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ON THE DIRICHLET PROBLEM FOR THE GENERALIZED n–LAPLACIAN: SINGULAR NONLINEARITY WITH THE EXPONENTIAL AND MULTIPLE EXPONENTIAL CRITICAL GROWTH RANGE

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Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain containing the origin. Applying the Mountain Pass Theorem and a singular version of the generalized Moser-Trudinger inequality we prove the existence of a non-trivial weak solution to the problem

$$u \in W_0^1 L^\Phi(\Omega) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = \frac{f(x,u)}{|x|^a} \quad \text{in } \Omega\,,$$

where $a \in [0,n)$, Φ is a Young function such that the space $W_0^1 L^\Phi(\Omega)$ is embedded into exponential or multiple exponential Orlicz space and $f(x,t)$ has the corresponding critical growth.

1. Introduction

Throughout the paper Ω is a bounded domain containing the origin in $\mathbb{R}^n$, $n \geqslant 2$, and ω_{n-1} denotes the surface of the unit sphere.

It is an often studied problem to find solutions to the Laplace equation

$$u \in W_0^{1,2}(\Omega) \qquad \text{and} \qquad -\Delta u = f(x,u) \quad \text{in } \Omega\,. \tag{1.1}$$

For $n \geqslant 3$ and f satisfying $\lim_{t\to\infty} \frac{f(x,t)}{t^q} = 0$ uniformly on Ω with $q < \frac{n+2}{n-2}$, there are many results using the compactness of the embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r \in [1, \frac{2n}{n-2})$ (see a review article by Lions [17] and the references given there). Problem (1.1) under condition $\lim_{t\to\infty} \frac{f(x,t)}{t^{\frac{n+2}{n-2}}} = 0$ becomes much more difficult thanks to the fact that the embedding of $W_0^{1,2}(\Omega)$ into $L^{\frac{2n}{n-2}}(\Omega)$ is no longer compact. This difficulty has been overcame by Brezis and Nirenberg [5]. Their method uses the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

When $n=2$, we do not only have the Sobolev embedding into $L^r(\Omega)$ for any $r \in [1,\infty)$ but there is also the Trudinger embedding [21] into the Orlicz space $\exp L^{\frac{n}{n-1}}(\Omega)$. In particular, there is so called Moser-Trudinger inequality by Moser [18]

$$\sup_{||u||_{W_0^{1,n}(\Omega)} \leqslant 1} \int_\Omega \exp(K|u|^{\frac{n}{n-1}}) \leqslant C(n, \mathscr{L}_n(\Omega)) \qquad \text{if and only if} \qquad K \leqslant n\omega_{n-1}^{\frac{1}{n-1}}\,.$$

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Therefore, in the literature, there is often used the variational approach by Brezis and Nirenberg [5] together with the Moser-Trudinger inequality to study the n-Laplace equation

$$u \in W_0^{1,n}(\Omega) \qquad \text{and} \qquad -\Delta_n u = f(x,u) \quad \text{in } \Omega\,, \tag{1.2}$$

where $\Delta_n u := \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ and $f(x,t) \approx \exp(b|t|^{\frac{n}{n-1}})$ for some $b > 0$. See for example Adimurthi [1], de Figueiredo, Miyagaki, Ruf [13] and do Ó [19].

In recent paper [8], above techniques are modified for a differential equation corresponding to the embedding of the Orlicz-Sobolev space $W_0L^n\log^\alpha L(\Omega)$, $\alpha < n-1$, into the Orlicz space $\exp L^{\frac{n}{n-1-\alpha}}(\Omega)$ (this embedding is due to Fusco, Lions, Sbordone [14] and Edmunds, Gurka, Opic [10]). The result is the existence of a non-trivial weak solution to the equation

$$u \in W_0L^\Phi(\Omega) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = f(x,u) \quad \text{in } \Omega,$$

with Φ being a Young function that behaves like $t^n\log^\alpha(t)$, $\alpha < n-1$, for large t and with the nonlinearity f having so called critical growth (corresponding to the choice of the Young function Φ).

The aim of this paper is to generalize above result in two ways. First, instead of considering a Young function corresponding to the embedding into exponential space we also deal with Young functions for which we have an embedding into multiple exponential spaces. Second, similarly as Adimurthi and Sandeep [2], we deal with the nonlinearity of the singular form $\frac{f(x,u)}{|x|^a}$, $a \in [0,n)$ (notice that we admit $a = 0$, hence our results cover the case with no singular weight on the right hand side). Our differential equation is then

$$u \in W_0L^\Phi(\Omega) \qquad \text{and} \qquad -\operatorname{div}\Big(\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = \frac{f(x,u)}{|x|^a} \quad \text{in } \Omega\,, \tag{1.3}$$

with Φ and f specified below.

On embedding into exponential and multiple exponential spaces

If $\ell \in \mathbb{N}$ and $\alpha < n-1$, we set

$$\gamma = \frac{n}{n-1-\alpha} > 0\,, \qquad B = 1 - \frac{\alpha}{n-1} = \frac{n}{(n-1)\gamma} > 0$$
$$\text{and} \qquad K_{\ell,n,\alpha} = \begin{cases} B^{\frac{1}{B}} n\omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell = 1 \\ B^{\frac{1}{B}} \omega_{n-1}^{\frac{\gamma}{n}} & \text{for } \ell \geqslant 2\,. \end{cases} \tag{1.4}$$

The space $W_0L^n\log^\alpha L(\Omega)$ of the Sobolev type, modeled on the Zygmund space $L^n\log^\alpha L(\Omega)$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(t^\gamma)$ for large t (see [14] and [10]). Moreover it is shown in [10] (see also [11]) that in the limiting case $\alpha = n-1$ we have the embedding into a double exponential space, i.e. the space $W_0L^n\log^{n-1}L\log^\alpha\log L(\Omega)$, $\alpha < n-1$, is continuously embedded into the Orlicz space with the Young function that behaves like $\exp(\exp(t^\gamma))$ for large t. Further in the limiting case $\alpha = n-1$ we have the embedding into triple exponential space and so on. The borderline case is always $\alpha = n-1$ and for $\alpha > n-1$ we have embedding into $L^\infty(\Omega)$. It is well-known that the Zygmund space $L^n \log^\alpha L(\Omega)$ coincides with the Orlicz space $L^\Phi(\Omega)$, where the Young function Φ satisfies

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^\alpha(t)} = 1\ ,$$

the space $L^n \log^{n-1} L \log^\alpha \log L(\Omega)$ coincides with $L^\Phi(\Omega)$ where

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log^{n-1}(t) \log^\alpha(\log(t))} = 1\ ,$$

and so on. For other results concerning these spaces we refer the reader to [10], [11] and [12].

The following notation enables us to work with the multiple exponential spaces comfortably. For $k \in \mathbb{N}$, let us write

$$\log_{[k]}(t) = \log(\log_{[k-1]}(t)), \qquad \text{where} \qquad \log_{[1]}(t) = \log(t)$$

and

$$\exp_{[k]}(t) = \exp(\exp_{[k-1]}(t)), \qquad \text{where} \qquad \exp_{[1]}(t) = \exp(t)\ .$$

Let $\ell \in \mathbb{N}$ and $\alpha < n-1$. Then we have above mentioned embedding results for any Young function Φ satisfying

$$\lim_{t\to\infty} \frac{\Phi(t)}{t^n \left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t)\right) \log_{[\ell]}^\alpha(t)} = 1 \tag{1.5}$$

(for $\ell = 1$ we read (1.5) as $\lim_{t\to\infty} \frac{\Phi(t)}{t^n \log_{[1]}^\alpha(t)} = 1$). As Ω is bounded, all Young functions satisfying (1.5) give the same Orlicz-Sobolev space.

Assumptions on Φ and f

We suppose that the function $\Phi : [0,\infty) \mapsto [0,\infty)$ is a C^1-Young function satisfying (1.5) and in addition we suppose that there are $C > 0$, $t_\Phi \geqslant 1$ and $\beta \in (0, \min(1,B))$ such that

$$\frac{1}{C} t^n \leqslant \Phi(t) \leqslant C t^n \qquad \text{for } t \in \left[0, \frac{1}{C}\right) \tag{1.6}$$

and

$$\Phi(t) \leqslant t^n \left(\prod_{j=1}^{\ell-1} \log_{[j]}^{n-1}(t)\right) \log_{[\ell]}^\alpha(t) \left(1 - \log_{[\ell]}^{-\beta}(t)\right) \qquad \text{for } t \in [t_\Phi, \infty)\ . \tag{1.7}$$

Notice that assumptions (1.5) and (1.6) together with the fact that Φ is a C^1-Young function imply the existence of $c_\Phi > 0$ such that

$$c_\Phi \Phi'(t) t \leqslant \Phi(t), \quad t > 0\ . \tag{1.8}$$

The function $f:\Omega\times\mathbb{R}\mapsto\mathbb{R}$ is supposed to satisfy the following conditions: There are $M>1$, $t_M>0$, $C_b>0$, $b>0$ and $\varepsilon_0>0$ such that

$$\begin{aligned}&f \text{ is uniformly continuous on } \Omega\times[-t_0,t_0] \text{ for every } t_0>0\,,\\ &f(x,0)=0 \quad\text{and}\quad tf(x,t)>0 \qquad \text{for all } x\in\Omega \text{ and } t\neq 0\,,\end{aligned} \tag{1.9}$$

$$0<F(x,t):=\int_0^t f(x,s)\,ds\leqslant M|t|^{1-\frac{1}{M}}|f(x,t)| \quad \text{provided } |t|>t_M \text{ and } x\in\Omega\,, \tag{1.10}$$

$$|f(x,t)|\leqslant C_b\exp_{[\ell]}(b\,|t|^{\gamma}) \quad \text{whenever } t\in\mathbb{R} \text{ and } x\in\Omega\,, \tag{1.11}$$

$$\limsup_{t\to 0}\frac{F(x,t)}{t^{n+\varepsilon_0}}<\infty \quad \text{uniformly on } \Omega\,, \tag{1.12}$$

and finally

$$\liminf_{t\to\infty}\frac{tf(x,t)}{\exp_{[\ell]}(b|t|^{\gamma})}>0 \qquad \text{uniformly on } \Omega\,. \tag{1.13}$$

The main result of this paper is the following theorem.

THEOREM 1.1. *Let $\ell\in\mathbb{N}$, $n\geqslant 2$, $\alpha<n-1$, $a\in[0,n)$ and let $\Omega\subset\mathbb{R}^n$ be a bounded domain containing the origin. Suppose that the C^1-Young function $\Phi:[0,\infty)\mapsto[0,\infty)$ satisfies* (1.5), (1.6) *and* (1.7). *Let $f:\Omega\times\mathbb{R}\mapsto\mathbb{R}$ be a function satisfying* (1.9), (1.10), (1.11), (1.12) *and* (1.13). *Then problem* (1.3) *has a non-trivial weak solution.*

Our approach is similar to [8], though the results related to the Moser-Trudinger inequality from [15] are replaced by those from [9] and [7]. For the convenience of the reader acquainted with [8] we organize the paper in the same way as [8] and we also use a similar notation.

Variational formulation

We define

$$J(u)=\int_\Omega\Phi\big(|\nabla u(x)|\big)\,dx-\int_\Omega\frac{F(x,u(x))}{|x|^a}\,dx\,, \qquad u\in W_0L^{\Phi}(\Omega)\,. \tag{1.14}$$

By Proposition 6.1 below, this is a C^1-functional on $W_0L^{\Phi}(\Omega)$ and its Fréchet derivative is

$$\langle J'(u),v\rangle=\int_\Omega\Phi'\big(|\nabla u(x)|\big)\frac{\nabla u(x)}{|\nabla u(x)|}\cdot\nabla v(x)\,dx-\int_\Omega\frac{f(x,u(x))\,v(x)}{|x|^a}\,dx\,, \tag{1.15}$$

$u,v\in W_0L^{\Phi}(\Omega)$, where the symbol $\langle J'(u),v\rangle$ denotes the value of the linear functional $J'(u)$ of v.

We say that $u\in W_0L^{\Phi}(\Omega)$ is a weak solution to problem (1.3) if

$$\langle J'(u),v\rangle=0 \qquad \text{for every } v\in W_0L^{\Phi}(\Omega)\,. \tag{1.16}$$

The paper is organized as follows. After preliminaries we state several versions of the generalized Moser-Trudinger inequality in the third section. In the fourth section, we verify the assumptions of the Mountain Pass Theorem (similarly as in [5] we use a version without the Palais-Smale condition). The fifth section is devoted to properties of the Palais-Smale sequence and then we apply the Mountain Pass Theorem in the proof of Theorem 1.1. Finally, we prove that the functional J is a C^1-functional in Section 6. In the last section, we give a remark concerning the sub-critical growth range (in the sense of [13]).

2. Preliminaries

The n-dimensional Lebesgue measure is denoted by $\mathscr{L}_n$. By $B(x_0,R)$ we denote an open Euclidean ball in $\mathbb{R}^n$ centered x_0 with the radius $R>0$. If $x_0=0$ we simply write $B(R)$.

For two functions $g,h: I\mapsto[0,\infty)$ we write $g\lesssim h$ on I, if there is $C>0$ such that $g(t)\leqslant Ch(t)$ for every $t\in I$. If $g\lesssim h$ on I and $h\lesssim g$ on I, we write $g\approx h$ on I. If $I=[0,\infty)$, we simply write $g\approx h$, etc. We write that $g<<h$ for t large, if $\lim_{t\to\infty}\frac{g(t)}{h(t)}=0$. If u is a measurable function on A, then by $u=0$ (or $u\neq 0$) we mean that u is equal (or not equal) to the zero function a.e. on A.

By C we denote a generic positive constant which may depend on ℓ, n, α, a, $\mathscr{L}_n(\Omega)$ and Φ. This constant may vary from expression to expression as usual.

By $\mathscr{M}(A)$ we denote the set of all Radon measures on a compact set A. We write that $\mu_k\overset{*}{\rightharpoonup}\mu$ in $\mathscr{M}(A)$ if $\int_A\psi\,d\mu_k\to\int_A\psi\,d\mu$ for every $\psi\in C(A)$.

We need the following property of the function $\exp_{[\ell]}$, $\ell\in\mathbb{N}$.

For every $p\in[1,\infty)$ and $\ell\in\mathbb{N}$ it can be easily proved that

$$\exp^p_{[\ell]}(t)\leqslant C\exp_{[\ell]}(pt)\qquad\text{on }[0,\infty)\,. \tag{2.1}$$

Young functions and Orlicz spaces.

A function $\Phi:[0,\infty)\to[0,\infty)$ is a Young function if Φ is increasing, convex, $\Phi(0)=0$ and $\lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty$.

Denote by $L^\Phi(A,d\mu)$ the Orlicz space corresponding to a Young function Φ on a set A with a measure μ. If $\mu=\mathscr{L}_n$ we simply write $L^\Phi(A)$. The space $L^\Phi(A,d\mu)$ is equipped with the Luxemburg norm

$$||u||_{L^\Phi(A,d\mu)}=\inf\Big\{\lambda>0:\int_A\Phi\Big(\frac{|u(x)|}{\lambda}\Big)\,d\mu(x)\leqslant 1\Big\}\,. \tag{2.2}$$

Given a differentiable Young function Φ we can define the generalized inverse function to $\phi(y)=\Phi'(y)$ by

$$\psi(s)=\inf\{y:\phi(y)>s\}\qquad\text{for}\quad s>0$$

and further we define the associated Young function Ψ by

$$\Psi(t)=\int_0^t\psi(s)\,ds\qquad\text{for}\quad t\geqslant 0\,.$$

The dual space to $L^{\Phi}(A,d\mu)$ can be identified as the Orlicz space $L^{\Psi}(A,d\mu)$. We further have generalized Hölder's inequality

$$\int_A |u(y)v(y)|\,d\mu(y) \leqslant 2||u||_{L^{\Phi}(A,d\mu)}||v||_{L^{\Psi}(A,d\mu)}\,. \tag{2.3}$$

Δ_2-condition

We say that a function Φ satisfies the Δ_2-condition, if there is $C_\Delta > 1$ such that

$$\Phi(2t) \leqslant C_\Delta \Phi(t) \qquad \text{whenever } t \geqslant 0\,.$$

It is not difficult to check the Δ_2-condition for our Young functions satisfying (1.5) and (1.6). Therefore one easily proves

$$\Phi(s+t) \leqslant C_\Delta\Phi(s) + C_\Delta\Phi(t)\,, \tag{2.4}$$

$$||u_k||_{L^{\Phi}(A,d\mu)} \overset{k\to\infty}{\to} 0 \quad \Longleftrightarrow \quad \int_A \Phi(|u_k|)\,d\mu(x) \overset{k\to\infty}{\to} 0\,, \tag{2.5}$$

for every $\xi \in (0,1)$ there is $\eta \in (0,1)$ such that

$$||u||_{L^{\Phi}(A,d\mu)} \leqslant 1-\eta \quad \Longrightarrow \quad \int_A \Phi(|u|)\,d\mu(x) \leqslant 1-\xi\,, \tag{2.6}$$

and for every $\eta \in (0,1)$ there is $\xi \in (0,1)$ such that

$$\int_A \Phi(|u|)\,d\mu(x) \leqslant 1-\xi \quad \Longrightarrow \quad ||u||_{L^{\Phi}(A,d\mu)} \leqslant 1-\eta\,. \tag{2.7}$$

We also need the following lemma from [8, Lemma 2.1].

LEMMA 2.1. *If Φ is a C^1-Young function satisfying the Δ_2-condition, then also Φ' satisfies the Δ_2-condition.*

We often use the following estimate together with the generalized Hölder's inequality.

LEMMA 2.2. *If a Young function Φ satisfies (1.5) and (1.6), then $\Psi(\Phi') \lesssim \Phi$.*

Proof. For $\ell = 1$, our lemma is the same as [8, Lemma 2.3]. Hence we can suppose that $\ell \geqslant 2$ in the sequel.

Let us find $E > \exp_{[\ell]}(1)$ large enough so that

$$\Phi_1(t) = t^n\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(E+t)\Big)\log_{[\ell]}^{\alpha}(E+t)$$

and

$$\tilde{\Psi}_1(t) = t^{\frac{n}{n-1}}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{-1}(E+t)\Big)\log_{[\ell]}^{-\frac{\alpha}{n-1}}(E+t)$$

are Young functions satisfying

$$\Phi_1(t) \approx t^n \quad \text{on } [0,C] \qquad \text{and} \qquad \tilde{\Psi}_1(t) \approx t^{\frac{n}{n-1}} \quad \text{on } [0,C]$$

for some $C > 0$. Plainly, Φ_1 and Ψ_1 are satisfying the Δ_2-condition. Next, if E is large enough, we have

$$\begin{aligned}
\Phi_1'(t) &= t^{n-1}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(E+t)\Big)\log_{[\ell]}^{\alpha}(E+t) \\
&\qquad \times \Big[n+\sum_{j=1}^{\ell-1}\frac{n-1}{\prod_{i=1}^{j}\log_{[i]}(E+t)}\frac{t}{t+E}+\frac{\alpha}{\prod_{i=1}^{[\ell]}\log_{[i]}(E+t)}\frac{t}{t+E}\Big] \\
&\approx t^{n-1}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(E+t)\Big)\log_{[\ell]}^{\alpha}(E+t) =: \varphi_2(t)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Psi}_1'(t) &= t^{\frac{1}{n-1}}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{-1}(E+t)\Big)\log_{[\ell]}^{-\frac{\alpha}{n-1}}(E+t) \\
&\qquad \times \Big[\frac{n}{n-1}-\sum_{j=1}^{\ell-1}\frac{1}{\prod_{i=1}^{j}\log_{[i]}(E+t)}\frac{t}{t+E}-\frac{\alpha}{n-1}\frac{1}{\prod_{i=1}^{[\ell]}\log_{[i]}(E+t)}\frac{t}{t+E}\Big] \\
&\approx t^{\frac{1}{n-1}}\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{-1}(E+t)\Big)\log_{[\ell]}^{-\frac{\alpha}{n-1}}(E+t) =: \psi_2(t)\,.
\end{aligned}$$

Further, by (1.5), (1.6) and by the fact that Φ is a Young function we have on $(0,\infty)$

$$\varphi(t) \geqslant \frac{\Phi(t)}{t} \geqslant \frac{1}{C}\varphi_2(t).$$

Hence there is C_1 large enough so that for every $t \in (0,\infty)$ we have

$$\begin{aligned}
\varphi(C_1\tilde{\Psi}_1'(t)) &\geqslant \frac{1}{C}\varphi_2(C_1\tilde{\Psi}_1'(t)) \geqslant \frac{1}{C}\varphi_2\Big(\frac{C_1}{C}\psi_2(t)\Big) \\
&= \frac{C_1^{n-1}}{C}\psi_2^{n-1}(t)\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}\Big(E+\frac{C_1}{C}\psi_2(t)\Big)\Big)\log_{[\ell]}^{\alpha}\Big(E+\frac{C_1}{C}\psi_2(t)\Big) \\
&\geqslant \frac{C_1^{n-1}}{C}\psi_2^{n-1}(t)\Big(\prod_{j=1}^{\ell-1}\frac{1}{C}\log_{[j]}^{n-1}(E+t)\Big)\frac{1}{C}\log_{[\ell]}^{\alpha}(E+t) \\
&= \frac{C_1^{n-1}}{C}t \geqslant t\,.
\end{aligned}$$

Since the associated function Ψ satisfies $\varphi(\psi(t)) = t$ for all $t \geqslant 0$, we obtain from above that

$$\Psi(t) \leqslant C_1\tilde{\Psi}_1(t)\,.$$

Finally

$$\begin{aligned}\Psi(\Phi'(t)) &\leqslant C_1\tilde{\Psi}_1(\Phi'(t)) \leqslant C_1\tilde{\Psi}_1(C\varphi_2(t))\\ &= C\varphi_2^{\frac{n}{n-1}}(t)\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{-1}(E+C\varphi_2(t))\Big)\log_{[\ell]}^{-\frac{\alpha}{n-1}}(E+C\varphi_2(t))\\ &\leqslant C\varphi_2^{\frac{n}{n-1}}(t)\Big(\prod_{j=1}^{\ell-1}C\log_{[j]}^{-1}(E+t)\Big)C\log_{[\ell]}^{-\frac{\alpha}{n-1}}(E+t)\\ &= Ct^n\Big(\prod_{j=1}^{\ell-1}\log_{[j]}^{n-1}(E+t)\Big)\log_{[\ell]}^{-\alpha}(E+t) = C\Phi_1(t) \leqslant C\Phi(t)\,.\end{aligned}$$

Thus, we are done. □

Next, we need to be able to estimate the norm by the modular and vice versa. Some rough estimates are given by the following lemma from [16, Lemma 3.8.4].

LEMMA 2.3. *Let* $u \in L^{\Phi}(\Omega)$.

(i) *If* $||u||_{L^{\Phi}(\Omega)} \leqslant 1$, *then* $\int_{\Omega}\Phi(|u|) \leqslant ||u||_{L^{\Phi}(\Omega)}$.

(ii) *If* $||u||_{L^{\Phi}(\Omega)} > 1$, *then* $\int_{\Omega}\Phi(|u|) \geqslant ||u||_{L^{\Phi}(\Omega)}$.

More careful estimates use the following lemma.

LEMMA 2.4. *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$||u||_{L^{\Phi}(\Omega)}^{n+\varepsilon} \leqslant \int_{\Omega}\Phi(|u|) \leqslant ||u||_{L^{\Phi}(\Omega)}^{n-\varepsilon} \qquad \textit{provided } ||u||_{L^{\Phi}(\Omega)} < \delta\,.$$

Proof. Since $t^{n-\varepsilon} << \Phi(t) << t^{n-\varepsilon}$ for large t, the proof is similar to [8, Proof of Lemma 2.4]. □

Orlicz-Sobolev spaces

Let A be an nonempty open bounded set in $\mathbb{R}^n$ and let Φ be a Young function satisfying (1.5). In this subsection we consider Orlicz spaces only with the Lebesgue measure. We define the Orlicz-Sobolev space $WL^{\Phi}(A)$ as the set

$$WL^{\Phi}(A) := \{u\colon u, |\nabla u| \in L^{\Phi}(A)\}$$

equipped with the norm

$$\|u\|_{WL^{\Phi}(A)} := \|u\|_{L^{\Phi}(A)} + \|\nabla u\|_{L^{\Phi}(A)}\ ,$$

where ∇u is the gradient of u and we use its Euclidean norm in $\mathbb{R}^n$.

We put $W_0L^\Phi(A)$ for the closure of $C_0^\infty(A)$ in $WL^\Phi(A)$. For this space we prefer to use throughout the paper the equivalent norm

$$\|u\|_{W_0L^\Phi(A)} := \|\nabla u\|_{L^\Phi(A)} .$$

The space $W_0L^\Phi(A)$ is a reflexive Banach space and it is compactly embedded into $L^\Phi(A)$.

We write that $u_k \rightharpoonup u$ in $W_0L^\Phi(A)$, if

$$\int_A \frac{\partial u_k}{\partial x_i} v\,dx \overset{k\to\infty}{\to} \int_A \frac{\partial u}{\partial x_i} v\,dx \quad \text{for every } v \in L^\Psi(A) \text{ and } i \in \{1,\ldots,n\} .$$

Tools from the Measure Theory

We make use of a version of a lemma from [13, Lemma 2.1]. The original version is stated for $\theta = 0$ and $a = 0$, but it can be easily seen that the proof given in [13] works also in our case after obvious minor modifications.

LEMMA 2.5. *Let $\theta \in [0,1)$ and $0 \leqslant a < n$. Let $\{u_k\}$ be a sequence of functions from $L^1(\Omega)$ converging to $u \in L^1(\Omega)$ a.e. in $\Omega \subset \mathbb{R}^n$. Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function bounded on $\Omega \times [-t_0, t_0]$ for every $t_0 > 0$. Suppose that $\frac{f(x,u_k)|u_k|^\theta}{|x|^a}$ and $\frac{f(x,u)|u|^\theta}{|x|^a}$ belong to $L^1(\Omega)$ and*

$$\int_\Omega \frac{|f(x,u_k)\,u_k|}{|x|^a} \leqslant C_1 .$$

Then $\frac{f(x,u_k)|u_k|^\theta}{|x|^a} \to \frac{f(x,u)|u|^\theta}{|x|^a}$ in $L^1(\Omega)$.

We also need the Generalized Lebesgue Dominated Convergence Theorem (see [20, Exercise 5.4.13]).

PROPOSITION 2.6. *Let $\{u_k\}$, $\{v_k\}$ be sequences of measurable functions on Ω such that $|u_k| \leqslant v_k$ for all $k \in \mathbb{N}$. Let u and v be measurable functions on Ω such that $u_k \to u$ a.e. in Ω and $v_k \to v$ a.e. in Ω. Then*

$$\lim_{k\to\infty} \int_\Omega v_k = \int_\Omega v \qquad \Longrightarrow \qquad \lim_{k\to\infty} \int_\Omega u_k = \int_\Omega u .$$

Tools from the Calculus of Variations

Our key instrument is the following version of the Mountain Pass Theorem by Ambrosetti and Rabinowitz [3].

THEOREM 2.7. *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$. Suppose that there exist a neighborhood U of 0 in X and $\xi \in \mathbb{R}$ satisfying the following conditions:*
(i) $J(0) < \xi$,

(ii) *$J(u) \geqslant \xi$ on the boundary of U,*
(iii) *there is $w \notin U$ such that $J(w) < \xi$.*
Set

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \geqslant \xi \ ,$$

where $\Gamma = \{g \in C([0,1],X) : g(0) = 0, g(1) = w\}$. Then there is a sequence $\{u_k\} \subset X$ such that

$$J(u_k) \to c \quad \text{and} \quad J'(u_k) \to 0 \text{ in } X^* \ . \tag{2.8}$$

The sequence satisfying (2.8) is called the Palais-Smale sequence and the constant c is a Palais-Smale level. Notice that that this version slightly differs from often used version of the Mountain Pass Theorem which requires the Palais-Smale condition (the Palais-Smale sequence has a subsequence convergent in the norm) and asserts that there is a critical point $x_0 \in X$ satisfying $J(x_0) = c$. We use this version of the Mountain Pass Theorem, because we need a bit less from the Palais-Smale sequence than the convergence in the norm. Our approach is taken from [5]. See [5, page 459] for the discussion concerning the proof of Theorem 2.7.

3. On the generalized Moser-Trudinger inequality

We often use the following embedding result from [7, Theorem 1.2 and Theorem 1.3].

THEOREM 3.1. *Let $\ell \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $a \in [0,n)$ and let Φ be a Young function satisfying (1.5). Suppose that $u \in W_0L^{\Phi}(\Omega)$.*
(i) *If $K \geqslant 0$, then*

$$\int_{\Omega} \frac{\exp_{[\ell]}(K|u(x)|^{\gamma})}{|x|^a} < \infty \ .$$

(ii) *If $\ell = 1$, $K \in [0, (1-\frac{a}{n})K_{1,n,\alpha})$ and $||\nabla u||_{L^{\Phi}(\Omega)} \leqslant 1$, then*

$$\int_{\Omega} \frac{\exp_{[\ell]}(K|u(x)|^{\gamma})}{|x|^a} \leqslant C \ .$$

(iii) *If $\ell \geqslant 2$, $K \in [0, K_{\ell,n,\alpha})$ and $||\nabla u||_{L^{\Phi}(\Omega)} \leqslant 1$, then*

$$\int_{\Omega} \frac{\exp_{[\ell]}(K|u(x)|^{\gamma})}{|x|^a} \leqslant C \ .$$

The constant C depends on $\ell, n, \alpha, a, \mathscr{L}_n(\Omega), K$ and Φ only (i.e. C is independent of the choice of u).

Let us note that when K is larger than the upper bound of the interval in Theorem 3.1(ii) and (iii), respectively, then there is no uniform estimate of the integral in Theorem 3.1(ii) and (iii), respectively. The counterexamples are given in [7]. In the borderline case ($K = (1-\frac{a}{n})K_{1,n,\alpha}$ for $\ell = 1$ and $K = K_{\ell,n,\alpha}$ for $\ell \geqslant 2$) we cannot say anything in general, for a detailed discussion see [7].

Next, using the same procedure as when obtaining [8, Proposition 3.2] from [8, Theorem 3.1] modifying the proof of Theorem 3.1(ii) and (iii) given in [7] we obtain the following "modular" version of Theorem 3.1.

PROPOSITION 3.2. *Let $\ell \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $b > 0$, $a \in [0,n)$ and let Φ be a Young function satisfying (1.5). Let $u_k \in W_0L^\Phi(\Omega)$ satisfy*

$$\int_\Omega \Phi(|\nabla u_k|) \leqslant c < \begin{cases} \left(\frac{K_{1,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{provided } \ell = 1 \\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{provided } \ell \geqslant 2\,. \end{cases}$$

Then there is $q > 1$ such that

$$\int_\Omega \left(\frac{\exp_{[\ell]}(b|u_k|^\gamma)}{|x|^a}\right)^q \leqslant C\,.$$

Finally, the proof of Theorem 3.1(ii) and (iii) can be modified (for more details see [8, Sketch of proof of Proposition 3.4]) so that we have a version of Theorem 3.1(ii) and (iii) for functions that are generally non-zero on the boundary of a fixed ball.

PROPOSITION 3.3. *Let $\ell \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $a \in [0,n)$ and let Φ be a Young function satisfying (1.5). Let $u_k \in W_0L^\Phi(\Omega)$ satisfy $||\nabla u_k||_{L^\Phi(\Omega)} \leqslant C_1$. Then for every $q \geqslant 1$ there is $\tau > 0$ with the following property:*
For every $x \in \Omega$ and $R > 0$ satisfying $B(x,2R) \subset \Omega$ we have

$$||\nabla u_k||_{L^\Phi(B(x,R))} < \tau \quad \Longrightarrow \quad \int_{B(x,R)} \frac{\exp_{[\ell]}(q|u_k|^\gamma)}{|x|^a} \leqslant C\,.$$

The constant $C \geqslant 0$ is independent of $k \in \mathbb{N}$ (it may depend on C_1, q, ℓ, n, α, a, Φ, x and R).

4. Assumptions of the Mountain Pass Theorem

In this section we check that our functional J has the Mountain Pass Geometry (i.e. assumptions (i), (ii) and (iii) from Theorem 2.7 are satisfied).

The following assertions follow easily from (1.9) and (1.10).
There is a positive constant C such that

$$F(x,t) \geqslant C\exp\left(C|t|^{\frac{1}{M}}\right), \quad |t| \geqslant t_M\,. \tag{4.1}$$

Given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$F(x,t) \leqslant \varepsilon f(x,t)t, \quad |t| \geqslant t_\varepsilon\,. \tag{4.2}$$

Now, we can start to check the assumptions of the Mountain Pass Theorem.

LEMMA 4.1. *If $u \in W_0L^{\Phi}(\Omega)$, $u \geqslant 0$ and $u \neq 0$, then*

$$J(tu) \overset{t\to\infty}{\to} -\infty .$$

Sketch of proof. Since Ω is bounded, we have by (4.1) (which can be extended by (1.9) to $\mathbb{R}\setminus(-t_0,t_0)$ for each $t_0>0$ fixed; with a different constant C, of course)

$$\frac{F(x,t)}{|x|^a} \geqslant \frac{1}{C}F(x,t) \geqslant \frac{1}{C}\exp\Big(C|t|^{\frac{1}{M}}\Big)$$

for $|t| \geqslant \tau$ for each $\tau > 0$ fixed. The rest of the proof is the same as [8, Proof of Lemma 4.1]. □

LEMMA 4.2. *There are $\rho > 0$ and $\xi > 0$ with the following property. If $u \in W_0L^{\Phi}(\Omega)$ with $||\nabla u||_{L^{\Phi}(\Omega)} = \rho$, then $J(u) \geqslant \xi$.*

Proof. Fix $q > n$. By assumptions (1.10), (1.11) and (1.12) we have

$$F(x,t) \leqslant C_1|t|^{n+\varepsilon_0} + C\exp_{[\ell]}(b|t|^{\gamma})|t|^q = F_1(t)+F_2(t) .$$

Next, from Theorem 3.1(ii) and (iii), respectively, we can easily see that if $u \in W_0L^{\Phi}(\Omega)$ is such that $||\nabla u||_{L^{\Phi}(\Omega)} \leqslant 1$, then

$$\int_\Omega \frac{|u|^{n+\varepsilon_0}}{|x|^a} \leqslant C_2 .$$

Hence by Lemma 2.4 with $\varepsilon \in (0,\varepsilon_0)$ we obtain for ρ small enough

$$\begin{aligned}\int_\Omega \frac{F_1(u)}{|x|^a} &\leqslant C_1\int_\Omega \frac{|u|^{n+\varepsilon_0}}{|x|^a} = C_1||\nabla u||^{n+\varepsilon_0}_{L^{\Phi}(\Omega)}\int_\Omega \frac{\Big|\frac{u}{||\nabla u||_{L^{\Phi}(\Omega)}}\Big|^{n+\varepsilon_0}}{|x|^a} \leqslant C_1C_2||\nabla u||^{n+\varepsilon_0}_{L^{\Phi}(\Omega)}\\ &\leqslant \frac{1}{4}||\nabla u||^{n+\varepsilon}_{L^{\Phi}(\Omega)} \leqslant \frac{1}{4}\int_\Omega \Phi(|\nabla u|) .\end{aligned} \tag{4.3}$$

Fix $p > 1$ such that $ap < n$. Next, if ρ is so small that $bp\rho^{\gamma} < (1-\frac{ap}{n})K_{\ell,n,\alpha}$, from Hölder's inequality, (2.1), Theorem 3.1(ii) and (iii), respectively, and from the fact that $W_0L^{\Phi}(\Omega)$ is continuously embedded into $L^s(\Omega)$, for every $s \in [1,\infty)$, we obtain

$$\begin{aligned}\int_\Omega \frac{F_2(u)}{|x|^a} &= C\int_\Omega \frac{\exp_{[\ell]}(b|u|^{\gamma})|u|^q}{|x|^a} \leqslant \Big(\int_\Omega \frac{\exp^p_{[\ell]}(b|u|^{\gamma})}{|x|^{ap}}\Big)^{\frac{1}{p}}\Big(\int_\Omega |u|^{qp'}\Big)^{\frac{1}{p'}}\\ &\leqslant C\int_\Omega \frac{\exp_{[\ell]}(bp||\nabla u||^{\gamma}_{L^{\Phi}(\Omega)}(\frac{|u|}{||\nabla u||_{L^{\Phi}(\Omega)}})^{\gamma})}{|x|^{ap}}||u||^q_{L^{qp'}}\\ &\leqslant C||u||^q_{W_0L^{\Phi}(\Omega)} = C||\nabla u||^q_{L^{\Phi}(\Omega)} .\end{aligned}$$

For $\rho > 0$ small enough we can further use Lemma 2.4 with $\varepsilon < q-n$ to infer

$$\int_\Omega \frac{F_2(u)}{|x|^a} \leqslant C||\nabla u||_{L^\Phi(\Omega)}^{q-n-\varepsilon}||\nabla u||_{L^\Phi(\Omega)}^{n+\varepsilon} \leqslant \frac{1}{4}\int_\Omega \Phi(|\nabla u|)\ . \tag{4.4}$$

Now, (4.3) and (4.4) give

$$J(u) = \int_\Omega \Phi(|\nabla u|) - \int_\Omega \frac{F(x,u)}{|x|^a} \geqslant \frac{1}{2}\int_\Omega \Phi(|\nabla u|)\ .$$

Finally, one can easily see from (2.5) that the modular is bounded away from zero, if the norm is equal to $\rho > 0$ and thus we are done. $\square$

REMARK 4.3. In case $a = 0$, we can replace assumption (1.12) by a weaker condition

$$\limsup_{t\to 0} \frac{F(x,t)}{C_S\Phi(|t|)} < 1 \quad \text{uniformly on } \Omega\ ,$$

where $C_S > 0$ is such that $C_S\int_\Omega \Phi(|u|) \leqslant \int_\Omega \Phi(|\nabla u|)$. With this assumption (which means $F(x,t) \lesssim t^n$ for small t) the proof of Lemma 4.2 is the same as the proof of [8, Lemma 4.2]. For $a \in (0,n)$, the author does not know whether there is $C_S > 0$ such that $C_S\int_\Omega \int \Phi(|u|)|x|^a \leqslant \int_\Omega \Phi(|\nabla u|)$, hence we use assumption (1.12) in this paper.

In the rest of the section we show that our Palais-Smale level is not too high. First, we need to construct a sequence of auxiliary functions concentrating around the origin with suitably controlled modulars.

LEMMA 4.4. *Suppose that $B(R) \subset \Omega$. If $\ell = 1$, let us define for each $k \in \mathbb{N}$*

$$w_k(x) = g_k(|x|)\ , \qquad \textit{where}$$

$$g_k(y) = \begin{cases} 0 & \textit{for } y \in [R,\infty) \\ (-\frac{2}{R}y+2)K_{1,n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(2) k^{\frac{1}{\gamma}-B}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [\frac{R}{2},R] \\ K_{1,n,\alpha}^{-\frac{1}{\gamma}} n^B \log^B(\frac{R}{y}) k^{\frac{1}{\gamma}-B}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [Re^{-\frac{k}{n}},\frac{R}{2}] \\ K_{1,n,\alpha}^{-\frac{1}{\gamma}} k^{\frac{1}{\gamma}}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [0,Re^{-\frac{k}{n}}]\ . \end{cases} \tag{4.5}$$

In case $\ell \geqslant 2$ we fix $T > \exp_{[\ell]}(1)$ and we define

$$w_k(x) = g_k(|x|)\ , \qquad \textit{where}$$

$$g_k(y) = \begin{cases} 0 & \textit{for } y \in [R,\infty) \\ (-\frac{2}{R}y+2)K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log^B_{[\ell]}(T+2) k^{\frac{1}{\gamma}-B}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [\frac{R}{2},R] \\ K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log^B_{[\ell]}(T+\frac{R}{y}) k^{\frac{1}{\gamma}-B}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [R\exp_{[\ell]}^{-\frac{1}{n}}(k),\frac{R}{2}] \\ K_{\ell,n,\alpha}^{-\frac{1}{\gamma}} \log^B_{[\ell]}\left(T+\exp_{[\ell]}^{\frac{1}{n}}(k)\right) k^{\frac{1}{\gamma}-B}\left(1+\frac{\log(k)}{k}\right)^{\frac{1}{\gamma}} & \textit{for } y \in [0,R\exp_{[\ell]}^{-\frac{1}{n}}(k)]\ . \end{cases} \tag{4.6}$$

Then for every $\theta_0 \geqslant 1$ *there is* $k_0 \in \mathbb{N}$ *such that*

$$\int_\Omega \Phi(\theta|\nabla w_k|) \leqslant \theta^n \qquad \text{for every } k \geqslant k_0 \text{ and } \theta \in [1,\theta_0]\,.$$

The proof of Lemma 4.4 for $\ell = 1$ and $\theta = 1$ is given in [6, Example 5.1]. For general $\theta \in [1,\theta_0]$ the proof requires a minor modification only. For $\ell \geqslant 2$, the proof is obtained modifying [9, Proof of Theorem 1.2]. Let us note that the proof uses assumption (1.7).

Now we can obtain the estimate concerning the Palais-Smale level.

LEMMA 4.5. *There is a non-trivial function* $w \in W_0L^\Phi(\Omega)$ *such that*

$$J(\theta w) < \begin{cases} \left(\frac{K_{\ell,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{provided } \ell = 1 \\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{provided } \ell \geqslant 2 \end{cases} \qquad \text{for every } \theta \in [0,\infty)\,.$$

Proof. Fix $R > 0$ such that $B(R) \subset \Omega$. By (1.13) we have

$$\liminf_{t\to\infty} \frac{tf(x,t)}{\exp_{[\ell]}(b|t|^\gamma)} > C \qquad \text{uniformly on } B(R)\,. \tag{4.7}$$

Our aim is to show that there is $k \in \mathbb{N}$ such that the assertion of the lemma holds for w_k given by Lemma 4.5. For the sake of contradiction suppose that for all $k \in \mathbb{N}$ we have

$$\sup\{J(\theta w_k) : \theta \in [0,\infty)\} \geqslant \begin{cases} \left(\frac{K_{\ell,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{provided } \ell = 1 \\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{provided } \ell \geqslant 2\,. \end{cases}$$

In view of Lemma 4.1 there are $\theta_k > 0$, $k \in \mathbb{N}$, such that

$$J(\theta_k w_k) = \max\{J(\theta w_k) : \theta \in [0,\infty)\}\,.$$

Since F is non-negative (see (1.9)), we arrive at

$$\begin{aligned} \int_\Omega \Phi(\theta_k|\nabla w_k|) &\geqslant J(\theta_k w_k) = \max\{J(\theta w_k) : \theta \in [0,\infty)\} \\ &\geqslant \begin{cases} \left(\frac{K_{\ell,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{provided } \ell = 1 \\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{provided } \ell \geqslant 2\,. \end{cases} \end{aligned} \tag{4.8}$$

Next, from Lemma 4.4 with $\theta = \theta_0 = 1$ we observe that there is $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$ we have

$$\int_\Omega \Phi(|\nabla w_k|) \leqslant 1\,. \tag{4.9}$$

Now, we claim that θ_k are bounded away from zero. Indeed, for $k \geqslant k_0$ such that $\theta_k \leqslant 1$ we have by (4.8), (4.9) and by the fact that Φ is a Young function (hence $\Phi(ts) \leqslant t\Phi(s)$ for every $t \in [0,1]$ and $s \geqslant 0$)

$$\theta_k \geqslant \theta_k \int_\Omega \Phi(|\nabla w_k|) \geqslant \int_\Omega \Phi(\theta_k|\nabla w_k|) \geqslant \begin{cases} \left(\frac{K_{\ell,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{provided } \ell = 1 \\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{provided } \ell \geqslant 2\,. \end{cases}$$

Further, as $\frac{d}{d\theta}J(\theta w_k)|_{\theta=\theta_k} = 0$, it follows that

$$\int_\Omega \Phi'(\theta_k|\nabla w_k|)|\nabla w_k| = \int_\Omega \frac{w_k f(x,\theta_k w_k)}{|x|^a}\,.$$

Multiplying both sides by $c_\Phi\theta_k$, using (1.8), (4.7) (recall that θ_k are bounded away from zero) and the definition of w_k we obtain $k_1 \geqslant k_0$ such that for all $k \geqslant k_1$ we have

$$\begin{aligned}\int_\Omega \Phi(\theta_k|\nabla w_k|) &\geqslant c_\Phi \int_\Omega \Phi'(\theta_k|\nabla w_k|)\theta_k|\nabla w_k| = c_\Phi \int_\Omega \frac{\theta_k w_k f(x,\theta_k w_k)}{|x|^a} \\ &\geqslant c_\Phi \int_{B(R\exp_{[\ell]}^{-\frac{1}{n}}(k))} \frac{\theta_k w_k f(x,\theta_k w_k)}{|x|^a} \geqslant C \int_{B(R\exp_{[\ell]}^{-\frac{1}{n}}(k))} \frac{\exp_{[\ell]}(b|\theta_k w_k|^\gamma)}{|x|^a}\,. \end{aligned} \tag{4.10}$$

In the rest of the proof we distinguish two cases.

Case $\ell = 1$.

Since

$$\int_{B(R\exp^{-\frac{1}{n}}(k))} \frac{1}{|x|^a}\,dx = C\int_0^{R\exp^{-\frac{1}{n}}(k))} r^{n-1-a}\,dr = C\exp\left(\left(\frac{a}{n}-1\right)k\right)$$

from (4.5) and (4.10) we obtain

$$\int_\Omega \Phi(\theta_k|\nabla w_k|) \geqslant C\exp\left(\left(\frac{b\theta_k^\gamma}{K_{1,n,\alpha}} + \frac{a}{n} - 1\right)k + \frac{b\theta_k^\gamma}{K_{1,n,\alpha}}\log(k)\right)\,. \tag{4.11}$$

Now, for each $k \in \mathbb{N}$ satisfying $\theta_k \geqslant 2$ let us find $s_k \in \mathbb{N}$ such that $\theta_k \in [2^{s_k}, 2^{s_k+1})$. Therefore the Δ_2-condition, (4.9) and (4.11) give us for every $k \geqslant k_1$ such that $\theta_k \geqslant 2$

$$\begin{aligned} C_\Delta^{s_k+1} &\geqslant C_\Delta^{s_k+1}\int_\Omega \Phi(|\nabla w_k|) \geqslant \int_\Omega \Phi(\theta_k|\nabla w_k|) \\ &\geqslant C\exp\left(\frac{b\theta_k^\gamma}{K_{1,n,\alpha}}k - k\right) \geqslant C\exp\left(\frac{b}{K_{1,n,\alpha}}2^{s_k\gamma}k - k\right)\,. \end{aligned}$$

Therefore s_k are bounded and thus there is $\theta_0 \geqslant 1$ such that $\theta_k \leqslant \theta_0$ for every $k \in \mathbb{N}$.

Hence we can use Lemma 4.4 to obtain $k_2 \geqslant k_1$ such that for every $k \geqslant k_2$ we have

$$\int_\Omega \Phi(\theta_k|\nabla w_k|) \leqslant \theta_k^n\,. \tag{4.12}$$

It has the following consequences. First, (4.8) and (4.12) give

$$\theta_k \geqslant \Big(\frac{K_{1,n,\alpha}}{b}\Big(1-\frac{a}{n}\Big)\Big)^{\frac{1}{\gamma}} =: D^{\frac{1}{\gamma}} \qquad \text{for } k \geqslant k_2\ . \tag{4.13}$$

Second, (4.11), (4.12) and (4.13) imply

$$\begin{aligned} C = \theta_0^n \geqslant \theta_k^n \geqslant \int_\Omega \Phi(\theta_k|\nabla w_k|) \\ &\geqslant C\exp\Big(\Big(\frac{b\theta_k^\gamma}{K_{1,n,\alpha}}+\frac{a}{n}-1\Big)k+\frac{b\theta_k^\gamma}{K_{1,n,\alpha}}\log(k)\Big) \\ &\geqslant C\exp\Big(\Big(\frac{bD}{K_{1,n,\alpha}}+\frac{a}{n}-1\Big)k+\frac{bD}{K_{1,n,\alpha}}\log(k)\Big) \\ &= C\exp\Big(\frac{bD}{K_{1,n,\alpha}}\log(k)\Big) \overset{k\to\infty}{\to} \infty\ . \end{aligned}$$

Thus, we have a contradiction and we are done in the case $\ell = 1$.

Case $\ell \geqslant 2$.
This time we have

$$\int_{B(R\exp_{[\ell]}^{-\frac{1}{n}}(k))} \frac{1}{|x|^a}\,dx = C\int_0^{R\exp_{[\ell]}^{-\frac{1}{n}}(k))} r^{n-1-a}\,dr = C\exp_{[\ell]}^{\frac{a}{n}-1}(k)$$

and further by [9, Proof of Theorem 4.1] we have for k large enough the estimate

$$\Big(\frac{\log_{[\ell]}(\exp_{[\ell]}^{\frac{1}{n}}(k))}{k}\Big)^{B\gamma}(k+\log(k)) \geqslant k+\frac{1}{2}\log(k)\ .$$

Hence (4.6) and (4.10) imply

$$\begin{aligned} \int_\Omega \Phi(\theta_k|\nabla w_k|) &\geqslant C\exp_{[\ell]}^{\frac{a}{n}-1}(k)\exp_{[\ell]}\Big(\frac{b\theta_k^\gamma}{K_{\ell,n,\alpha}}\log_{[\ell]}^{B\gamma}(T+\exp_{[\ell]}^{\frac{1}{n}}(k))k^{1-B\gamma}\Big(1+\frac{\log(k)}{k}\Big)\Big) \\ &\geqslant C\exp_{[\ell]}^{\frac{a}{n}-1}(k)\exp_{[\ell]}\Big(\frac{b\theta_k^\gamma}{K_{\ell,n,\alpha}}k\Big(1+\frac{\log(k)}{2k}\Big)\Big)\ . \end{aligned} \tag{4.14}$$

Next, we obtain in the same way as in the case $\ell = 1$ that θ_k are bounded. Thus, we have by Lemma 4.4 and (4.8) for k large enough

$$\int_\Omega \Phi(\theta_k|\nabla w_k|) \leqslant \theta_k^n \qquad \text{and} \qquad \theta_k \geqslant \Big(\frac{K_{\ell,n,\alpha}}{b}\Big)^{\frac{1}{\gamma}} =: D^{\frac{1}{\gamma}}\ .$$

This together with (4.14) implies

$$\begin{aligned} C &\geqslant \theta_k^n \geqslant \int_\Omega \Phi(\theta_k|\nabla w_k|) \\ &\geqslant C\exp_{[\ell]}^{\frac{a}{n}-1}(k)\exp_{[\ell]}\Big(\frac{b\theta_k^\gamma}{K_{\ell,n,\alpha}}k\Big(1+\frac{\log(k)}{2k}\Big)\Big) \\ &\geqslant C\exp_{[\ell]}^{-1}(k)\exp_{[\ell]}\Big(\frac{bD}{K_{\ell,n,\alpha}}k\Big(1+\frac{\log(k)}{2k}\Big)\Big) \\ &= C\exp_{[\ell]}^{-1}(k)\exp_{[\ell]}\Big(k+\frac{\log(k)}{2}\Big) \overset{k\to\infty}{\to} \infty\,. \end{aligned}$$

This is a contradiction. Hence we are done in both cases $\ell = 1$ and $\ell \geqslant 2$. □

5. Properties of the Palais-Smale sequence

In this section we study the properties of the Palais-Smale sequence. Our aim is to show that it contains a subsequence with the gradients converging a.e. in Ω (see Lemma 5.2) and that the limit (in the sense of (5.5)) is a weak solution to problem (1.3) (see Lemma 5.3).

Let $\{u_k\}$ be a Palais-Smale sequence from $W_0L^\Phi(\Omega)$, that is by (2.8),

$$J(u_k) = \int_\Omega \Phi\big(|\nabla u_k|\big) - \int_\Omega \frac{F(x,u_k)}{|x|^a} \overset{k\to\infty}{\to} c\,, \tag{5.1}$$

and (see (1.15)) there are $\varepsilon_k \to 0$ such that for every $v \in W_0L^\Phi(\Omega)$ we have

$$|\langle J'(u_k),v\rangle| = \Big|\int_\Omega \Phi'\big(|\nabla u_k|\big)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla v - \int_\Omega \frac{f(x,u_k)v}{|x|^a}\Big| \leqslant \varepsilon_k\|\nabla v\|_{L^\Phi(\Omega)}\,. \tag{5.2}$$

LEMMA 5.1. *There is a constant $C > 0$ independent of $k \in \mathbb{N}$ such that*

$$\|\nabla u_k\|_{L^\Phi(\Omega)} \leqslant C\,, \qquad \int_\Omega \Phi\big(|\nabla u_k|\big) \leqslant C \tag{5.3}$$

and

$$0 \leqslant \int_\Omega \frac{f(x,u_k)u_k}{|x|^a} \leqslant C\,. \tag{5.4}$$

Proof. We obtain from (4.2) and (5.1) that, for any $\varepsilon > 0$,

$$\int_\Omega \Phi\big(|\nabla u_k|\big) \leqslant C + \int_\Omega \frac{F(x,u_k)}{|x|^a} \leqslant C_\varepsilon + \varepsilon\int_\Omega \frac{f(x,u_k)u_k}{|x|^a}\,.$$

Hence, using (5.2) with $v = u_k$ and (1.8) we arrive at

$$\begin{aligned} \int_\Omega \Phi\big(|\nabla u_k|\big) &\leqslant C_\varepsilon + \varepsilon\Big(\int_\Omega \Phi'\big(|\nabla u_k|\big)|\nabla u_k| + \varepsilon_k\|\nabla u_k\|_{L^\Phi(\Omega)}\Big) \\ &\leqslant C_\varepsilon + \varepsilon C\int_\Omega \Phi\big(|\nabla u_k|\big) + \varepsilon\varepsilon_k\|\nabla u_k\|_{L^\Phi(\Omega)}\,. \end{aligned}$$

Together with Lemma 2.3(ii) it implies that $\|\nabla u_k\|_{L^\Phi(\Omega)} \leqslant C$ and $\int_\Omega \Phi\big(|\nabla u_k|\big) \leqslant C$. The remaining estimate now follows from (5.2) (with $v = u_k$, see also (1.8)). The integral in (5.4) is non-negative by (1.9). □

By (5.3) there is a function $u \in W_0L^\Phi(\Omega)$ (passing to a suitable subsequence of $\{u_k\}$ if necessary) such that

$$\begin{aligned} u_k &\rightharpoonup u && \text{in } W_0L^\Phi(\Omega)\,, \\ u_k &\to u && \text{in } L^\Phi(\Omega)\,, \\ u_k &\to u && \text{in } L^r(\Omega) \text{ for every } r \in [1,\infty)\,, \\ u_k &\to u && \text{a.e. in } \Omega\,. \end{aligned} \tag{5.5}$$

By (1.11) and Theorem 3.1(i) we have $\frac{f(x,u)}{|x|^a}, \frac{f(x,u_k)}{|x|^a} \in L^1(\Omega)$. Since we also have (5.4), Lemma 2.5 with $\theta = 0$ implies

$$\lim_{k\to\infty} \int_\Omega \frac{f(x,u_k)}{|x|^a} = \int_\Omega \frac{f(x,u)}{|x|^a}\,. \tag{5.6}$$

Moreover, from (5.4) and Lemma 2.5 we also obtain $\frac{f(x,u_k)}{|x|^a}|u_k|^{1-\frac{1}{M}} \to \frac{f(x,u)}{|x|^a}|u|^{1-\frac{1}{M}}$ in $L^1(\Omega)$ and thus by (1.10) and Proposition 2.6 we see that

$$\lim_{k\to\infty} \int_\Omega \frac{F(x,u_k)}{|x|^a} = \int_\Omega \frac{F(x,u)}{|x|^a}\,. \tag{5.7}$$

LEMMA 5.2. *Passing to a subsequence we have*

$$\nabla u_k \to \nabla u \quad \textit{a.e. on } \Omega\,. \tag{5.8}$$

Sketch of proof. The proof is very long, technical and most of it can be taken from [8, Proof of Lemma 5.2]. Therefore we focus on the differences only.

In the same way as in [8, Proof of Lemma 5.2] it can be shown that it is enough to prove that

$$\int_\Omega \psi_\varepsilon \Big(\Phi'(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|} - \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)\cdot(\nabla u_k - \nabla u) \overset{k\to\infty}{\to} 0\,,$$

where $\psi_\varepsilon \in C^1(\bar{\Omega})$ satisfying $0 \leqslant \psi_\varepsilon \leqslant 1$ and $\psi_\varepsilon = 0$ on a suitably chosen small exceptional open set B_ε. Next, above integral is estimated by $I_1 + I_2 + I_3 + I_4 + I_5$, where

$$I_3 = \int_\Omega \psi_\varepsilon \frac{f(x,u_k)}{|x|^a}(u_k - u)$$

and I_1, I_2, I_4 and I_5 are the same as in [8, Proof of Lemma 5.2] and tend to zero. We want to show that we also have $I_3 \to 0$. The first step is to prove that for fixed $p \in (1, \frac{n}{a})$ we have

$$\int_{\Omega\setminus B_\varepsilon} \Big(\frac{|f(x,u_k)|}{|x|^a}\Big)^p \leqslant C\,. \tag{5.9}$$

This is shown in a similar way as in [8, Proof of Lemma 5.2, Step 6] using the compactness of $\bar{\Omega}\setminus B_\varepsilon$, Proposition 3.3, (1.11) and (2.1).

Finally, Hölder's inequality, (5.5) and (5.9) imply

$$|I_3| \leqslant \int_{\Omega\setminus B_\varepsilon} \left|\frac{f(x,u_k)}{|x|^a}(u_k-u)\right| \leqslant \left(\int_{\Omega\setminus B_\varepsilon}\left(\frac{|f(x,u_k)|}{|x|^a}\right)^p\right)^{\frac{1}{p}} ||u_k-u||_{L^{p'}(\Omega)} \overset{k\to\infty}{\to} 0\,. \qquad \square$$

LEMMA 5.3. *The function $u\in W_0L^\Phi(\Omega)$ given by* (5.5) *is a weak solution to problem* (1.3)*, i.e. we have* (1.16).

Sketch of proof. It is enough to modify [8, Proof of Lemma 5.3] in the following way. First, instead of $f(x,u)$ we always deal with $\frac{f(x,u)}{|x|^a}$. Second, when proving that $\psi_k\to v$ in $W_0L^\Phi(\Omega)$ implies

$$\left|\int_\Omega \frac{f(x,u)}{|x|^a}(v-\psi_k)\right| \overset{k\to\infty}{\to} 0\,,$$

we use Hölder's inequality raising $\frac{f(x,u)}{|x|^a}$ to the power $p\in(1,\frac{n}{a})$. Then we apply (1.11), (2.1) and Theorem 3.1(i) to show the boundedness of $||\frac{f(x,u)}{|x|^a}||_{L^p(\Omega)}$. The fact that $W_0L^\Phi(\Omega)$ is continuously embedded into $L^{p'}(\Omega)$ is used to show that $||v-\psi_k||_{L^{p'}(\Omega)}\to 0$. $\square$

Proof of Theorem 1.1. Since we have $J(0)=0$, Lemmata 4.1, 4.2 and Proposition 6.1, we can apply the Mountain Pass Theorem (Theorem 2.7) which together with Lemma 4.5 gives us a Palais-Smale sequence $\{u_k\}\subset W_0L^\Phi(\Omega)$ approaching a Palais-Smale level c such that

$$0<c<\begin{cases}\left(\frac{K_{1,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{for } \ell=1\\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{for } \ell\geqslant 2\,.\end{cases} \tag{5.10}$$

Passing to a subsequence we can further suppose that we have (5.5).

By Lemma 5.3 we know that the function $u\in W_0L^\Phi(\Omega)$ given by (5.5) is a weak solution to (1.3) and thus it remains to show that u is non-trivial. For the sake of contradiction suppose that we have $u=0$. From (5.1), (5.7), $u=0$ and from (5.10) we obtain $\tilde{c}>c$ such that for k sufficiently large we have

$$\int_\Omega \Phi(|\nabla u_k|)\leqslant \tilde{c}<\begin{cases}\left(\frac{K_{1,n,\alpha}}{b}\left(1-\frac{a}{n}\right)\right)^{\frac{n}{\gamma}} & \text{for } \ell=1\\ \left(\frac{K_{\ell,n,\alpha}}{b}\right)^{\frac{n}{\gamma}} & \text{for } \ell\geqslant 2\,.\end{cases}$$

Hence Proposition 3.2 together with estimate (1.11) and Hölder's inequality give us

$$\int_\Omega \frac{f(x,u_k)u_k}{|x|^a} \overset{k\to\infty}{\to} 0\,.$$

Therefore (5.2) with $v = u_k$ and (5.3) imply

$$\int_\Omega \Phi'(|\nabla u_k|)|\nabla u_k| \overset{k\to\infty}{\to} 0\,.$$

Next, as Φ is a Young function, we have $\Phi(t) \leqslant t\Phi'(t)$ for every $t \geqslant 0$ and thus we obtain from above

$$\int_\Omega \Phi(|\nabla u_k|) \overset{k\to\infty}{\to} 0\,.$$

However, in view of (5.1) and (5.7) (recall $F(x,0) = 0$ by the definition) this contradicts $c > 0$. Hence u is non-trivial. □

6. Functional J is C^1

PROPOSITION 6.1. *For the functional J defined by* (1.14) *we have $J \in C^1(W_0L^\Phi(\Omega),\mathbb{R})$ and its Fréchet derivative is* (1.15).

Sketch of proof. The proof using the approach from [4, Proof of Theorem A.V] is similar as the one given in [8, Section 6]. Therefore we sketch it.

It is shown in [8, Lemma 6.2] that the functional

$$J_1(u) = \int_\Omega \Phi(|\nabla u|)\,, \qquad u \in W_0L^\Phi(\Omega)$$

satisfies $J_1 \in C^1(W_0L^\Phi(\Omega),\mathbb{R})$ and

$$\langle J_1'(u),\varphi\rangle = \int_\Omega \Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\varphi\,, \qquad u,\varphi \in W_0L^\Phi(\Omega)\,.$$

In fact, paper [8] concerns Young functions satisfying (1.5) with $\ell = 1$ only, but the proof is still valid for any $\ell \in \mathbb{N}$.

Next, we show that

$$J_2(u) = \int_\Omega \frac{F(x,u)}{|x|^a}\,, \qquad u \in W_0L^\Phi(\Omega)$$

satisfies $J_2 \in C^1(W_0L^\Phi(\Omega),\mathbb{R})$ and

$$\langle J_2'(u),\varphi\rangle = \int_\Omega \frac{f(x,u)\varphi}{|x|^a}\,, \qquad u,\varphi \in W_0L^\Phi(\Omega)\,.$$

This time we have to modify the proof from [8] a bit. Let us start showing that J_2 is Gateaux differentiable everywhere on $W_0L^\Phi(\Omega)$. For a.e. $x \in \Omega$ it is easy to see that the point wise limit satisfies

$$\lim_{t\to 0}\frac{\frac{F(x,u(x)+t\varphi(x))}{|x|^a} - \frac{F(x,u(x))}{|x|^a}}{t} = \frac{f(x,u(x))}{|x|^a}\varphi(x)\,. \tag{6.1}$$

Moreover, we may use the Mean Value Theorem and (1.11) to obtain

$$\left|\frac{\frac{F(x,u+t\varphi)}{|x|^a}-\frac{F(x,u)}{|x|^a}}{t}\right| \leqslant \frac{|f(x,u+\xi\varphi)|}{|x|^a}|\varphi| \leqslant C\frac{\exp_{[\ell]}(\beta(|u|+|\varphi|)^\gamma)}{|x|^a}|\varphi| \leqslant C\Big(\frac{\exp_{[\ell]}(\beta 2^\gamma|u|^\gamma)}{|x|^a}+\frac{\exp_{[\ell]}(\beta 2^\gamma|\varphi|^\gamma)}{|x|^a}\Big)|\varphi|\,. \tag{6.2}$$

Now, fix $p\in(1,\frac{n}{a})$ (hence $ap<n$). Let $p'=\frac{p}{p-1}$. We know that $\varphi\in L^{p'}(\Omega)$ and thus we may apply Hölder's inequality on the right-hand side of (6.2) and using (2.1) together with Theorem 3.1(i) we easily obtain that the right-hand side of (6.2) is integrable. Thus, it follows from the Lebesgue Dominated Convergence Theorem applied to (6.1) that

$$\lim_{t\to 0}\frac{J_2(u+t\varphi)-J_2(u)}{t}=\int_\Omega\frac{f(x,u)\varphi}{|x|^a}\qquad\text{for every } u,\ \varphi\in W_0L^\Phi(\Omega)\,.$$

This is the Gateaux differentiability everywhere on $W_0L^\Phi(\Omega)$.

To prove that $J_2'(u)$ is continuous, let $u_k\to u$ in $W_0L^\Phi(\Omega)$. Passing to a subsequence, we can suppose that $u_k\to u$ a.e. in Ω and moreover that there is a majorant $V\in W_0L^\Phi(\Omega)$, i.e. $|u_k|\leqslant V$ for every k. The existence of a common majorant is shown in a standard way dealing with a subsequence (still denoted $\{u_k\}$) satisfying $||u_k-u||_{W_0L^\Phi(\Omega)}\leqslant 2^{-k}$ and setting $V=|u|+\sum_{k=1}^\infty|u_k-u|$.

Next, fix $p\in(1,\sqrt{\frac{n}{a}})$ (hence $ap^2<n$). From (1.11), (2.1) and Theorem 3.1(i) we obtain

$$\int_\Omega\Big(\frac{|f(x,u_k)|}{|x|^a}\Big)^{p^2}\leqslant C\int_\Omega\frac{\exp_{[\ell]}(p^2\beta|u_k|^\gamma)}{|x|^{ap^2}}\leqslant C\int_\Omega\frac{\exp_{[\ell]}(p^2\beta|V|^\gamma)}{|x|^{ap^2}}<\infty\,.$$

The boundedness in $L^{p^2}(\Omega)$ and the point wise convergence a.e. implies that $\frac{f(x,u_k)}{|x|^a}\overset{L^p(\Omega)}{\to}\frac{f(x,u)}{|x|^a}$. This, the continuous embedding of $W_0L^\Phi(\Omega)$ into $L^{p'}(\Omega)$ and Hölder's inequality finally imply

$$\begin{aligned}||J_2'(u_k)-J_2'(u)||_{C(W_0L^\Phi(\Omega),\mathbb{R})}&=\sup_{||\nabla\varphi||_{L^\Phi(\Omega)}\leqslant 1}\Big|\int_\Omega\Big(\frac{f(x,u_k)}{|x|^a}-\frac{f(x,u)}{|x|^a}\Big)\varphi\Big|\\&\leqslant\Big|\Big|\frac{f(x,u_k)}{|x|^a}-\frac{f(x,u)}{|x|^a}\Big|\Big|_{L^p(\Omega)}\sup_{||\nabla\varphi||_{L^\Phi(\Omega)}\leqslant 1}||\varphi||_{L^{p'}(\Omega)}\overset{k\to\infty}{\to}0\,.\end{aligned}$$

□

7. Sub-critical case

We can use our methods to obtain the existence of a non-trivial weak solution to (1.3) also in the sub-critical case. It is, instead of (1.11) we have

$$\begin{aligned}&\text{for every } b>0 \text{ there is } C_b>0 \text{ such that}\\&|f(x,t)|\leqslant C_b\exp_{[\ell]}(b|t|^\gamma)\quad\text{whenever } t\in\mathbb{R} \text{ and } x\in\Omega\,.\end{aligned}\tag{7.1}$$

In this case we do not need to assume (1.7) and (1.13).

THEOREM 7.1. *Let $\ell \in \mathbb{N}$, $n \geqslant 2$, $\alpha < n-1$, $a \in [0,n)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain containing the origin. Suppose that the C^1-Young function $\Phi : [0,\infty) \mapsto [0,\infty)$ satisfies* (1.5) *and* (1.6)*. Let $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying* (1.9)*,* (1.10)*,* (7.1) *and* (1.12)*. Then problem* (1.3) *has a non-trivial weak solution.*

Proof. Since assumptions (1.7) and (1.13) were used in the proof of Lemma 4.5 only, we can use all our partial results but Lemma 4.5.

Fix $w \in W_0L^{\Phi}(\Omega)$ such that $w \geqslant 0$ and $w \neq 0$. By Lemma 4.1 and non-negativity of F (see (1.9)), we observe that

$$\sup_{t\in[0,\infty)} J(tw) < \infty\,. \tag{7.2}$$

Since we have $J(0) = 0$, Lemmata 4.1, 4.2 and Proposition 6.1, we can apply the Mountain Pass Theorem (Theorem 2.7) which together with (7.2) gives us a Palais-Smale sequence $\{u_k\} \subset W_0L^{\Phi}(\Omega)$ approaching a Palais-Smale level $c \in (0,\infty)$.

Moreover, we can find $b_0 > 0$ small enough so that $c \in (0, ((1-\frac{a}{n})\frac{K_{\ell,n,\alpha}}{b_0})^{\frac{n}{\gamma}})$. Finally, since assumption (7.1) implies inequality (1.11) with $b = b_0$, we conclude the proof in the same way as the proof of Theorem 1.1. □

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