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NOTE ON KATO'S TYPE INEQUALITY

FLAVIUS LUCIAN PATER

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Abstract. The main purpose of this note is to prove a Kato's type inequality for a generalized Schrödinger operator $\mathscr{A}$. As an application, we present the L^∞-uniqueness of $\mathscr{A}$.

1. Framework and main result

Kato's inequality in its classical form say the following (see [3, Lemma A, p. 138]): if $u \in L^1_{loc}(\mathbb{R}^d)$ is such that the distributional Laplacian satisfies $\Delta u \in L^1_{loc}(\mathbb{R}^d)$, then the inequality

$$\Delta|u| \geqslant \mathrm{sign}(u)\Delta u$$

holds in the sense of distributions, i.e.

$$\langle \xi, \Delta|u|\rangle \geqslant \langle \xi, \mathrm{sign}(u)\Delta u\rangle$$

holds for all $0 \leqslant \xi \in C_0^\infty(\mathbb{R}^d)$, the space of all infinitely differentiable functions with compact support.

A few years later, Simon [15] realized that the Kato inequality for Δ is related to positivity of the semigroup generated by Δ in $L^2(\mathbb{R}^d)$. An abstract Kato inequality originates from the distributional inequality and it was established by Arendt [1] and by Schep [14] that the abstract Kato inequality (when properly formulated) for a generator $\mathscr{A}$ together with an additional condition is equivalent to the positivity of C_0-semigroup generated by $\mathscr{A}$.

Let ϕ be a continuous strictly positive function on $\mathbb{R}^d$, $d \geqslant 1$, such that $\nabla\phi \in L^2_{loc}(\mathbb{R}^d, dx)$ in the Schwartz distribution sense and denote by $\cdot$ the iner product in $\mathbb{R}^d$ and by $|x| = \sqrt{x \cdot x}$ the euclidian norm. Consider the generalized Schrödinger operator

$$\mathscr{A} := \frac{\Delta}{2} + \frac{\nabla\phi}{\phi} \cdot \nabla$$

with domain $C_0^\infty(\mathbb{R}^d)$, the space of infinitely differentiable functions with compact support in $\mathbb{R}^d$. This operator has been used to describe the random path of mouvement of a quantum system in the equilibrium measure $\mu_\phi := \phi^2 dx$ (see [11]).

The main result of this paper is

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THEOREM 1.1. *Let $\lambda > 0$ large enough and $u \in L^1(\mathbb{R}^d, \mu_\phi)$ such that*

$$\langle u, (\mathscr{A} - \lambda I) f \rangle_{\mu_\phi} = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d)$$

where

$$\langle f, g \rangle_{\mu_\phi} := \int_{\mathbb{R}^d} f g \, \phi^2 dx.$$

Then $\phi^2 u \in H^{1,2}_{loc}(\mathbb{R}^d, dx) \cap L^\infty_{loc}(\mathbb{R}^d, dx)$ and we have

$$-\frac{1}{2} \int_{\mathbb{R}^d} \nabla \xi \cdot \nabla(\phi^2 |u|) \, dx + \int_{\mathbb{R}^d} \frac{\nabla \phi}{\phi} \cdot \nabla \xi \, \phi^2 |u| \, dx \geqslant \int_{\mathbb{R}^d} \lambda \xi \phi^2 |u| \, dx \tag{1}$$

for all positive compactly supported functions $\xi \in H^{1,2}(\mathbb{R}^d, dx)$.

Proof. Let $\lambda > 0$ large enough and $u \in L^1(\mathbb{R}^d, \mu_\phi)$ such that

$$\langle u, (\mathscr{A} - \lambda I) f \rangle_{\mu_\phi} = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Then

$$\frac{1}{2} \int_{\mathbb{R}^d} \Delta f \, \phi^2 u dx + \int_{\mathbb{R}^d} \frac{\nabla \phi}{\phi} \cdot \nabla f \, \phi^2 u dx = \int_{\mathbb{R}^d} \lambda f \, \phi^2 u dx, \qquad \forall f \in C_0^\infty(\mathbb{R}^d).$$

By the ellipticity regularity result in [2, Lemma 2, p. 341], $\phi^2 u \in H^{1,2}_{loc}(\mathbb{R}^d, dx) \cap L^\infty_{loc}(\mathbb{R}^d, dx)$ and an integration by parts yields

$$-\frac{1}{2} \int_{\mathbb{R}^d} \nabla f \cdot \nabla(\phi^2 u) \, dx + \int_{\mathbb{R}^d} \frac{\nabla \phi}{\phi} \cdot \nabla f \phi^2 u \, dx = \int_{\mathbb{R}^d} \lambda f \phi^2 u \, dx \tag{2}$$

for all $f \in H^{1,2}(\mathbb{R}^d, dx)$ with compact support.

Now we can follow [2]. Since $\phi^2 u \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$, $\phi^2 |u| \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$ as well. For $\varepsilon > 0$, let $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be given by

$$\psi_\varepsilon(x) = \begin{cases} \operatorname{sgn}(x), & \text{if } |x| > \varepsilon \\ \frac{x}{\varepsilon}, & \text{if } |x| \leqslant \varepsilon. \end{cases}$$

Obviously, ψ_ε is Lipschitz continuous, whence $\psi_\varepsilon(\phi^2 u) \in H^{1,2}_{loc}(\mathbb{R}^d, dx)$ and

$$\nabla \psi_\varepsilon(\phi^2 u) = \psi_\varepsilon'(\phi^2 u) \cdot \nabla(\phi^2 u) = \frac{1}{\varepsilon} \chi_{\{\phi^2 |u| \leqslant \varepsilon\}} \nabla(\phi^2 u) \quad \text{dx -a.e.}$$

Fix a positive function $\xi \in C_0^\infty(\mathbb{R}^d)$. Setting $f := \xi \psi_\varepsilon(\phi^2 u)$ in (2), we obtain

$$-\frac{1}{2} \int_{\mathbb{R}^d} \nabla(\xi \psi_\varepsilon(\phi^2 u)) \cdot \nabla(\phi^2 u) \, dx + \int_{\mathbb{R}^d} \frac{\nabla \phi}{\phi} \cdot \nabla(\xi \psi_\varepsilon(\phi^2 u)) \phi^2 u \, dx$$
$$= \int_{\mathbb{R}^d} \lambda \xi \psi_\varepsilon(\phi^2 u) \phi^2 u \, dx$$

so that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \lambda \xi \, \psi_\varepsilon(\phi^2 u)\phi^2 u \, dx \\
&= -\frac{1}{2} \int_{\{\phi^2|u|>\varepsilon\}} \nabla\xi \cdot \nabla(\phi^2 u) \, \operatorname{sgn}(\phi^2 u) \, dx - \frac{1}{2\varepsilon} \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \\
&\qquad + \int_{\{\phi^2|u|>\varepsilon\}} \frac{\nabla\phi}{\phi} \cdot \nabla\xi \, \operatorname{sgn}(\phi^2 u)\phi^2 u \, dx + \frac{1}{\varepsilon} \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi \frac{\nabla\phi}{\phi} \cdot \nabla(\phi^2 u)\phi^2 u \, dx.
\end{aligned}
$$

To estimate the right hand side of this equality, note that

$$
\int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi \frac{\nabla\phi}{\phi} \cdot \nabla(\phi^2 u)\phi^2 u \, dx \leqslant (C_\varepsilon)^{\frac{1}{2}} \left(\int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \right)^{\frac{1}{2}},
$$

where

$$
C_\varepsilon := \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi \left| \frac{\nabla\phi}{\phi} \right|^2 \phi^4 u^2 \, dx \leqslant \varepsilon \left\| \chi_{\{\phi^2|u|\leqslant\varepsilon\}} \xi \left| \frac{\nabla\phi}{\phi} \right|^2 \right\|_{L^\infty(\mathbb{R}^d, d\mu_\phi)} \|u\|_{L^1(\mathbb{R}^d, d\mu_\phi)}.
$$

Remark that $\varepsilon^{-1} C_\varepsilon$ converge to 0 as ε tends to 0. We obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi \frac{\nabla\phi}{\phi} \cdot \nabla(\phi^2 u)\phi^2 u \, dx - \frac{1}{2\varepsilon} \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \\
&\leqslant \frac{1}{\varepsilon} (C_\varepsilon)^{\frac{1}{2}} \left(\int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \right)^{\frac{1}{2}} - \frac{1}{2\varepsilon} \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \\
&= \frac{1}{2\varepsilon} \left[2(C_\varepsilon)^{\frac{1}{2}} \left(\int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \right)^{\frac{1}{2}} - \int_{\{\phi^2|u|\leqslant\varepsilon\}} \xi |\nabla(\phi^2 u)|^2 \, dx \right] \\
&\leqslant \frac{1}{2\varepsilon} C_\varepsilon \longrightarrow 0 \text{ if } \varepsilon \to 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \lambda \xi \, \psi_\varepsilon(\phi^2 u)\phi^2 u \, dx \\
&\leqslant -\frac{1}{2} \int_{\{\phi^2 u>\varepsilon\}} \nabla\xi \cdot \nabla(\phi^2 u) \, \operatorname{sgn}(\phi^2 u) \, dx + \int_{\{\phi^2 u>\varepsilon\}} \frac{\nabla\phi}{\phi} \cdot \nabla\xi \, \operatorname{sgn}(\phi^2 u)\phi^2 u \, dx + \frac{1}{2\varepsilon} C_\varepsilon.
\end{aligned}
$$

Now, by letting $\varepsilon \to 0$ in this inequality, we obtain (3), because

$$\lim_{\varepsilon\to 0} \int\limits_{\{\phi^2|u|>\varepsilon\}} \nabla\xi\cdot\nabla(\phi^2 u)\,\mathrm{sgn}(\phi^2 u)\,dx = \int\limits_{\mathbb{R}^d} \nabla\xi\cdot\nabla(\phi^2|u|)\,dx$$

by dominated convergence, and the other integrals in the above inequality converge similarly. Hence (3) holds for all positive functions $\xi \in C_0^\infty(\mathbb{R}^d)$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in

$$\left\{ f \in H^{1,2}(\mathbb{R}^d, dx) \,\middle|\, \text{support of } f \text{ is compact} \right\},$$

it follows that (3) is true for all positive compactly supported functions $\xi \in H^{1,2}(\mathbb{R}^d, dx)$. □

A similar result was announced in [10]. As we can see in next section, Kato's inequality play a key rolle in the study of uniqueness problems.

2. Application

The uniqueness of generalized Schrödinger operator

$$\mathscr{A} f := \frac{\Delta}{2} f + \frac{\nabla\phi}{\phi}\cdot\nabla f \quad \forall f \in C_0^\infty(\mathbb{R}^d)$$

in $L^2(\mathbb{R}^d, \mu_\phi)$ is defined as the *essential self-adjointness* of $\mathscr{A}$, i.e. the closure of $\mathscr{A}$ in $L^2(\mathbb{R}^d, \mu_\phi)$ coincides with the generator $\mathscr{L}^\phi_{(2)}$ of the C_0-semigroup $\left\{P_t^\phi\right\}_{t\geqslant 0}$ given by

$$P_t^\phi f(x) := \mathbb{E}^{\mathbb{P}_x} f(W_t) \exp\left\{ \int_0^t \frac{\nabla\phi}{\phi}(W_s)dW_s - \frac{1}{2}\int_0^t \left|\frac{\nabla\phi}{\phi}\right|^2 (W_s)ds \right\}$$

where $(W_t)_{t\geqslant 0}$ is the standard Brownian Motion in $\mathbb{R}^d$ defined on a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d}\right)$ with $\mathbb{P}_x(W_0 = x) = 1$ for any initial point $x \in \mathbb{R}^d$.

In the classical situation where $p \in [1,\infty)$, the $L^p\left(\mathbb{R}^d, \mu_\phi\right)$-uniqueness of $\mathscr{A}$ is defined as following: the closure of $\mathscr{A}$ in $L^p\left(\mathbb{R}^d, \mu_\phi\right)$ coincides with the generator $\mathscr{L}^\phi_{(p)}$ of $\left\{P_t^\phi\right\}_{t\geqslant 0}$. For example, the essential self-adjointness of $\mathscr{A}$ was studied by Wielens [16] and the $\left(L^1(\mathbb{R}^d, \mu_\phi), \|\cdot\|_1\right)$-uniqueness of $\mathscr{A}$ has been studied by Wu [18].

Following [19], consider on $L^\infty(\mathbb{R}^d, \mu_\phi)$ *the topology of uniform convergence on compact subsets of* $(L^1(\mathbb{R}^d, \mu_\phi), \|\,.\,\|_1)$, denoted by $\mathscr{C}(L^\infty, L^1)$.

Recall that $\mathscr{A}$ is said to be a *pre-generator* in $L^\infty(\mathbb{R}^d, \mu_\phi)$, if there exists some C_0-semigroup on $\left(L^\infty(\mathbb{R}^d, \mu_\phi), \mathscr{C}\left(L^\infty, L^1\right)\right)$ such that its generator $\mathscr{L}$ extends $\mathscr{A}$.

Moreover, we say that $\mathscr{A}$ is $\left(L^\infty(\mathbb{R}^d, \mu_\phi), \mathscr{C}\left(L^\infty, L^1\right)\right)$*-unique*, if $\mathscr{A}$ is closable and its closure $\overline{\mathscr{A}}$ with respect to the topology $\mathscr{C}\left(L^\infty, L^1\right)$ is the generator of some C_0-semigroup on $\left(L^\infty(\mathbb{R}^d, \mu_\phi), \mathscr{C}\left(L^\infty, L^1\right)\right)$.

This uniqueness notion has been used by Röckner and Zhang [12], [13], Wu [17], Wu and Zhang [19], Lemle [4], [5], [6], [7], Lemle and Wu [8], [9] and others in different contexts. In a more general setting, the main result concerning uniqueness of pre-generators can be find in [19, Theorem 2.1, p. 570] or [9, Theorem 1.1, p. 486].

Our goal in this section is to prove the $\left(L^\infty(\mathbb{R}^d,\mu_\phi),\mathscr{C}\left(L^\infty,L^1\right)\right)$-uniqueness of the operator $(\mathscr{A},C_0^\infty(\mathbb{R}^d))$.

For this purpose, first we must remark that the operator $(\mathscr{A},C_0^\infty(\mathbb{R}^d))$ is a pre-generator in $\left(L^\infty(\mathbb{R}^d,\mu_\phi),\mathscr{C}\left(L^\infty,L^1\right)\right)$. Indeed, by [18, Lemma 1.2, p. 556] we can see that $\left\{P_t^\phi\right\}_{t\geqslant 0}$ is a C_0-semigroup in $\left(L^1(\mathbb{R}^d,\mu_\phi),\|\cdot\|_1\right)$ and its generator $\mathscr{L}_{(1)}^\phi$ extends $\mathscr{A}$. By [19, Theorem 1.4, p. 564] it follows that $\left\{P_t^\phi\right\}_{t\geqslant 0}$ is a C_0-semigroup in $\left(L^\infty(\mathbb{R}^d,\mu_\phi),\mathscr{C}\left(L^\infty,L^1\right)\right)$ and its generator $\mathscr{L}_{(\infty)}^\phi=\left(\mathscr{L}_{(1)}^\phi\right)^*$ extends the operator $\mathscr{A}$ meaning that $\mathscr{A}$ is a pre-generator on $\left(L^\infty(\mathbb{R}^d,\mu_\phi),\mathscr{C}\left(L^\infty,L^1\right)\right)$.

Finaly, by [19, Theorem 2.1, p. 570], for the $L^\infty(\mathbb{R}^d,\mu_\phi)$-uniqueness of $\mathscr{A}$ it is enough to show that if for some $\lambda>0$, $u\in L^1(\mathbb{R}^d,\mu_\phi)$ satisfies

$$((\mathscr{A})^*-\lambda I)u=0$$

in the sense of distributions, then $u=0$.

Let $\lambda>0$ large enough and $u\in L^1(\mathbb{R}^d,\mu_\phi)$ such that

$$\langle u,(\mathscr{A}-\lambda I)f\rangle_{\mu_\phi}=0, \qquad \forall f\in C_0^\infty(\mathbb{R}^d).$$

By Theorem 1.1, it follows $\phi^2u\in H_{loc}^{1,2}(\mathbb{R}^d,dx)\cap L_{loc}^\infty(\mathbb{R}^d,dx)$ and we have

$$-\frac{1}{2}\int_{\mathbb{R}^d}\nabla\xi\cdot\nabla(\phi^2|u|)\,dx+\int_{\mathbb{R}^d}\frac{\nabla\phi}{\phi}\cdot\nabla\xi\,\phi^2|u|\,dx\geqslant\int_{\mathbb{R}^d}\lambda\xi\phi^2|u|\,dx \tag{3}$$

for all positive compactly supported functions $\xi\in H^{1,2}(\mathbb{R}^d,dx)$. But Kato's inequality (3) is equivalent to

$$\int_{\mathbb{R}^d}|u|\left(\frac{\Delta}{2}\xi+\frac{\nabla\phi}{\phi}\cdot\nabla\xi\right)d\mu_\phi\geqslant\int_{\mathbb{R}^d}|u|\lambda\xi\,d\mu_\phi\geqslant 0.$$

This means that $|u|$ is an $\mathscr{A}$-subharmonic function. By [19, Proposition 6.4 (i), p. 607], it follows that $|u|$ must be a constant function and then $u=0$.

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Flavius Lucian Pater
Department of Mathematics
"Politehnica" University of Timişoara
300006 Timişoara, Romania
e-mail: flavius.pater@mat.upt.ro