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INEQUALITIES FOR CONVEX FUNCTIONS AND DOUBLY STOCHASTIC MATRICES

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Abstract. We generalize some results on convex functions presented in papers L. Bougoffa, New inequalities about convex functions, J. Inequal. Pure Appl. Math., **7** (4), (2006) Art. 148, and L.-C. Wang, X.-F. Ma and L.-H. Liu, A note on some new refinements of Jensen's inequality for convex functions, J. Inequal. Pure Appl. Math., **10** (2), (2009) Art. 48. To this end, we use majorization of vectors, doubly stochastic matrices and circular matrices.

1. Motivation

In [1], L. Bougoffa established the following results.

THEOREM A. [1, Theorem 1.2] If f is a convex function and $x_1,x_2,\dots,x_n$ lie in its domain, then

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \geqslant \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right)+\dots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right]. \tag{1}$$

THEOREM B. [1, Theorem 1.4] If f is a convex function and $a_1,a_2,\dots,a_n$ lie in its domain, then

$$(n-1)[f(b_1)+\dots+f(b_n)] \leqslant n[f(a_1)+\dots+f(a_n)-f(a)] \tag{2}$$

where $a = \frac{a_1+a_2+\dots+a_n}{n}$ *and* $b_i = \frac{na-a_i}{n-1}$, $i=1,2,\dots,n$.

In [7], Wang et. al. gave some refinements of Theorems A and B (see [7, Theorems 2.1 and 2.2]).

The purpose of the present paper is to show further generalizations of Theorems A and B by applying Majorization Theorem, doubly stochastic matrices and circular matrices, and to extend the above-mentioned refinements to a more general framework.

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2. Making use of doubly stochastic matrices

We denote by $z_{[1]} \geqslant z_{[2]} \geqslant \ldots \geqslant z_{[n]}$ the entries of a vector $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n$ arranged in decreasing order.

We say that a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ is *majorized* by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ (in symbols, $\mathbf{y} \prec_m \mathbf{x}$) if $\sum_{k=1}^{i} y_{[k]} \leqslant \sum_{k=1}^{i} x_{[k]}$ for all $i = 1, 2, \ldots, n$ with equality for $i = n$ (see [6, p. 8]).

A real function $F : A \to \mathbb{R}$ on a set $A \subset \mathbb{R}^n$ is called *Schur-convex* on A if

$$\mathbf{y} \prec_m \mathbf{x} \quad \text{implies} \quad F(\mathbf{y}) \leqslant F(\mathbf{x}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in A$$

(see [6, p. 80]).

Majorization Theorem asserts that if $f : I \to \mathbb{R}$ is a convex function and $\mathbf{x}, \mathbf{y} \in I^n$ then

$$\mathbf{y} \prec_m \mathbf{x} \quad \text{implies} \quad \sum_{i=1}^{n} f(y_i) \leqslant \sum_{i=1}^{n} f(x_i) \tag{3}$$

(see [6, p. 92]). In other words, if $f : I \to \mathbb{R}$ is a convex function then the induced function

$$F(\mathbf{x}) = \sum_{i=1}^{n} f(x_i) \quad \text{for } \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in I^n$$

is Schur-convex on I^n.

A real $n \times n$ matrix $S = (s_{ij})$ is called *doubly stochastic* (in short, *d. s.*) if

(i) $s_{ij} \geqslant 0$ for $i, j = 1, 2, \ldots, n$,

(ii) $\sum_{j=1}^{n} s_{ij} = 1$ for $i = 1, 2, \ldots, n$,

(iii) $\sum_{i=1}^{n} s_{ij} = 1$ for $j = 1, 2, \ldots, n$.

It is known that if S is doubly stochastic then

$$S\mathbf{x} \prec_m \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \tag{4}$$

(see [6, p. 33]).

In the sequel, we use the notation

$$\overline{z} = \frac{z_1 + z_2 + \ldots + z_n}{n} \quad \text{for } z_1, z_2, \ldots, z_n \in \mathbb{R}.$$

The following result holds.

THEOREM 2.1. *Let f be a convex function on an interval $I \subset \mathbb{R}$ and $S = (s_{ij})$ be a doubly stochastic $n \times n$ matrix.*

If $x_1, x_2, \ldots, x_n \in I$, and $(y_1, y_2, \ldots, y_n)^T = S(x_1, x_2, \ldots, x_n)^T$, i.e., $y_i = \sum_{j=1}^{n} s_{ij} x_j$ for $i = 1, 2, \ldots, n$, then for any $a, b \in \mathbb{R}$ with $a + b = 1$,

(i)
$$\sum_{i=1}^{n} f(x_i) \geqslant a\sum_{i=1}^{n} f(y_i) + bnf(\overline{y}) \quad \textit{for } a>0,\ b>0, \tag{5}$$

(ii)
$$\sum_{i=1}^{n} f(y_i) \leqslant a\sum_{i=1}^{n} f(x_i) + bnf(\overline{x}) \quad \textit{for } a>0,\ b<0, \tag{6}$$

(iii)
$$\sum_{i=1}^{n} f(x_i) \geqslant a\sum_{i=1}^{n} f(y_i) + bnf(\overline{y}) \quad \textit{for } a<0,\ b>0, \tag{7}$$

(iv)
$$\sum_{i=1}^{n} f(y_i) \geqslant a\sum_{i=1}^{n} f(x_i) + bnf(\overline{x}) \quad \textit{for } a<0,\ b>0. \tag{8}$$

Proof. (Based on the proof of [1, Theorem 1.2].)
Since $a+b=1$, it is easy to check that for any $z_1, z_2, \dots, z_n \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} f(z_i) = a\sum_{i=1}^{n} f(z_i) + bn\sum_{i=1}^{n} \frac{1}{n} f(z_i). \tag{9}$$

Next, by Jensen's inequality, we have

$$f(\overline{z}) \leqslant \sum_{i=1}^{n} \frac{1}{n} f(z_i). \tag{10}$$

Hence we get

$$bnf(\overline{z}) \leqslant bn\sum_{i=1}^{n} \frac{1}{n} f(z_i) \quad \text{for } b>0, \tag{11}$$

$$bnf(\overline{z}) \geqslant bn\sum_{i=1}^{n} \frac{1}{n} f(z_i) \quad \text{for } b<0. \tag{12}$$

Now, by combining (9) with (11)–(12), we establish

$$\sum_{i=1}^{n} f(z_i) \geqslant a\sum_{i=1}^{n} f(z_i) + bnf(\overline{z}) \quad \text{for } b>0, \tag{13}$$

$$\sum_{i=1}^{n} f(z_i) \leqslant a\sum_{i=1}^{n} f(z_i) + bnf(\overline{z}) \quad \text{for } b<0. \tag{14}$$

On the other hand, by (4) we have $\mathbf{y} \prec_m \mathbf{x}$, and hence $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$ and $\overline{y} = \overline{x}$. Therefore, by (3), we obtain

$$\sum_{i=1}^{n} f(y_i) \leqslant \sum_{i=1}^{n} f(x_i). \tag{15}$$

Now, by using (15) and (13) for $\mathbf{z} = \mathbf{y}$, we derive

$$\sum_{i=1}^{n} f(x_i) \geqslant a \sum_{i=1}^{n} f(y_i) + bnf(\overline{y}) \quad \text{for } b > 0,$$

and from (15) and (14) for $\mathbf{z} = \mathbf{x}$, we find that

$$\sum_{i=1}^{n} f(y_i) \leqslant a \sum_{i=1}^{n} f(x_i) + bnf(\overline{x}) \quad \text{for } b < 0,$$

completing the proof of (5), (6) and (7).

To see (8), observe that for $a < 0$ and $b > 0$ we have

$$\begin{aligned}\sum_{i=1}^{n} f(y_i) &= a \sum_{i=1}^{n} f(y_i) + b \sum_{i=1}^{n} f(y_i) \geqslant a \sum_{i=1}^{n} f(y_i) + bnf(\overline{y}) \\ &\geqslant a \sum_{i=1}^{n} f(x_i) + bnf(\overline{x}),\end{aligned}$$

the first and second inequalities being consequences of Jensen's inequality and of (3), respectively. □

REMARK 2.2. Observe that (13)–(14) are reformulations of Jensen inequality (10). For $b = 0$ inequalities in (13)–(14) are trivial.

REMARK 2.3. A detailed version of Theorem 2.1 can be presented as follows.

By denoting

$$A = \sum_{i=1}^{n} f(x_i) \quad \text{and} \quad B = \sum_{i=1}^{n} f(y_i),$$

$$C = a \sum_{i=1}^{n} f(x_i) + bnf(\overline{x}) \quad \text{and} \quad D = a \sum_{i=1}^{n} f(y_i) + bnf(\overline{y}),$$

and by writing inequalities $\alpha \geqslant \gamma$ and $\beta \leqslant \delta$ in the arrow forms $\alpha \rightarrow \gamma$ (or $\begin{smallmatrix}\alpha\\ \downarrow\\ \gamma\end{smallmatrix}$) and $\beta \leftarrow \delta$ (or $\begin{smallmatrix}\beta\\ \uparrow\\ \delta\end{smallmatrix}$), respectively, we have

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \qquad \text{for } a > 0 \text{ and } b > 0,$$

$$\begin{array}{ccc} A & \longleftarrow & C \\ \downarrow & & \downarrow \\ B & \longleftarrow & D \end{array} \qquad \text{for } a > 0 \text{ and } b < 0,$$

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \uparrow \\ B & \longrightarrow & D \end{array} \qquad \text{for } a < 0 \text{ and } b > 0.$$

(Since $a + b = 1$, the case $a < 0$ and $b < 0$ is impossible.)

If $a+b=1$ then $\frac{1}{a}+\frac{(-b)}{a}=1$ for $a\neq 0$ and $\frac{1}{b}+\frac{(-a)}{b}=1$ for $b\neq 0$. For this reason we are allowed to replace (a,b) by $(\frac{1}{a},\frac{-b}{a})$ or $(\frac{1}{b},\frac{-a}{b})$ in Theorem 2.1.

COROLLARY 2.4. *Let f be a convex function on an interval $I\subset\mathbb{R}$ and $S=(s_{ij})$ be a doubly stochastic $n\times n$ matrix.*

If $x_1,x_2,\dots,\ x_n\in I$, and $(y_1,y_2,\dots,y_n)^T=S(x_1,x_2,\dots,x_n)^T$, then for any $a,b\in\mathbb{R}$ with $a+b=1$ and $a\neq 0$,

(i)
$$\sum_{i=1}^{n} f(y_i)\leqslant \frac{1}{a}\sum_{i=1}^{n} f(x_i)+\left(\frac{-b}{a}\right)nf(\overline{x}) \quad \text{for } a>0 \text{ and } b>0,$$

(ii)
$$\sum_{i=1}^{n} f(x_i)\geqslant \frac{1}{a}\sum_{i=1}^{n} f(y_i)+\left(\frac{-b}{a}\right)nf(\overline{y}) \quad \text{for } a>0 \text{ and } b<0,$$

(iii)
$$\sum_{i=1}^{n} f(x_i)\geqslant \frac{1}{a}\sum_{i=1}^{n} f(y_i)+\left(\frac{-b}{a}\right)nf(\overline{y}) \quad \text{for } a<0 \text{ and } b>0,$$

(iv)
$$\sum_{i=1}^{n} f(y_i)\geqslant \frac{1}{a}\sum_{i=1}^{n} f(x_i)+\left(\frac{-b}{a}\right)nf(\overline{x}) \quad \text{for } a<0 \text{ and } b>0.$$

By using Theorem 2.1 and Corollary 2.4 for $\mathbf{x}=\mathbf{y}$ and $S=I$ (the $n\times n$ identity matrix), we obtain the following.

COROLLARY 2.5. *Let f be a convex function on an interval $I\subset\mathbb{R}$ and $x_1,x_2,\dots,$ $x_n\in I$.*

Then for any $a,b\in\mathbb{R}$ with $a+b=1$,

(i) *if $a>0,\ b>0$ then*
$$\frac{1}{a}\sum_{i=1}^{n} f(x_i)+\left(\frac{-b}{a}\right)nf(\overline{x})\geqslant \sum_{i=1}^{n} f(x_i)\geqslant a\sum_{i=1}^{n} f(x_i)+bnf(\overline{x}),$$

(ii) *if $a>0,\ b<0$ then*
$$a\sum_{i=1}^{n} f(x_i)+bnf(\overline{x})\geqslant \sum_{i=1}^{n} f(x_i)\geqslant \frac{1}{a}\sum_{i=1}^{n} f(x_i)+\left(\frac{-b}{a}\right)nf(\overline{x}),$$

(iii) *if $a<0,\ b>0$ then*
$$\sum_{i=1}^{n} f(x_i)\geqslant \max\left\{a\sum_{i=1}^{n} f(x_i)+bnf(\overline{x}),\frac{1}{a}\sum_{i=1}^{n} f(x_i)+\left(\frac{-b}{a}\right)nf(\overline{x})\right\}.$$

REMARK 2.6. It is readily seen that Theorem A is a specialization of Theorem 2.1, part (ii), for numbers $a=\frac{n}{n-1}$ and $b=-\frac{1}{n-1}$ and for vectors $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$ and

$$\mathbf{y}=S\mathbf{x}=\left(\frac{x_1+x_2}{2},\frac{x_2+x_3}{2},\ldots,\frac{x_n+x_1}{2}\right)^T, \tag{16}$$

where S is $n\times n$ matrix given by

$$S=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \ldots & \ldots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \ldots & \ldots & 0 & \frac{1}{2} \end{pmatrix}. \tag{17}$$

Likewise, in order to obtain Theorem A, one can apply Theorem 2.1, part (i), for $a=\frac{n-1}{n}$ and $b=\frac{1}{n}$ with (16)–(17).

Observe that (17) is a circular matrix.

3. Application of circular matrices

Remind that the *circular matrix (circulant)* induced by a real sequence $(c_1,c_2,\ldots,c_n)$ is the $n\times n$ matrix whose first row is $(c_1,c_2,\ldots,c_n)$ and the other rows are obtained by succesive cyclic permutations of the first row, i.e.,

$$S=\begin{pmatrix} c_1 & c_2 & c_3 & \ldots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & \ldots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_n & c_1 & \ddots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ c_3 & c_4 & c_5 & \ldots & c_1 & c_2 \\ c_2 & c_3 & c_4 & \ldots & c_n & c_1 \end{pmatrix}. \tag{18}$$

In the forthcoming corollary we apply Theorem 2.1 to the circular matrix (18) with nonnegative entries summing to one in each row and column (doubly stochastic circular matrix; see [6, pp. 62-64]).

COROLLARY 3.1. *Let f be a convex function on an interval $I\subset\mathbb{R}$ and $x_1,x_2,\ldots,x_n\in I$, and let S be an $n\times n$ circular matrix defined by (18) with $\sum_{i=1}^{n} c_i=1$ and $c_i\geqslant 0$ for $i=1,2,\ldots,n$.*

Then for any $a,b\in\mathbb{R}$ with $a+b=1$ and $b>0$, we have

$$\sum_{i=1}^{n} f(x_i)-bnf\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)\geqslant a\left[f(\langle\mathbf{r}_1,\mathbf{x}\rangle)+f(\langle\mathbf{r}_2,\mathbf{x}\rangle)+\ldots+f(\langle\mathbf{r}_n,\mathbf{x}\rangle)\right], \tag{19}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$, $\mathbf{r}_i^T$ denotes the ith row of S and $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T$.

Given two real numbers α and β, we now focus on the circular matrix

$$S_{k,n-k}(\alpha,\beta)=\frac{1}{k\alpha+(n-k)\beta}\begin{pmatrix}\alpha & \dots & \dots & \alpha & \beta & \dots & \dots & \beta\\ \beta & \alpha & \dots & \dots & \alpha & \beta & \dots & \beta\\ \vdots & \ddots & \ddots & \dots & \dots & \ddots & \ddots & \vdots\\ \alpha & \dots & \alpha & \beta & \dots & \dots & \beta & \alpha\end{pmatrix} \tag{20}$$

with the first row $(\underbrace{\alpha,\dots,\alpha}_{k\ \text{times}},\ \underbrace{\beta,\dots,\beta}_{n-k\ \text{times}})$.

COROLLARY 3.2. *If f is a convex function on an interval $I\subset\mathbb{R}$ and $x_1,x_2,\dots,$ $x_n\in I$, then for any $a,b\in\mathbb{R}$ with $a+b=1$ and $b>0$, we have*

$$\begin{aligned}&\sum_{i=1}^{n} f(x_i)-bnf\left(\frac{x_1+x_2+\dots+x_n}{n}\right)\\ &\geqslant a\left[f\left(\frac{\alpha(x_1+x_2+\dots+x_k)+\beta(x_{k+1}+\dots+x_n)}{k\alpha+(n-k)\beta}\right)\right.\\ &\quad+f\left(\frac{\alpha(x_2+x_3+\dots+x_{k+1})+\beta(x_{k+2}+\dots+x_n+x_1)}{k\alpha+(n-k)\beta}\right)\\ &\quad\left.+\dots+f\left(\frac{\alpha(x_1+\dots+x_{k-1}+x_n)+\beta(x_k+\dots+x_{n-1})}{k\alpha+(n-k)\beta}\right)\right].\end{aligned}$$

For $\alpha=1$ and $\beta=0$, (20) becomes

$$S_{k,n-k}(1,0)=\frac{1}{k}\begin{pmatrix}1 & \dots & 1 & 0 & \dots & \dots & 0\\ 0 & 1 & \dots & 1 & 0 & \dots & 0\\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots\\ 1 & \dots & 1 & 0 & \dots & 0 & 1\end{pmatrix}$$

with the first row $(\underbrace{1,\dots,1}_{k\ \text{times}},\ \underbrace{0,\dots,0}_{n-k\ \text{times}})$, and then Corollary 3.2 reduces to

COROLLARY 3.3. *If f is a convex function on an interval $I\subset\mathbb{R}$ and $x_1,x_2,\dots,$ $x_n\in I$, then for any $a,b\in\mathbb{R}$ with $a+b=1$ and $b>0$, we have*

$$\begin{aligned}&\sum_{i=1}^{n} f(x_i)-bnf\left(\frac{x_1+x_2+\dots+x_n}{n}\right)\\ &\geqslant a\left[f\left(\frac{x_1+x_2+\dots+x_k}{k}\right)+f\left(\frac{x_2+x_3+\dots+x_{k+1}}{k}\right)+\dots+f\left(\frac{x_1+\dots+x_{k-1}+x_n}{k}\right)\right].\end{aligned}$$

For $\alpha=0$ and $\beta=1$, it follows from (20) that

$$S_{k,n-k}(0,1)=\frac{1}{n-k}\begin{pmatrix}0 & \dots & 0 & 1 & \dots & \dots & 1\\ 1 & 0 & \dots & 0 & 1 & \dots & 1\\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 1 & \dots & 1 & 0\end{pmatrix}$$

with the first row $(\underbrace{0,\ldots,0}_{k \text{ times}},\underbrace{1,\ldots,1}_{n-k \text{ times}})$, and then Corollary 3.2 implies the following.

COROLLARY 3.4. *If f is a convex function on an interval $I \subset \mathbb{R}$ and $x_1, x_2, \ldots, x_n \in I$, then for any $a, b \in \mathbb{R}$ with $a+b=1$ and $b>0$, we have*

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - bnf\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \\
&\geqslant a\left[f\left(\frac{x_{k+1}+x_{k+2}+\ldots+x_n}{n-k}\right)+f\left(\frac{x_1+x_{k+2}+\ldots+x_n}{n-k}\right)\right. \\
&\quad \left. +\ldots+f\left(\frac{x_k+x_{k+1}+\ldots+x_{n-1}}{n-k}\right)\right].
\end{aligned}$$

For instance, for $k=1$,

$$S_{1,n-1}(0,1) = \frac{1}{n-1}\begin{pmatrix} 0 & 1 & 1 & \ldots & \ldots & 1 \\ 1 & 0 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{pmatrix}$$

and (21) can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - bnf\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \qquad (21)\\
&\geqslant a\left[f\left(\frac{x_2+x_3+\ldots+x_n}{n-1}\right)+f\left(\frac{x_1+x_3+\ldots+x_n}{n-1}\right)+\ldots+f\left(\frac{x_1+x_2+\ldots+x_{n-1}}{n-1}\right)\right].
\end{aligned}$$

By putting $a=\frac{n-1}{n}$ and $b=\frac{1}{n}$ into (21), we get

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \\
&\geqslant \frac{n-1}{n}\left[f\left(\frac{x_2+x_3+\ldots+x_n}{n-1}\right)+f\left(\frac{x_1+x_3+\ldots+x_n}{n-1}\right)\right. \\
&\quad \left. +\ldots+f\left(\frac{x_1+x_2+\ldots+x_{n-1}}{n-1}\right)\right].
\end{aligned}$$

Denote $a_i = x_i$ and $b_i = \frac{x_1+\ldots+x_{i-1}+x_{i+1}+\ldots+x_n}{n-1}$. Since $b_i = \frac{\sum_{j=1}^{n} x_j - x_i}{n-1} = \frac{n\bar{x}-x_i}{n-1}$, it is readily seen that (22) reduces to inequality (2) in Theorem B.

4. Refinements of Jensen's inequality

Throughout this section $\prec$ stands for the majorization ordering on $\mathbb{R}^2$. The vector of ones in $\mathbb{R}^2$ is denoted by $\mathbf{1}$. By $\langle \cdot, \cdot \rangle$ we denote the standard inner product of $\mathbb{R}^2$, i.e.,

$$\langle \mathbf{p}, \mathbf{q} \rangle = ac + bd \quad \text{for any } \mathbf{p} = (a,b)^T \text{ and } \mathbf{q} = (c,d)^T \text{ in } \mathbb{R}^2. \tag{22}$$

A vector $\mathbf{p} = (a,b)^T \in \mathbb{R}^2$ is said to be

(i) *positive* if $a \geqslant 0$ and $b \geqslant 0$,

(ii) *increasing* (*decreasing*) if $a \leqslant b$ (resp. $a \geqslant b$),

(iii) *probabilistic* if $a \geqslant 0$, $b \geqslant 0$ and $a + b = 1$.

We are now in a position to give some refinements of Jensen's inequality (cf. [2, 3, 5]).

THEOREM 4.1. *Let f be a convex function on an interval $I \subset \mathbb{R}$ and $S = (s_{ij})$ be a doubly stochastic $n \times n$ matrix. Let $x_1, x_2, \ldots, x_n \in I$, and $(y_1, y_2, \ldots, y_n)^T = S(x_1, x_2, \ldots, x_n)^T$.*

Let $a_i, b_i, c_j, d_j \in \mathbb{R}$ for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, m$, be such that

$$1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{k-1} \geqslant a_k \geqslant \frac{1}{2}, \tag{23}$$

$$b_i = 1 - a_i \quad \text{for } i = 1, 2, \ldots, k, \tag{24}$$

$$\frac{1}{2} \geqslant c_m \geqslant c_{m-1} \geqslant \ldots \geqslant c_2 \geqslant c_1 \geqslant 0, \tag{25}$$

$$d_j = 1 - c_j \quad \text{for } j = 1, 2, \ldots, m. \tag{26}$$

Then

$$\begin{aligned}
\sum_{i=1}^n f(x_i) &\geqslant \sum_{i=1}^n f(y_i) \geqslant a_1 \sum_{i=1}^n f(y_i) + b_1 n f(\overline{y}) \geqslant a_2 \sum_{i=1}^n f(y_i) + b_2 n f(\overline{y}) \\
&\geqslant \ldots \geqslant a_{k-1} \sum_{i=1}^n f(y_i) + b_{k-1} n f(\overline{y}) \geqslant a_k \sum_{i=1}^n f(y_i) + b_k n f(\overline{y}) \\
&\geqslant c_m \sum_{i=1}^n f(y_i) + d_m n f(\overline{y}) \geqslant c_{m-1} \sum_{i=1}^n f(y_i) + d_{m-1} n f(\overline{y}) \\
&\geqslant \ldots \geqslant c_2 \sum_{i=1}^n f(y_i) + d_2 n f(\overline{y}) \geqslant c_1 \sum_{i=1}^n f(y_i) + d_1 n f(\overline{y}) \\
&\geqslant n f(\overline{y}) = n f(\overline{x}).
\end{aligned} \tag{27}$$

In consequence,

$$\begin{aligned}
\sum_{i=1}^n f(x_i) - n f(\overline{x}) &\geqslant \sum_{i=1}^n f(y_i) - n f(\overline{y}) \\
&\geqslant (a_i - c_j) \sum_{i=1}^n f(y_i) + (b_i - d_j) n f(\overline{y}) \geqslant 0
\end{aligned} \tag{28}$$

for $i=1,2,\dots,k,\ j=1,2,\dots,m.$

Proof. We set

$$A=\sum_{i=1}^{n} f(x_i)\ ,\quad B=\sum_{i=1}^{n} f(y_i)\ ,\quad C=nf(\overline{x})\ \text{ and }\ D=nf(\overline{y}), \tag{29}$$

where $\overline{x}=\frac{x_1+x_2+\ldots+x_n}{n}$ and $\overline{y}=\frac{y_1+y_2+\ldots+y_n}{n}$.

Since $\mathbf{y}\prec\mathbf{x}$, we have $A\geqslant B$ and $C=D$ by Majorization Theorem (see (3)).

For $i=1,2,\dots,k$ and $j=1,2,\dots,m$, we define vectors $\mathbf{p}_i=(a_i,b_i)^T$ and $\mathbf{q}_j=(c_j,d_j)^T$ in $\mathbb{R}^2$.

By virtue of (23)–(26) we have

$$\mathbf{p}_1\succ\mathbf{p}_2\succ\ldots\succ\mathbf{p}_{k-1}\succ\mathbf{p}_k\succ\mathbf{p}_{k+1}, \tag{30}$$

$$\mathbf{q}_1\succ\mathbf{q}_2\succ\ldots\succ\mathbf{q}_{m-1}\succ\mathbf{q}_m\succ\mathbf{q}_{m+1} \tag{31}$$

with $\mathbf{p}_{k+1}=\mathbf{q}_{m+1}=(1/2,1/2)^T$.

Denote $\mathbf{v}=(B,D)^T$. Since $B\geqslant D$ by Jensen's inequality, it follows by an easy calculation that the functional

$$\mathbf{p}\to\langle\mathbf{p},\mathbf{v}\rangle\quad\text{for }\mathbf{p}\in\mathbb{R}^2$$

is Schur-convex on the set of decreasing vectors. Likewise, the functional

$$\mathbf{q}\to\langle\mathbf{q},\mathbf{v}\rangle\quad\text{for }\mathbf{q}\in\mathbb{R}^2$$

is Schur-concave on the set of increasing vectors.

Therefore from (30)–(31) we get

$$\begin{aligned}&\langle\mathbf{p}_1,\mathbf{v}\rangle\geqslant\langle\mathbf{p}_2,\mathbf{v}\rangle\geqslant\ldots\geqslant\langle\mathbf{p}_{k-1},\mathbf{v}\rangle\geqslant\langle\mathbf{p}_k,\mathbf{v}\rangle\geqslant\langle\mathbf{p}_{k+1},\mathbf{v}\rangle\\&\geqslant\langle\mathbf{q}_{m+1},\mathbf{v}\rangle\geqslant\langle\mathbf{q}_m,\mathbf{v}\rangle\geqslant\langle\mathbf{q}_{m-1},\mathbf{v}\rangle\geqslant\ldots\geqslant\langle\mathbf{q}_2,\mathbf{v}\rangle\geqslant\langle\mathbf{q}_1,\mathbf{v}\rangle.\end{aligned} \tag{32}$$

Because $B\geqslant D$ we find that

$$\mathbf{v}\leqslant B\mathbf{1}\quad\text{and}\quad D\mathbf{1}\leqslant\mathbf{v},$$

where $\leqslant$ denotes the standard componentwise ordering on $\mathbb{R}^2$. Furthermore, the functionals $\mathbf{z}\to\langle\mathbf{p}_1,\mathbf{z}\rangle$ and $\mathbf{z}\to\langle\mathbf{q}_1,\mathbf{z}\rangle$ are $\leqslant$-increasing, since $\mathbf{p}_1$ and $\mathbf{q}_1$ are positive. In addition, $\mathbf{p}_1$ and $\mathbf{q}_1$ are probabilistic. For this reason, we conclude that

$$B=\langle\mathbf{p}_1,B\mathbf{1}\rangle\geqslant\langle\mathbf{p}_1,\mathbf{v}\rangle\quad\text{and}\quad\langle\mathbf{q}_1,\mathbf{v}\rangle\geqslant\langle\mathbf{q}_1,D\mathbf{1}\rangle=D. \tag{33}$$

So, by combining (32) and (33) we obtain (27) via (22) and (29).

Finally, inequality (28) is an easy consequence of (27). This completes the proof. □

We now use Theorem 4.1 for the following decreasing vectors $\mathbf{p}_i = (a_i, b_i)^T$ and increasing vectors $\mathbf{q}_j = (c_j, d_j)^T$, where

$$a_i = \frac{n+k-1-i}{n+k-i} \quad \text{and} \quad b_i = \frac{1}{n+k-i} \quad \text{for } i = 1,2,\ldots,k, \tag{34}$$

$$c_j = \frac{1}{m+3-j} \quad \text{and} \quad d_j = \frac{m+2-j}{m+3-j} \quad \text{for } j = 1,2,\ldots,m. \tag{35}$$

It follows that (23)–(26) hold for a_i, b_i, c_j and d_j defined by (34) and (35). Therefore we obtain the following extension of [7, Theorems 2.1 and 2.2].

COROLLARY 4.2. *Let f be a convex function on an interval $I \subset \mathbb{R}$ and $S = (s_{ij})$ be a doubly stochastic $n \times n$ matrix. Let $x_1, x_2, \ldots, x_n \in I$ and $(y_1, y_2, \ldots, y_n)^T = S(x_1, x_2, \ldots, x_n)^T$.*

Then

$$\begin{aligned}
\sum_{i=1}^n f(x_i) \geqslant \sum_{i=1}^n f(y_i) &\geqslant \frac{n+k-2}{n+k-1} \sum_{i=1}^n f(y_i) + \frac{1}{n+k-1} n f(\overline{y}) \\
&\geqslant \frac{n+k-3}{n+k-2} \sum_{i=1}^n f(y_i) + \frac{1}{n+k-2} n f(\overline{y}) \\
&\geqslant \ldots \geqslant \frac{n}{n+1} \sum_{i=1}^n f(y_i) + \frac{1}{n+1} n f(\overline{y}) \\
&\geqslant \frac{n-1}{n} \sum_{i=1}^n f(y_i) + \frac{1}{n} n f(\overline{y}) \\
&\geqslant \frac{1}{2} \sum_{i=1}^n f(y_i) + \frac{1}{2} n f(\overline{y}) \\
&\geqslant \frac{1}{3} \sum_{i=1}^n f(y_i) + \frac{2}{3} n f(\overline{y}) \\
&\geqslant \ldots \geqslant \frac{1}{m+1} \sum_{i=1}^n f(y_i) + \frac{m}{m+1} n f(\overline{y}) \\
&\geqslant \frac{1}{m+2} \sum_{i=1}^n f(y_i) + \frac{m+1}{m+2} n f(\overline{y}) \\
&\geqslant n f(\overline{y}) = n f(\overline{x}).
\end{aligned} \tag{36}$$

In consequence,

$$\begin{aligned}
&\sum_{i=1}^n f(x_i) - n f(\overline{x}) \geqslant \sum_{i=1}^n f(y_i) - n f(\overline{y}) \\
&\geqslant \left| \left(\frac{n+k-3}{n+k-2} - \frac{1}{m+1} \right) \sum_{i=1}^n f(y_i) - \left(\frac{m}{m+1} - \frac{1}{n+k-2} \right) n f(\overline{y}) \right|.
\end{aligned} \tag{37}$$

In order to see (36), it is sufficient to apply Theorem 4.1 with (34)–(35).
In addition, (37) follows easily from (36).

REMARK 4.3. In [4] there is another extension of [7, Theorems 2.1]. In addition, a similar method to that used above is applied in [4, Example 3].

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