beta/2 = \frac{SB_{M,N}|\overset{\frown}{PC}|}{\sqrt{N\frac{M+N}{2}}}, \tag{8}$$

$$|EF| = 2|BC|\tan\beta/2 = \frac{SB_{M,N}|\widehat{PC}|}{\frac{M+N}{2}}. \tag{9}$$

This in turn leads to identities

$$SB_{M,N}\cdot\beta = N\sin\beta = M\tan\beta = 2\sqrt{N\frac{M+N}{2}}\sin\beta/2 = 2\frac{M+N}{2}\tan\beta/2, \tag{10}$$

that we shall explore in the next sections.

3. Obvious inequalities

The inequalities in this section between the means M,N and their borchardtisation follow immediately from (10) and monotonicity of functions $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$. The geometric interpretation shows that the constants are optimal.

THEOREM 3.1. *The inequalities hold*

$$M \leqslant SB_{M,N} \leqslant \frac{\tan\alpha}{\alpha}M, \tag{11}$$

(in case $\alpha = \frac{\pi}{2}$ this indicates lack of the upper bound)

$$\frac{\sin\alpha}{\alpha}N \leqslant SB_{M,N} \leqslant N, \tag{12}$$

$$\frac{\sin\alpha/2}{\alpha/2}\sqrt{N\frac{M+N}{2}} \leqslant SB_{M.N} \leqslant \sqrt{N\frac{M+N}{2}}, \tag{13}$$

$$\frac{M+N}{2} \leqslant SB_{M.N} \leqslant \frac{\tan\alpha/2}{\alpha/2}\cdot\frac{M+N}{2}. \tag{14}$$

Let us apply this result to Seiffert means: the inequality (15) was established first by Seiffert in [16]. The right-hand sides of (16), (18) and the left-hand sides of (17) and (19) follow also immediately from the monotonicity of sequences defining Schwab-Borchardt means (see Sándor's papers [9, 10]). The right-hand side of (17) was proved by P. Hästö in [3, Cor. 1.11].

COROLLARY 3.1. *The Seiffert mean P satisfies the inequalities*

$$G \leqslant P,$$

$$\frac{2}{\pi}A \leqslant P \leqslant A, \tag{15}$$

$$\frac{2\sqrt{2}}{\pi}\sqrt{A\frac{A+G}{2}} \leqslant P \leqslant \sqrt{A\frac{A+G}{2}}, \tag{16}$$

$$\frac{A+G}{2} \leqslant P \leqslant \frac{4}{\pi}\cdot\frac{A+G}{2}. \tag{17}$$

COROLLARY 3.2. *The Seiffert mean T satisfies the inequalities*

$$A \leqslant T \leqslant \frac{4}{\pi}A,$$

$$\frac{2\sqrt{2}}{\pi}Q \leqslant T \leqslant Q,$$

$$\frac{4\sqrt{2-\sqrt{2}}}{\pi}\sqrt{Q\frac{Q+A}{2}} \leqslant T \leqslant \sqrt{Q\frac{Q+A}{2}}, \tag{18}$$

$$\frac{Q+A}{2} \leqslant T \leqslant \frac{8(\sqrt{2}-1)}{\pi} \cdot \frac{Q+A}{2}. \tag{19}$$

4. Arithmetic interpolations

The set of obvious inequalities

$$|PQ| < |PC| < |\overset{\frown}{PC}| < |EF| < |AC| \tag{20}$$

allows us to consider different kind of interpolations, like $|\overset{\frown}{PC}| - t|PQ| - (1-t)|AC|$, $|\overset{\frown}{PC}| - |PC|^t|EF|^{1-t}$ etc. and look for these t for which the interpolations preserve sign. As we shall see, the formulas (5)–(9) will turn them into optimal bounds for $SB_{M,N}$ in terms of arithmetic, geometric or harmonic interpolations of surrounding means.

In most cases we shall face the situation similar to the one described in the next lemma (monotonicity and convexity may vary).

LEMMA 4.1. *Suppose $f_t : [0,\pi/2] \to \mathbb{R}$, $t \in [0,1]$ is a family of functions satisfying the following assumptions:*

- *f_t increases with t,*
- *$f_t(0) = f_t'(0) = 0$ for every t,*
- *there exists t_0 such that $f_t(x)$ are concave in x for every $t \leqslant t_0$,*
- *if $t > t_0$, then $f_t(x)$ is convex for small x and has at most one inflection point.*

Let $0 < \alpha \leqslant \pi/2$. Then

- *$f_t(x) \leqslant 0$ holds for all $x \in [0,\alpha]$ if and only if $t \leqslant t_0$*
- *$f_t(x) \geqslant 0$ holds for all $x \in [0,\alpha]$ if and only if $f_t(\alpha) \geqslant 0$. In particular, if $f_{t_\alpha}(\alpha) = 0$, then f_t is nonnegative for all $t \geqslant t_\alpha$.*

We leave its simple proof (draw a picture and look) to the reader.

THEOREM 4.1. *We have the following optimal bounds*

$$(1-h(\alpha))N + h(\alpha)M \leqslant SB_{M.N} \leqslant \frac{2}{3}N + \frac{1}{3}M,$$

where $h(x) = \frac{x-\sin x}{x(1-\cos x)}$.

Proof. Using (10) we obtain

$$SB_{M.N}-(1-t)N-tM \cong \frac{1}{\beta}-\frac{1-t}{\sin\beta}-\frac{t}{\tan\beta}$$
$$=\frac{1}{\beta\sin\beta}(\sin\beta-(1-t)\beta-t\beta\cos\beta).$$

The functions $\varphi_t(x)=\sin x-(1-t)x-tx\cos x$ satisfy $\varphi_t(0)=\varphi_t'(0)=0$ and

$$\varphi_t''(x)=t\sin x\left(\frac{x}{\tan x}-\frac{1-2t}{t}\right).$$

Note that φ_t increases in t. Since $\frac{x}{\tan x}$ decreases from 1 to 0, we see that for $t\leqslant 1/3$ the function φ_t is concave and therefore negative. We also see that for $t>1/2$ it is convex in $(0,\pi/2)$. In case $1/2<t<2/3$, φ_t is convex for small x and has one inflection point, so by Lemma 4.1, $SB_{M.N}\leqslant(1-t)N+tM$ holds for $t\leqslant\frac{1}{3}$ and since the right-hand side decreases in t, it attains its best bound at $t=1/3$.

On the other hand, the condition $\varphi_t(\alpha)\geqslant 0$ is equivalent to

$$t\geqslant\frac{\alpha-\sin\alpha}{\alpha(1-\cos\alpha)}=h(\alpha).$$

This gives the left-hand side of our statement. □

COROLLARY 4.1. *For Seiffert means we have*

$$\frac{2}{\pi}A+\frac{\pi-2}{\pi}G\leqslant P\leqslant\frac{2}{3}A+\frac{1}{3}G, \tag{21}$$
$$(1-r_1)Q+r_1A\leqslant T\leqslant\frac{2}{3}Q+\frac{1}{3}A, \tag{22}$$

where $r_1=\dfrac{2(\pi-2\sqrt{2})}{(2-\sqrt{2})\pi}\approx .340341385.$

The right-hand sides of (21) and (22) are due to Sándor ([9, 10]).

THEOREM 4.2. *The constants in the inequalities below are optimal*

$$(1-h(\alpha/2))\sqrt{N\frac{M+N}{2}}+h(\alpha/2)\frac{M+N}{2}\leqslant S_{M,N}\leqslant\frac{2}{3}\sqrt{N\frac{M+N}{2}}+\frac{1}{3}\frac{M+N}{2}$$

where $h(x)$ *is defined in Theorem 4.1.*

Proof. We use (10) to obtain

$$SB_{M.N}-(1-t)\sqrt{N\frac{M+N}{2}}-t\frac{M+N}{2}\cong\frac{1}{\beta}-\frac{1-t}{2\sin\beta/2}-\frac{t}{2\tan\beta/2}$$
$$=\frac{1}{\beta\sin\beta/2}\varphi_t(\beta/2),$$

where φ is the function defined in the proof of Theorem 4.1. The proof concludes as above. □

COROLLARY 4.2.

$$(1-r_1)\sqrt{A\frac{A+G}{2}}+r_1\frac{A+G}{2}\leqslant P\leqslant\frac{2}{3}\sqrt{A\frac{A+G}{2}}+\frac{1}{3}\frac{A+G}{2}, \tag{23}$$

$$(1-r_2)\sqrt{Q\frac{Q+A}{2}}+r_2\frac{Q+A}{2}\leqslant T\leqslant\frac{2}{3}\sqrt{Q\frac{Q+A}{2}}+\frac{1}{3}\frac{Q+A}{2}, \tag{24}$$

where r_1 *is the same as in the previous corollary and* $r_2=\frac{\pi-8\sin(\pi/8)}{\pi(1-\cos(\pi/8))}\approx 0.335056$.

The right-hand side inequality in (23) was proved by Sándor in [9]. Note that the right-hand side of (24) can be obtained from (23) using methods described in [10].

5. Geometric interpolations

Next two theorems provide the best bounds by weighted geometric means.

THEOREM 5.1. *The inequalities hold*

$$N^{2/3}M^{1/3}\leqslant SB_{M,N}\leqslant N^{1-k(\alpha)}M^{k(\alpha)},$$

where $k(x)=\frac{\log\sin x-\log x}{\log\cos x}$. *The constants cannot be improved.*

Proof. Equations (10) imply that

$$\begin{aligned}SB_{M.N}-N^{1-t}M^t &\cong \frac{1}{\beta}-\frac{1}{\sin^{1-t}\beta\tan^t\beta}\\ &=\frac{1}{\beta\sin\beta\tan^t\beta}\left(\sin\beta\cos^{-t}\beta-\beta\right).\end{aligned}$$

The functions $\xi_t(x)=\sin x\cos^{-t}x-x$, $t>0$ satisfy $\xi_t(0)=\xi_t'(0)=0$ and increase in t. Their second derivative equals

$$\xi_t''(x)=-(1-t)^2\sin x\cos^{-t-2}x\left(\cos^2 x-\frac{t(t+1)}{(1-t)^2}\right).$$

The quotient $\frac{t(t+1)}{(1-t)^2}$ is greater than 1 if $t>1/3$, which yields convexity of $\xi_t(x)$. On the other hand, for $0<t<1/3$ the expression in brackets is positive for small x and changes sing once as x varies, therefore, by slight modification of Lemma 4.1 we conclude that $SB_{M.N}\geqslant N^{2/3}M^{1/3}$ for $t\geqslant 1/3$, and $SB_{M.N}\leqslant N^{1-t}M^t$ on $(0,\alpha)$ if and only if $\sin\alpha\cos^{-t}\alpha-\alpha\leqslant 0$, which is equivalent to

$$t<\frac{\log\sin\alpha-\log\alpha}{\log\cos\alpha}=k(\alpha). \qquad \square$$

COROLLARY 5.1. *The Seiffert means satisfy*

$$A^{2/3}G^{1/3} \leqslant P, \tag{25}$$

$$Q^{2/3}A^{1/3} \leqslant T \leqslant Q^{1-r_3}A^{r_3}, \tag{26}$$

where $r_3 = \dfrac{\log \frac{\pi^2}{8}}{\log 2} \approx .302992259$.

The left-hand sides of (25) and (26) are due to Sándor ([9, 10]).

THEOREM 5.2. *The inequalities hold*

$$N^{1/3}\left(\frac{M+N}{2}\right)^{2/3} \leqslant SB_{M,N} \leqslant N^{\frac{1-k(\alpha/2)}{2}}\left(\frac{M+N}{2}\right)^{\frac{1+k(\alpha/2)}{2}},$$

where $k(x)$ *is defined in Theorem 5.1. The constants cannot be improved.*

Proof. Let $0 < t < 1$. Again by (10) we have

$$SB_{M,N} - N^{1-t}\left(\frac{M+N}{2}\right)^t \cong \frac{1}{\beta} - \frac{1}{2^t \sin^{1-t}\beta \tan^t \beta/2}$$
$$= \frac{1}{\beta \sin\beta/2 \cos^{1-2t}\beta/2}\xi_{2t-1}(\beta/2),$$

where ξ_t is defined in the proof of previous theorem, and we conclude that $SB_{M,N} - N^{1-t}\left(\frac{M+N}{2}\right)^t$ is positive if $t \geqslant 2/3$ and negative if $t < \frac{1+k(\alpha/2)}{2}$. □

COROLLARY 5.2. *For the Seiffert means we have*

$$A^{1/3}\left(\frac{A+G}{2}\right)^{2/3} \leqslant P \leqslant A^{r_3}\left(\frac{A+G}{2}\right)^{1-r_3}, \tag{27}$$

$$Q^{1/3}\left(\frac{Q+A}{2}\right)^{2/3} \leqslant T \leqslant Q^{r_4}\left(\frac{Q+A}{2}\right)^{1-r_4}, \tag{28}$$

where $r_3 = \dfrac{\log(4/\pi)}{\log 2} \approx .348503871$ *and* $r_4 = \dfrac{\log(\frac{\pi}{8}\cot(\frac{\pi}{8}))}{2\log(\cos(\frac{\pi}{8}))} \approx .336842548$.

The left-hand sides of (27) and (28) are also due to Sándor ([9, 10]).

6. Harmonic interpolations

This section is devoted to bounds of borchardtisation by the weighted harmonic mean of its originators.

THEOREM 6.1. *The following inequalities hold*

$$\frac{1-m(\alpha)}{M}+\frac{m(\alpha)}{N}\leqslant\frac{1}{SB_{M,N}}\leqslant\frac{1/3}{M}+\frac{2/3}{N},$$

where $m(x)=\frac{\tan x-x}{\tan x-\sin x}$.

Proof. As usual we can write

$$\frac{1}{SB_{M.N}}-\frac{t}{N}-\frac{1-t}{M}\cong\beta-t\sin\beta-(1-t)\tan\beta.$$

The functions $\phi_t(x)=x-t\sin x-(1-t)\tan x$ increase in t, $\phi_t(0)=\phi_t'(0)=0$. Their second derivative $\phi_t''(x)=\frac{t\sin x}{\cos^3 x}\left(\cos^3 x-\frac{2(1-t)}{t}\right)$ is negative for $t\leqslant 2/3$. In case $t>2/3$ the function ϕ_t is convex for small x and has one inflection point. The inequality $\phi_t(\alpha)\geqslant 0$ is equivalent to

$$t\geqslant\frac{\tan\alpha-\alpha}{\tan\alpha-\sin\alpha}=m(\alpha),$$

and application of Lemma 4.1 completes the proof. □

COROLLARY 6.1. *The Seiffert means satisfy*

$$\begin{aligned}\frac{1}{A}\leqslant\frac{1}{P}&\leqslant\frac{1/3}{G}+\frac{2/3}{A},\\ \frac{1-r_5}{A}+\frac{r_5}{Q}\leqslant\frac{1}{T}&\leqslant\frac{1/3}{A}+\frac{2/3}{Q},\end{aligned}\tag{29}$$

where $r_5=\dfrac{4-\pi}{4-2\sqrt{2}}\approx 0.732696501$.

The right-hand side of (29) is due to Seiffert ([16]).

THEOREM 6.2. *The following inequalities hold*

$$\frac{1-m(\alpha/2)}{\frac{M+N}{2}}+\frac{m(\alpha/2)}{\sqrt{N\frac{M+N}{2}}}\leqslant\frac{1}{SB_{M,N}}\leqslant\frac{1/3}{\frac{M+N}{2}}+\frac{2/3}{\sqrt{N\frac{M+N}{2}}},$$

where $m(x)$ *is defined as above.*

We leave the proof to the reader.

COROLLARY 6.2. *For the Seiffert means we have*

$$\frac{1-r_5}{\frac{A+G}{2}}+\frac{r_5}{\sqrt{A\frac{A+G}{2}}}\leqslant\frac{1}{P}\leqslant\frac{1/3}{\frac{A+G}{2}}+\frac{2/3}{\sqrt{A\frac{A+G}{2}}},$$
$$\frac{1-r_6}{\frac{Q+A}{2}}+\frac{r_6}{\sqrt{Q\frac{Q+A}{2}}}\leqslant\frac{1}{T}\leqslant\frac{1/3}{\frac{Q+A}{2}}+\frac{2/3}{\sqrt{Q\frac{Q+A}{2}}},$$

where r_5 *is defined in the previous corollary and* $r_6=m(\pi/8)\approx .6823467$.

The reader familiar with our method can easily prove the following theorem

THEOREM 6.3. *The following bounds are optimal*

$$\frac{n(\alpha)}{N}+\frac{1-n(\alpha)}{\frac{M+N}{2}}\leqslant\frac{1}{SB_{M,N}}\leqslant\frac{1/3}{N}+\frac{2/3}{\frac{M+N}{2}},$$

where $n(x)=\frac{2\tan x/2-x}{2\tan x/2-\sin x}$.

COROLLARY 6.3. *For the Seiffert means we have the bounds*

$$\frac{2-\frac{\pi}{2}}{A}+\frac{\frac{\pi}{2}-1}{\frac{A+G}{2}}\leqslant\frac{1}{P}\leqslant\frac{1/3}{A}+\frac{2/3}{\frac{A+G}{2}} \tag{30}$$
$$\frac{r_7}{Q}+\frac{1-r_7}{\frac{Q+A}{2}}\leqslant\frac{1}{T}\leqslant\frac{1/3}{Q}+\frac{2/3}{\frac{Q+A}{2}}, \tag{31}$$

where $r_7=\dfrac{8(\sqrt{2}-1)-\pi}{6\sqrt{2}-8}\approx 0.354672268$.

The proof of the right-hand side of (30) using (27) and the AG inequality can be found in [11].

7. Q and A_{-2} interpolations

Inequalities (20) between elements of the triangle are valid also for their squares. This enables us to prove the next estimates.

THEOREM 7.1. *The inequalities hold*

$$\sqrt{l(\alpha)M^2+(1-l(\alpha))N^2}\leqslant SM_{M,N}\leqslant\sqrt{\frac{M^2+2N^2}{3}},$$
$$\sqrt{\frac{M+N}{2}}\sqrt{\frac{l(\alpha/2)}{2}M+\left(1-\frac{l(\alpha/2)}{2}\right)N}\leqslant SM_{M,N}\leqslant\sqrt{\frac{M+N}{2}}\sqrt{\frac{M+5N}{6}},$$

where $l(x)=\frac{1}{\sin^2 x}-\frac{1}{x^2}$. *The bounds cannot be improved.*

Proof. To prove the first part consider the difference

$$SB_{M,N}^2 - tM^2 - (1-t)N^2 \cong \frac{1}{\beta^2} - \frac{t}{\tan^2\beta} - \frac{1-t}{\sin^2\beta} = t - l(\beta).$$

The function $l(x)$ increases from $1/3$ to $l(\alpha)$, which concludes the proof.

To prove the second part replace M with $\frac{M+N}{2}$ and N with $\sqrt{N\frac{M+N}{2}}$. □

COROLLARY 7.1. *For the Seiffert means the inequalities hold*

$$\sqrt{\frac{\pi^2-4}{\pi^2}G^2 + \frac{4}{\pi^2}A^2} \leqslant P \leqslant \sqrt{\frac{G^2+2A^2}{3}}, \tag{32}$$

$$\sqrt{\frac{2\pi^2-16}{\pi^2}A^2 + \frac{16-\pi^2}{\pi^2}Q^2} \leqslant T \leqslant \sqrt{\frac{A^2+2Q^2}{3}},$$

$$\sqrt{\frac{G+A}{2}}\sqrt{\frac{\pi^2-8}{\pi^2}G + \frac{8}{\pi^2}A} \leqslant P \leqslant \sqrt{\frac{G+A}{2}}\sqrt{\frac{G+5A}{6}},$$

$$\sqrt{\frac{A+Q}{2}}\sqrt{r_8A + (1-r_8)Q} \leqslant T \leqslant \sqrt{\frac{A+Q}{2}}\sqrt{\frac{A+5Q}{6}},$$

where $r_8 \approx 0.171935686$.

Sándor and Triff in [12] refined (32) by showing that between P and the right-hand side of (32) is quite a place to fit the identric mean $I(x,y) = e^{-1}(x^x/y^y)^{1/(x-y)}$.

Applying our method to reciprocals of squares, we obtain

THEOREM 7.2. *The constants in the inequalities are sharp*

$$\sqrt{\frac{p(\alpha)}{M^2} + \frac{1-p(\alpha)}{N^2}} \leqslant \frac{1}{SB_{M,N}} \leqslant \sqrt{\frac{1/3}{M^2} + \frac{2/3}{N^2}},$$

$$\sqrt{\frac{1}{\frac{M+N}{2}}\left(\frac{p(\alpha/2)}{\frac{M+N}{2}} + \frac{1-p(\alpha/2)}{N}\right)} \leqslant \frac{1}{SB_{M,N}} \leqslant \sqrt{\frac{1}{\frac{M+N}{2}}\left(\frac{1/3}{\frac{M+N}{2}} + \frac{2/3}{N}\right)},$$

where $p(x) = \frac{x^2-\sin^2 x}{\tan^2 x - \sin^2 x}$.

The proof is similar to the previous one and we leave it and corrolaries to the reader.

8. Interpolations by other means

That M and N are means, does not imply that M^2/N or N^2/M are means (take min and max as counterexamples), although they often are. For example, Theorem 7.1 states that $2P^2/(A+G)$ is a mean. It is well known that the harmonic mean $H(x,y) = \frac{2}{1/x+1/y}$ equals G^2/A and the contraharmonic mean $C(x,y) = \frac{x^2+y^2}{x+y}$ can be written as

Q^2/A. This motivates us to search for interpolations of $SB_{M,N}$ using M^2/N and N^2/M as bounds.

Note that in our geometric parlance $M^2/N = M\cos\beta$ and $N^2/M = N/\cos\beta$.

THEOREM 8.1. *The bounds below cannot be improved*

$$q(\alpha)\frac{M^2}{N} + (1-q(\alpha))N \leqslant SB_{M.N} \leqslant \frac{5}{6}N + \frac{1}{6}\frac{M^2}{N},$$

where $q(x) = \frac{x-\sin x}{x\sin^2 x}$.

Proof. The proof goes along known line

$$\begin{aligned} SB_{M.N} - (1-t)N - t\frac{M^2}{N} &\cong \frac{1}{\beta} - \frac{1-t}{\sin\beta} - \frac{t\cos\beta}{\tan\beta} \\ &= \frac{1}{\beta\sin\beta}\left(\sin\beta - (1-t)\beta - t\beta\cos^2\beta\right) \\ &= \frac{1}{\beta\sin\beta}\left(\sin\beta - \beta + t\beta\sin^2\beta\right). \end{aligned}$$

The functions $\rho_t(x) = \sin x - x + tx\sin^2 x$ vanish with their derivatives at $x = 0$. Because their second derivatives equal

$$\rho_t''(x) = t\sin x\left(\frac{1}{t} - 2\frac{\sin 2x + x\cos 2x}{\sin x}\right),$$

it is enough to show that $\frac{\sin 2x + x\cos 2x}{\sin x}$ decreases to apply Lemma 4.1.

We shall do it in three steps:

- $\frac{\sin 2x}{\sin x} = 2\cos x$ decreases,
- if $x > \pi/4$, then both $-\cos 2x$ and $\frac{x}{\sin x}$ are positive and increase, thus $\frac{x\cos 2x}{\sin x}$ is negative and decreases,
- for $x < \pi/4$ and $0 < s < 1$ the function $\frac{\cos sx}{\cos 2x}$ increases, hence so does $\int_0^1 \frac{\cos sx}{\cos 2x}ds = \frac{\sin x}{x\cos 2x}$. Its reciprocal thus decreases. □

COROLLARY 8.1. *Since* $\frac{G^2}{A} = H$ *and* $\frac{A^2}{Q} = \frac{A_{-2}+Q}{2}$, *the Seiffert means satisfy*

$$\frac{\pi-2}{\pi}H + \frac{2}{\pi}A \leqslant P \leqslant \frac{1}{6}H + \frac{5}{6}A, \tag{33}$$

$$\frac{\pi-2\sqrt{2}}{\pi}A_{-2} + \frac{2\sqrt{2}}{\pi}Q \leqslant T \leqslant \frac{1}{12}A_2 + \frac{11}{12}Q.$$

The bounds (33) *were proven in* [2].

The bounds for geometric interpolations (see [17]) follow immediately from Theorem 5.1, and for the harmonic interpolations we obtain

THEOREM 8.2. *There are sharp bounds*

$$\frac{1-r(\alpha)}{M^2/N}+\frac{r(\alpha)}{N}\leqslant\frac{1}{SB_{M,N}}\leqslant\frac{1/6}{M^2/N}+\frac{5/6}{N},$$

where $r(x)=\frac{\sin x-x\cos^2 x}{\sin^3 x}$.

Proof. We have

$$\frac{1}{SB_{M.N}}-\frac{t}{N}-\frac{1-t}{M^2/N}\cong\beta-t\sin\beta-(1-t)\frac{\sin\beta}{\cos^2\beta},$$

and we apply Lemma 4.1 to functions $\mu_t(x)=x-t\sin x-(1-t)\frac{\sin x}{\cos^2 x}$. □

Applying this reasoning to M and N^2/M we obtain

THEOREM 8.3. *The inequalities hold*

$$s(\alpha)M+(1-s(\alpha))\frac{N^2}{M}\leqslant SB_{M.N}\leqslant\frac{2}{3}M+\frac{1}{3}\frac{N^2}{M},$$

where $s(x)=\frac{x-\sin x\cos x}{x\sin^2 x}$.

and

THEOREM 8.4. *The inequalities hold*

$$\frac{1-u(\alpha)}{N^2/M}+\frac{u(\alpha)}{M}\leqslant\frac{1}{SB_{M,N}}\leqslant\frac{1/3}{N^2/M}+\frac{2/3}{M},$$

where $u(x)=\frac{x-\sin x\cos x}{\tan x-\sin x\cos x}$.

COROLLARY 8.2. *The following inequalities between arithmetic, contraharmonic and Seiffert* T *means hold*

$$\left(2-\frac{4}{\pi}\right)A+\left(\frac{4}{\pi}-1\right)C\leqslant T\leqslant\frac{2}{3}A+\frac{1}{3}C,$$

$$\frac{2-\frac{\pi}{2}}{C}+\frac{\frac{\pi}{2}-1}{A}\leqslant\frac{1}{T}\leqslant\frac{1/3}{C}+\frac{2/3}{A}.$$

All the theorems show that the geometric method described here is much more efficient than straightforward approach presented in many current papers ([2, 17, 18]) and it does not require assumption about homogeneity of means.

Note also that our method, similarly to the sequential method of Sándor can lead to further refinements: consider a right triangle formed by the lines BE, EF and the bissectrice of $\angle B$. Then we can apply our method taking $\frac{N+M}{2}$ and $\sqrt{N\frac{N+M}{2}}$ as a starting point.

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