Mathematical
Inequalities
& Applications
Volume 16, Number 1 (2013), 207–220
doi:10.7153/mia-16-15

APPLICATIONS OF HAUSDORFF MEASURE OF NONCOMPACTNESS IN THE SPACES OF GENERALIZED MEANS

M. MURSALEEN AND ABDULLAH K. NOMAN

(*Communicated by J. Pečarić*)

Abstract. In this paper, we derive some identities for the Hausdorff measures of noncompactness of certain matrix operators on the sequence spaces $X(r,s)$ of generalized means. Further, we apply the Hausdorff measure of noncompactness to obtain the necessary and sufficient conditions for such operators to be compact.

1. Introduction and preliminaries

In this section, we give some definitions, notations and preliminary results which form the back ground of the present work.

Let w denote the space of all sequences $x=(x_k)_{k=0}^{\infty}$ real or complex. Let ℓ_∞, c and c_0 be the spaces of all bounded, convergent and null sequences, respectively. Further, by cs and ℓ_p $(1 \leqslant p < \infty)$, we denote the spaces of all sequences associated with convergent and p-absolutely convergent series, respectively.

A sequence space X is called a *BK space* if it is a Banach space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denotes the complex field and $p_n(x) = x_n$ for all $x=(x_k)\in X$ and every $n\in\mathbb{N}$. A BK space $X \supset \phi$ is said to have AK if every sequence $x=(x_k)\in X$ has a unique representation $x=\sum_{k=0}^{\infty} x_k e^{(k)}$; where ϕ denotes the set of all finite sequences that terminate in zeros, $e=(1,1,1,\ldots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the k^{th} place for each $k\in\mathbb{N}:=\{0,1,2,\ldots\}$.

The sequence spaces ℓ_∞, c and c_0 are BK spaces with the same sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k\in\mathbb{N}$. Further, the space ℓ_p is a BK space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^{\infty}|x_k|^p)^{1/p}$, where $1 \leqslant p < \infty$. Moreover, the BK spaces c_0 and ℓ_p $(1 \leqslant p < \infty)$ have AK (cf. [6, 17]).

Let S_X denote the unit sphere in a normed linear space X. If $X \supset \phi$ is a BK space and $a=(a_k)\in w$, then we write

$$\|a\|_X^* = \sup_{x\in S_X}\left|\sum_{k=0}^{\infty} a_k x_k\right| \tag{1.1}$$

Mathematics subject classification (2010): 46B15, 46B45, 46B50.

Keywords and phrases: BK space, generalized means, matrix transformation, Hausdorff measure of noncompactness, compact operator.

provided the expression on the right is defined and finite which is the case whenever $a \in X^{\beta}$, where

$$X^{\beta} = \{a = (a_k) \in w : \ ax = (a_k x_k) \in cs \textit{ for all } \ x = (x_k) \in X\}$$

is the β-dual of a subset X of w.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix. We write A_n for the sequence in the n^{th} row of A, that is $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$ then we define the A-*transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}) \tag{1.2}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

For arbitrary subsets X and Y of w, we write (X,Y) for the class of all infinite matrices that map X into Y. Thus $A \in (X,Y)$ if and only if $A_n \in X^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

For any subset X of w, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \{x \in w : Ax \in X\}.$$

An infinite matrix $T = (t_{nk})$ is called a *triangle* if $t_{nn} \neq 0$ and $t_{nk} = 0$ for all $k > n$ $(n \in \mathbb{N})$. The study of matrix domains of triangles in sequence spaces has a special importance due to the various properties which they have. For example, if X is a BK space then X_T is also a BK space with the norm given by $\|x\|_{X_T} = \|Tx\|_X$ for all $x \in X_T$ [17, Theorem 4.3.12].

The following results are very important in our study.

LEMMA 1.1. [6, Theorem 1.29] Let $1 < p < \infty$ and $q = p/(p-1)$. Then, we have $\ell_{\infty}^{\beta} = c^{\beta} = c_0^{\beta} = \ell_1$, $\ell_1^{\beta} = \ell_{\infty}$ and $\ell_p^{\beta} = \ell_q$. Furthermore, let X denote any of the spaces c_0, c, ℓ_{∞}, ℓ_1 or ℓ_p. Then, we have $\|a\|_X^* = \|a\|_{X^{\beta}}$ for all $a \in X^{\beta}$, where $\|\cdot\|_{X^{\beta}}$ is the natural norm on the dual space X^{β}.

LEMMA 1.2. [6, Theorem 1.23 (a)] Let X and Y be BK spaces. Then, we have $(X,Y) \subset \mathscr{B}(X,Y)$, that is, every matrix $A \in (X,Y)$ defines an operator $L_A \in \mathscr{B}(X,Y)$ by $L_A(x) = Ax$ for all $x \in X$, where $\mathscr{B}(X,Y)$ denotes the set of all bounded (continuous) linear operators $L : X \to Y$.

LEMMA 1.3. [6, Lemma 2.2] Let $X \supset \phi$ be a BK space and Y be any of the spaces c_0, c or ℓ_{∞}. If $A \in (X,Y)$, then

$$\|L_A\| = \|A\|_{(X,\ell_{\infty})} = \sup_n \|A_n\|_X^* < \infty.$$

Also, let $\mathscr{F}$ be the collection of all nonempty and finite subsets of $\mathbb{N} = \{0,1,2,\ldots\}$, throughout. Then, we have the following result:

LEMMA 1.4. [7, Proposition 3.3] Let $X \supset \phi$ be a BK space. If $A \in (X, \ell_1)$, then

$$\|A\|_{(X,\ell_1)} \leqslant \|L_A\| \leqslant 4 \cdot \|A\|_{(X,\ell_1)},$$

where

$$\|A\|_{(X,\ell_1)} = \sup_{N \in \mathscr{F}} \left\| \sum_{n \in N} A_n \right\|_X^* < \infty.$$

By $\mathscr{M}_X$, we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in \mathscr{M}_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) := \inf\{\varepsilon > 0 : Q \subset \cup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \varepsilon (i = 1, 2, ...), n \in \mathbb{N}_0\}.$$

The function $\chi : \mathscr{M}_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness* [15].

The basic properties of the Hausdorff measure of noncompactness can be found in [6] and [15].

To compute the Hausdorff measure of noncompactness of a bounded subset of the BK space ℓ_p $(1 \leqslant p < \infty)$, we may use the following result [15, Theorem 2.8].

LEMMA 1.5. Let $1 \leqslant p < \infty$ and $Q \in \mathscr{M}_{\ell_p}$. If $P_m : \ell_p \to \ell_p$ $(m \in \mathbb{N})$ is the operator defined by $P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_p$, then we have

$$\chi(Q) = \lim_{m \to \infty} \left(\sup_{x \in Q} \|(I - P_m)(x)\|_{\ell_p} \right),$$

where I is the identity operator on ℓ_p.

The idea of *compact operators* between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows:

Let X and Y be Banach spaces. Then, a linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and $L(Q)$ is a totally bounded subset of Y for every $Q \in \mathscr{M}_X$. Equivalently, we say that L is compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y . An operator $L \in \mathscr{B}(X,Y)$ is said to be *of finite rank* if $\dim R(L) < \infty$, where $R(L)$ denotes the range space of L. An operator of finite rank is clearly compact.

The most effective way in the characterization of compact operators between the Banach spaces is by applying the Hausdorff measure of noncompactness. This can be achieved as follows:

Let X and Y be Banach spaces and $L \in \mathscr{B}(X;Y)$. Then, the Hausdor measure of noncompactness of L, denoted by $\| L \|_\chi$, can be given by

$$\| L \|_\chi = \chi(L(S_X)) \tag{1.3}$$

and we have

$$L \text{ is compact if and only if } \| L \|_\chi = 0. \tag{1.4}$$

Recent developments on this particular topic can be found in (cf. [1], [2], [4], [8]–[14]).

2. The sequence spaces of generalized means

Throughout this paper, let $r \in \mathscr{U}$ and $s \in \mathscr{U}_o$, where

$$\mathscr{U} = \{u = (u_k) \in w : u_k \neq 0 \text{ for all } k\} \text{ and } \mathscr{U}_o = \{u = (u_k) \in w : u_0 \neq 0\}.$$

For any sequence $x = (x_n) \in w$, we define the sequence $\bar{x} = (\bar{x}_n)$ of generalized means of x by

$$\bar{x}_n = \frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} x_k; \quad (n \in \mathbb{N}). \tag{2.1}$$

Further, we define the infinite matrix $\bar{A}(r,s)$ of generalized means by

$$\bar{A}(r,s)_{nk} = \begin{cases} s_{n-k}/r_n; & (0 \leqslant k \leqslant n), \\ 0; & (k > n) \end{cases} \tag{2.2}$$

for all $n,k \in \mathbb{N}$. Then, by using the notation of (1.2), it follows by (2.1) that $\bar{x}$ is the $\bar{A}(r,s)$-transform of x, that is $\bar{x} = \bar{A}(r,s)x$ for all $x \in w$.

Moreover, it is obvious by (2.2) that $\bar{A}(r,s)$ is a triangle. Thus, it has a unique inverse $\bar{A}(r,s)^{-1}$ which is also a triangle. More precisely, we put $D_0^{(s)} = 1/s_0$ and

$$D_n^{(s)} = \frac{1}{s_0^{n+1}} \begin{vmatrix} s_1 & s_0 & 0 & 0 & \cdots & 0 \\ s_2 & s_1 & s_0 & 0 & \cdots & 0 \\ s_3 & s_2 & s_1 & s_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 \end{vmatrix}; \quad (n = 1,2,3,\ldots).$$

Then, the entries of $\bar{A}(r,s)^{-1}$ are given by

$$\bar{A}(r,s)_{nk}^{-1} = \begin{cases} (-1)^{n-k} D_{n-k}^{(s)} r_k; & (0 \leqslant k \leqslant n), \\ 0; & (k > n) \end{cases} \tag{2.3}$$

for all $n,k \in \mathbb{N}$, that is $\bar{A}(r,s)^{-1} = \bar{A}(s',r)$, where $s' = (s'_n)$ such that $s'_n = (-1)^n D_n^{(s)}$ for all $n \in \mathbb{N}$. Therefore, we have by (2.1) that

$$x_n = \sum_{k=0}^{n} (-1)^{n-k} D_{n-k}^{(s)} r_k \bar{x}_k; \quad (n \in \mathbb{N}). \tag{2.4}$$

REMARK 2.1. It is worth mentioning that the general forms of the well-known matrices of Nörlund, Cesàro, Euler and weighted means can be obtained as special cases of the matrix $\bar{A}(r,s)$ of generalized means (see [11, Example 2.1]). Also, let a and b be non-zero complex numbers. Then, by taking $r = e$ and $s = (a,b,0,0,\ldots)$, the matrix $\bar{A}(r,s)$ is reduced to the generalized difference matrix $B(a,b)$ studied in [2, 3, 4].

For an arbitrary subset X of w, we define the set $X(r,s)$ which is a special case of $X(r,s,t)$ (cf. [11]) as the matrix domain of the triangle $\bar{A}(r,s)$ in X, that is

$$X(r,s) = \left\{ x = (x_k) \in w : \bar{x} = \left(\frac{1}{r_n} \sum_{k=0}^{n} s_{n-k} x_k \right)_{n=0}^{\infty} \in X \right\}.$$

It is obvious that $X(r,s)$ is a sequence space whenever X is a sequence space, and we call it the sequence space of generalized means. Further, if X is a BK space then $\bar{X} = X(r,s)$ is also a BK space with the norm given by

$$\|x\|_{\bar{X}} = \|\bar{x}\|_X; \quad (x \in \bar{X}). \tag{2.5}$$

REMARK 2.2. Let X be a BK space and $\bar{X} = X(r,s)$. Then, it is trivial that $x \in \bar{X}$ if and only if $\bar{x} \in X$. Moreover, we have by (2.5) that $x \in S_{\bar{X}}$ if and only if $\bar{x} \in S_X$. In fact, the linear operator $L : \bar{X} \to X$ defined by $L(x) = \bar{x}$ $(x \in \bar{X})$ is bijective and norm preserving by (2.1), (2.4) and (2.5).

The β-duals of the spaces of generalized means have been determined and some related matrix classes characterized. We refer the reader to [11] for relevant terminology.

Furthermore, by taking into account that the inverse of $\bar{A}(r,s)$ is given by (2.3), we have the following lemma which is immediate by [11, Theorem 4.5].

LEMMA 2.3. *Let X be a BK space with AK or $X = \ell_\infty$. If $a = (a_k) \in (X(r,s))^\beta$, then $\tilde{a} = (\tilde{a}_k) \in X^\beta$ and we have*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \tilde{a}_k \bar{x}_k \tag{2.6}$$

for all $x = (x_k) \in \bar{X}$ with $\bar{x} = \bar{A}(r,s)x$, where

$$\tilde{a}_k = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k}^{(s)} r_k a_j; \quad (k \in \mathbb{N}). \tag{2.7}$$

Now, we prove the following results which will be needed in the sequel.

LEMMA 2.4. *Let X be a BK space with AK or $X = \ell_\infty$ and let $\bar{X} = X(r,s)$. Then, we have*

$$\|a\|_{\bar{X}}^* = \|\tilde{a}\|_X^*$$

for all $a = (a_k) \in \bar{X}^\beta$, where $\tilde{a} = (\tilde{a}_k)$ is the sequence defined by (2.7).

Proof. Let $a = (a_k) \in \bar{X}^\beta$. Then, it follows by Lemma 2.3 that $\tilde{a} = (\tilde{a}_k) \in X^\beta$ and the equality (2.6) holds for all sequences $x = (x_k) \in \bar{X}$ and $\bar{x} = (\bar{x}_k) \in X$ which are connected by the relation $\bar{x} = \bar{A}(r,s)x$. Further, we have by Remark 2.2 that $x \in S_{\bar{X}}$ if and only if $\bar{x} \in S_X$. Therefore, we derive from (1.1) and (2.6) that

$$\|a\|_{\bar{X}}^{*} = \sup_{x \in S_{\bar{X}}} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{\bar{x} \in S_X} \left| \sum_{k=0}^{\infty} \tilde{a}_k \bar{x}_k \right| = \|\tilde{a}\|_X^{*}.$$

This concludes the proof. □

REMARK 2.5. Let $\bar{c}_0 = c_0(r,s)$, $\bar{\ell}_\infty = \ell_\infty(r,s)$, $\bar{\ell}_1 = \ell_1(r,s)$ and $\bar{\ell}_p = \ell_p(r,s)$ for $1 < p < \infty$. Then, by combining Lemmas 1.1 and 2.4, we have the following:

(a) If $a \in (\bar{c}_0)^\beta$, then $\|a\|_{\bar{c}_0}^{*} = \sum_{k=0}^{\infty} |\tilde{a}_k| < \infty$.

(b) If $a \in (\bar{\ell}_\infty)^\beta$, then $\|a\|_{\bar{\ell}_\infty}^{*} = \sum_{k=0}^{\infty} |\tilde{a}_k| < \infty$.

(c) If $a \in (\bar{\ell}_1)^\beta$, then $\|a\|_{\bar{\ell}_1}^{*} = \sup_k |\tilde{a}_k| < \infty$.

(d) If $a \in (\bar{\ell}_p)^\beta$, then $\|a\|_{\bar{\ell}_p}^{*} = (\sum_{k=0}^{\infty} |\tilde{a}_k|^q)^{1/q} < \infty$, where $q = p/(p-1)$.

Throughout this paper, if $A = (a_{nk})$ is an infinite matrix, we define the *associated matrix* $\tilde{A} = (\tilde{a}_{nk})$ by

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} D_{j-k}^{(s)} r_k a_{nj}; \quad (n,k \in \mathbb{N}) \tag{2.8}$$

provided the series on the right converge for all $n,k \in \mathbb{N}$ which is the case whenever $A_n \in X(r,s)^\beta$ for all $n \in \mathbb{N}$, where X is a BK space with AK or $X = \ell_\infty$ [11, Theorem 4.5]. Then, we have:

LEMMA 2.6. *Let X be a BK space with AK or $X = \ell_\infty$, $\bar{X} = X(r,s)$, Y a sequence space and $A = (a_{nk})$ an infinite matrix. If $A \in (\bar{X}, Y)$, then $\tilde{A} \in (X,Y)$ such that $Ax = \tilde{A}\bar{x}$ for all $x \in \bar{X}$ with $\bar{x} = \bar{A}(r,s)x$, where $\tilde{A} = (\tilde{a}_{nk})$ is the associated matrix defined by* (2.8).

Proof. Suppose that $A \in (\bar{X}, Y)$ and let $x \in \bar{X}$. Then $A_n \in \bar{X}^\beta$ for all $n \in \mathbb{N}$. Thus, it follows by Lemma 2.3 that $\tilde{A}_n \in X^\beta$ for all $n \in \mathbb{N}$ and the equality $Ax = \tilde{A}\bar{x}$ holds which yields that $\tilde{A}\bar{x} \in Y$, where $\bar{x}$ is the sequence of generalized means of x, i.e., $\bar{x} = \bar{A}(r,s)x$. Further, it is obvious by (2.4) and Remark 2.2 that every $\bar{x} \in X$ is the sequence of generalized means of some $x \in \bar{X}$. Hence, we deduce that $\tilde{A} \in (X,Y)$. This completes the proof. □

Finally, we conclude this section by the following results on operator norms.

THEOREM 2.7. *Let X be a BK space with AK or $X = \ell_\infty$, $\bar{X} = X(r,s)$, $A = (a_{nk})$ an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})$ the associated matrix. If A is in any of the classes $(\bar{X}, \ell_\infty)$, $(\bar{X}, c)$ or $(\bar{X}, c_0)$, then*

$$\|L_A\| = \|A\|_{(\bar{X},\ell_\infty)} = \sup_n \|\tilde{A}_n\|_X^{*} < \infty.$$

Proof. This is immediate by combining Lemmas 1.3 and 2.4. □

THEOREM 2.8. *Let X be a BK space with AK or $X = \ell_\infty$ and let $\bar{X} = X(r,s)$. If $A \in (\bar{X}, \ell_1)$, then*

$$\|A\|_{(\bar{X},\ell_1)} \leqslant \|L_A\| \leqslant 4 \cdot \|A\|_{(\bar{X},\ell_1)},$$

where

$$\|A\|_{(\bar{X},\ell_1)} = \sup_{N \in \mathscr{F}} \left\| \sum_{n \in N} \tilde{A}_n \right\|_X^* < \infty.$$

Proof. This result follows from Lemmas 1.4 and 2.4. □

REMARK 2.9. The special cases of Theorems 2.7 and 2.8 when X is any of the spaces c_0, ℓ_∞ or ℓ_p $(1 \leqslant p < \infty)$ can be obtained by means of Lemma 1.1.

THEOREM 2.10. *Let $\bar{\ell}_1 = \ell_1(r,s)$ and $1 \leqslant p < \infty$. If $A \in (\bar{\ell}_1, \ell_p)$, then*

$$\|L_A\| = \|A\|_{(\bar{\ell}_1,\ell_p)} = \sup_k \left(\sum_{n=0}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} < \infty.$$

Proof. The proof is elementary and left to the reader (see [11, Corollary 5.10]). □

REMARK 2.11. The characterizations of matrix classes considered in this paper can be found in [11, Corollaries 5.4; 5.7]. Thus, we shall omit these characterizations and only deal with the operator norms and the Hausdorff measures of noncompactness of some matrix operators which are given by infinite matrices in such classes.

3. Compact operators on the spaces of generalized means

In this section, we derive some identities for the Hausdorff measures of noncompactness of certain matrix operators on the spaces of generalized means and apply our results to obtain the necessary and sufficient (or only sufficient) conditions for such operators to be compact.

We recall the following lemma [10, Theorem 3.7] which is very useful in establishing the results of this section.

LEMMA 3.1. *Let $X \supset \phi$ be a BK space. Then, we have*

(a) *If $A \in (X, \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|A_n\|_X^*.$$

(b) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^*.$$

(c) *If* X *has* AK *or* $X = \ell_\infty$ *and* $A \in (X,c)$*, then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \|A_n - \alpha\|_X^* \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|A_n - \alpha\|_X^*,$$

where $\alpha = (\alpha_k)$ *with* $\alpha_k = \lim_{n\to\infty} a_{nk}$ *for all* $k \in \mathbb{N}$.

Now, let $A = (a_{nk})$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})$ the associated matrix defined by (2.8). Then, by combining Lemmas 2.4, 2.6 and 3.1, we have the following result:

THEOREM 3.2. *Let* X *be a BK space with AK or* $X = \ell_\infty$ *and* $\bar{X} = X(r,s)$*. Then, we have*

(a) *If* $A \in (\bar{X}, \ell_\infty)$*, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|\tilde{A}_n\|_X^* \tag{3.1}$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty} \|\tilde{A}_n\|_X^* = 0. \tag{3.2}$$

(b) *If* $A \in (\bar{X}, c_0)$*, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|\tilde{A}_n\|_X^* \tag{3.3}$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \|\tilde{A}_n\|_X^* = 0. \tag{3.4}$$

(c) *If* $A \in (\bar{X}, c)$*, then*

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_X^* \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_X^* \tag{3.5}$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_X^* = 0, \tag{3.6}$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ *with* $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ *for all* $k \in \mathbb{N}$.

Proof. It is obvious that (3.2), (3.4) and (3.6) are respectively obtained from (3.1), (3.3) and (3.5) by using (1.4). Thus, we have to prove (3.1), (3.3) and (3.5).

Since $\bar{X}$ is a *BK* space, we deduce by means of Lemma 2.4 that (3.1) and (3.3) are immediate by parts (a) and (b) of Lemma 3.1, respectively.

To prove (3.5), we have $A \in (\bar{X}, c)$ and hence $\tilde{A} \in (X,c)$ by Lemma 2.6. Therefore, it follows by part (c) of Lemma 3.1 that

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_X^* \leqslant \|L_{\tilde{A}}\|_\chi \leqslant \limsup_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_X^*, \tag{3.7}$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ and $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

Now, let us write $S = S_X$ and $\bar{S} = S_{\bar{X}}$, for short. Then, we obtain by (1.3) and Lemma 1.2 that

$$\|L_A\|_\chi = \chi(L_A(\bar{S})) = \chi(A\bar{S}) \tag{3.8}$$

and

$$\|L_{\tilde{A}}\|_\chi = \chi(L_{\tilde{A}}(S)) = \chi(\tilde{A}S). \tag{3.9}$$

Further, we have by Remark 2.2 that $x \in \bar{S}$ if and only if $\bar{x} \in S$, and since $Ax = \tilde{A}\bar{x}$ by Lemma 2.6, we deduce that $A\bar{S} = \tilde{A}S$. This leads us with (3.8) and (3.9) to the consequence that $\|L_A\|_\chi = \|L_{\tilde{A}}\|_\chi$. Hence, we get (3.5) from (3.7). This completes the proof. □

It is worth mentioning that the condition in (3.2) is only a sufficient condition for the operator L_A to be compact, where $A \in (\bar{X}, \ell_\infty)$ and X is a BK space with AK or $X = \ell_\infty$. More precisely, the following example will show that it is possible for L_A to be compact while $\lim_{n\to\infty} \|\tilde{A}_n\|_X^* \neq 0$. Hence, in general, we have just 'if' in (3.2) of Theorem 3.2 (a).

EXAMPLE 3.3. Let X denote any of the spaces c_0, ℓ_∞ or ℓ_p $(1 \leqslant p < \infty)$ and let $\bar{X} = X(r,s)$. Also, let us define the matrix $A = (a_{nk})$ by $a_{n0} = s_0/r_0$ and $a_{nk} = 0$ for $k \geqslant 1$ $(n \in \mathbb{N})$. Then, we have for every $x = (x_k) \in \bar{X}$ that $Ax = (s_0x_0/r_0)e$ and hence $A \in (\bar{X}, \ell_\infty)$. Further, it is obvious that L_A is of finite rank and so L_A is compact. On the other hand, by using (2.8), it can easily be seen that $\tilde{A}_n = e^{(0)}$ for all $n \in \mathbb{N}$. Thus, we obtain by Lemma 1.1 that $\|\tilde{A}_n\|_X^* = 1$ for all $n \in \mathbb{N}$ which implies that $\lim_{n\to\infty} \|\tilde{A}_n\|_X^* = 1$.

Moreover, as an immediate consequence of Theorem 3.2, we have the following corollary in which we write $\bar{\ell}_\infty = \ell_\infty(r,s)$.

COROLLARY 3.4. *If either $A \in (\bar{\ell}_\infty, c_0)$ or $A \in (\bar{\ell}_\infty, c)$, then the operator L_A is compact.*

Proof. Let $A \in (\bar{\ell}_\infty, c_0)$. Then, we have by Lemma 2.6 that $\tilde{A} \in (\ell_\infty, c_0)$ which implies that $\lim_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk}|) = 0$ [16], that is, $\lim_{n\to\infty} \|\tilde{A}_n\|_{\ell_\infty}^* = 0$ by Lemma 1.1. This leads us with Theorem 3.2 (b) to the consequence that L_A is compact. Similarly, if $A \in (\bar{\ell}_\infty, c)$ then $\tilde{A} \in (\ell_\infty, c)$ and hence $\lim_{n\to\infty}(\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|) = 0$ which can be written as $\lim_{n\to\infty} \|\tilde{A}_n - \tilde{\alpha}\|_{\ell_\infty}^* = 0$, where $\tilde{\alpha} = (\tilde{\alpha}_k)$ and $\tilde{\alpha}_k = \lim_{n\to\infty} \tilde{a}_{nk}$ for all $k \in \mathbb{N}$. Therefore, we deduce from Theorem 3.2 (c) that L_A is compact. □

Throughout, let $\mathscr{F}_m$ $(m \in \mathbb{N})$ be the subcollection of $\mathscr{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than m, that is

$$\mathscr{F}_m = \{N \in \mathscr{F} : n > m \text{ for all } n \in N\}; \quad (m \in \mathbb{N}).$$

Then, we have the following [10, Theorem 3.11]:

LEMMA 3.5. *Let $X \supset \phi$ be a BK space. If $A \in (X, \ell_1)$, then*

$$\lim_{m\to\infty}\left(\sup_{N\in\mathscr{F}_m}\left\|\sum_{n\in N}A_n\right\|_X^*\right) \leqslant \|L_A\|_\chi \leqslant 4\cdot \lim_{m\to\infty}\left(\sup_{N\in\mathscr{F}_m}\left\|\sum_{n\in N}A_n\right\|_X^*\right).$$

THEOREM 3.6. *Let X be a BK space with AK and $\bar{X} = X(r,s)$. If $A \in (\bar{X}, \ell_1)$, then*

$$\lim_{m\to\infty}\|A\|_{(\bar{X},\ell_1)}^{(m)} \leqslant \|L_A\|_\chi \leqslant 4\cdot \lim_{m\to\infty}\|A\|_{(\bar{X},\ell_1)}^{(m)} \tag{3.10}$$

and

$$L_A \text{ is compact if and only if } \lim_{m\to\infty}\|A\|_{(\bar{X},\ell_1)}^{(m)} = 0, \tag{3.11}$$

where

$$\|A\|_{(\bar{X},\ell_1)}^{(m)} = \sup_{N\in\mathscr{F}_m}\left\|\sum_{n\in N}\tilde{A}_n\right\|_X^*; \quad (m\in\mathbb{N}).$$

Proof. It is obvious that (3.10) is obtained by combining Lemmas 2.4 and 3.5. Also, by using (1.4), we get (3.11) from (3.10). □

Now, we may note that Theorems 3.2 and 3.6 have several consequences when X is any of the spaces c_0 or ℓ_p $(1 \leqslant p < \infty)$. For instance, by using Lemma 1.1, we have the following corollaries:

COROLLARY 3.7. *Let $\bar{c}_0 = c_0(r,s)$. Then, we have*

(a) *If $A \in (\bar{c}_0, \ell_\infty)$, then*

$$0 \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk}|\right)$$

and

$$L_A \text{ is compact if } \lim_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk}|\right) = 0.$$

(b) *If $A \in (\bar{c}_0, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk}|\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk}|\right) = 0.$$

(c) *If $A \in (\bar{c}_0, c)$, then*

$$\frac{1}{2}\cdot\limsup_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|\right) \leqslant \|L_A\|_\chi \leqslant \limsup_{n\to\infty}\left(\sum_{k=0}^\infty |\tilde{a}_{nk} - \tilde{\alpha}_k|\right)$$

and

$$L_A \text{ is compact if and only if } \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\tilde{a}_{nk}-\tilde{\alpha}_k|\right)=0,$$

where $\tilde{\alpha}_k=\lim_{n\to\infty}\tilde{a}_{nk}$ *for all* $k\in\mathbb{N}$.

COROLLARY 3.8. *Let* $1<p<\infty$, $q=p/(p-1)$ *and* $\overline{\ell}_p=\ell_p(r,s)$. *If* $A\in(\overline{\ell}_p,\ell_1)$, *then*

$$\lim_{m\to\infty}\|A\|^{(m)}_{(\overline{\ell}_p,\ell_1)}\leqslant\|L_A\|_\chi\leqslant 4\cdot\lim_{m\to\infty}\|A\|^{(m)}_{(\overline{\ell}_p,\ell_1)}$$

and

$$L_A \text{ is compact if and only if } \lim_{m\to\infty}\|A\|^{(m)}_{(\overline{\ell}_p,\ell_1)}=0,$$

where

$$\|A\|^{(m)}_{(\overline{\ell}_p,\ell_1)}=\sup_{N\in\mathscr{F}_m}\left(\sum_{k=0}^{\infty}\left|\sum_{n\in N}\tilde{a}_{nk}\right|^q\right)^{1/q};\quad (m\in\mathbb{N}).$$

Now, we prove the following result:

THEOREM 3.9. *Let* $\overline{\ell}_1=\ell_1(r,s)$ *and* $1\leqslant p<\infty$. *If* $A\in(\overline{\ell}_1,\ell_p)$, *then*

$$\|L_A\|_\chi=\lim_{m\to\infty}\left(\sup_k\left(\sum_{n=m}^{\infty}|\tilde{a}_{nk}|^p\right)^{1/p}\right)\tag{3.12}$$

and

$$L_A \text{ is compact if and only if } \lim_{m\to\infty}\left(\sup_k\left(\sum_{n=m}^{\infty}|\tilde{a}_{nk}|^p\right)\right)=0.\tag{3.13}$$

Proof. Let us remark that the limit in (3.12) exists by Theorem 2.10.

Now, we write $\overline{S}=S_{\overline{\ell}_1}$. Then, we have by Lemma 1.2 that $L_A(\overline{S})=A\overline{S}\in\mathscr{M}_{\ell_p}$. Thus, it follows from (1.3) and Lemma 1.5 that

$$\|L_A\|_\chi=\chi(A\overline{S})=\lim_{m\to\infty}\left(\sup_{x\in\overline{S}}\|(I-P_m)(Ax)\|_{\ell_p}\right),\tag{3.14}$$

where $P_m:\ell_p\to\ell_p$ $(m\in\mathbb{N})$ is the operator defined by $P_m(x)=(x_0,x_1,\ldots,x_m,0,0,\ldots)$ for all $x=(x_k)\in\ell_p$ and I is the identity operator on ℓ_p.

On the other hand, let $x\in\overline{\ell}_1$ be given. Then $\overline{x}\in\ell_1$ and since $A\in(\overline{\ell}_1,\ell_p)$, we obtain from Lemma 2.6 that $\tilde{A}\in(\ell_1,\ell_p)$ and $Ax=\tilde{A}\overline{x}$. Thus, we have for every $m\in\mathbb{N}$

that

$$\begin{aligned}\|(I-P_m)(Ax)\|_{\ell_p} &= \|(I-P_m)(\tilde{A}\bar{x})\|_{\ell_p} \\ &= \Big(\sum_{n=m+1}^{\infty}|\tilde{A}_n(\bar{x})|^p\Big)^{1/p} \\ &= \Big(\sum_{n=m+1}^{\infty}\Big|\sum_{k=0}^{\infty}\tilde{a}_{nk}\bar{x}_k\Big|^p\Big)^{1/p} \\ &\leqslant \sum_{k=0}^{\infty}\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}\bar{x}_k|^p\Big)^{1/p} \\ &\leqslant \|\bar{x}\|_{\ell_1}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}\Big) \\ &= \|x\|_{\bar{\ell}_1}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}\Big).\end{aligned}$$

This yields that

$$\sup_{x\in\bar{S}}\|(I-P_m)(Ax)\|_{\ell_p} \leqslant \sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}; \quad (m\in\mathbb{N}).$$

Therefore, we deduce from (3.14) that

$$\|L_A\|_\chi \leqslant \lim_{m\to\infty}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}\Big). \tag{3.15}$$

To prove the converse inequality, let $b^{(k)}\in\bar{\ell}_1$ be such that $\bar{A}(r,s)b^{(k)}=e^{(k)}$ $(k\in\mathbb{N})$, that is, $e^{(k)}$ is the sequence of generalized means of $b^{(k)}$ for each $k\in\mathbb{N}$ (see [11, Corollary 3.5]). Then, we have by Lemma 2.6 that $Ab^{(k)}=\tilde{A}e^{(k)}=(\tilde{a}_{nk})_{n=0}^{\infty}$ for every $k\in\mathbb{N}$.

Now, let $B=\{b^{(k)}:k\in\mathbb{N}\}$. Then $B\subset\bar{S}$ and hence $AB\subset A\bar{S}$ which implies that $\chi(AB)\leqslant\chi(A\bar{S})=\|L_A\|_\chi$.

Further, it follows by applying Lemma 1.5 that

$$\begin{aligned}\chi(AB) &= \lim_{m\to\infty}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|A_n(b^{(k)})|^p\Big)^{1/p}\Big) \\ &= \lim_{m\to\infty}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}\Big).\end{aligned}$$

Thus, we obtain that

$$\lim_{m\to\infty}\Big(\sup_k\Big(\sum_{n=m+1}^{\infty}|\tilde{a}_{nk}|^p\Big)^{1/p}\Big) \leqslant \|L_A\|_\chi. \tag{3.16}$$

Hence, we get (3.12) by combining (3.15) and (3.16). This completes the proof, since (3.13) is immediate by (1.4) and (3.12). □

Finally, we end this section with the following example which shows that the limit in (3.12) may not be zero, that is, there exist matrix operators in the class $\mathscr{B}(\overline{\ell}_1, \ell_p)$ which are not compact, where $1 \leqslant p < \infty$.

EXAMPLE 3.10. Let $A = (a_{nk})$ be the infinite matrix defined by (2.2), that is $A = \overline{A}(r,s)$. Since $\overline{\ell}_1$ is the matrix domain of A in ℓ_1, we have $A \in (\overline{\ell}_1, \ell_1)$ and hence $A \in (\overline{\ell}_1, \ell_p)$ for $1 \leqslant p < \infty$. Further, it is trivial to see that the associated matrix $\tilde{A}$ is the identity matrix, that is $\tilde{a}_{nn} = 1$ and $\tilde{a}_{nk} = 0$ for $k \neq n$ $(n \in \mathbb{N})$. Now, let $m \in \mathbb{N}$ be given. Then, we have for every $k \in \mathbb{N}$ that

$$\sum_{n=m}^{\infty} |\tilde{a}_{nk}|^p = \begin{cases} 1; & (k \geqslant m), \\ 0; & (k < m). \end{cases}$$

This implies that

$$\sup_k \left(\sum_{n=m}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} = 1; \quad (m \in \mathbb{N})$$

which leads us with (3.12) of Theorem 3.9 to the consequence that $\|L_A\|_\chi = 1$ and hence L_A is not compact.

REMARK 3.11. Many applications and special cases of Theorems 3.2, 3.6 and 3.9 can be found in [4, 5, 7, 8, 9, 12, 13, 14] for some particular sequences r and s.

Acknowledgement

The authors are very much grateful to the learned referee for his/ her kind suggestions which improved the presentation of the paper.

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(Received July 26, 2011)

M. Mursaleen
Department of Mathematics
Aligarh Muslim University
Aligarh 202002, India
e-mail: mursaleenm@gmail.com

Abdullah K. Noman
Department of Mathematics
Faculty of Education and Science
Al Bayda University, Yemen
e-mail: akanoman@gmail.com