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# ON THE MAXIMAL OPERATORS OF VILENKIN–FEJÉR MEANS ON HARDY SPACES

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*Abstract.* The main aim of this paper is to prove that when $0<p<1/2$ the maximal operator $\widetilde{\sigma}_p^{*}f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{(n+1)^{1/p-2}}$ is bounded from the martingale Hardy space $H_p$ to the space $L_p$, where $\sigma_n$ is $n$-th Fejér mean with respect to bounded Vilenkin system.

## 1. Introduction

In one-dimensional case the weak type inequality for maximal operator of Fejér means for trigonometric system can be found in Zygmund [21], in Schipp [11] for Walsh system and in Pál, Simon [10] for bounded Vilenkin system. Fujii [4] and Simon [13] verified that $\sigma^{*,w}$ is bounded from $H_1$ to $L_1$, where $\sigma^{*,w}$ denotes the maximal operator of Fejér means of Walsh-Fourier series. Weisz [18] generalized this result and proved the boundedness of $\sigma^{*,w}$ from the martingale Hardy space $H_p$ to the space $L_p$, for $1/2<p\leqslant 1$. Simon [12] gave a counterexample, which shows that boundedness of $\sigma^{*,w}$ does not hold for $0<p<1/2$. The counterexample for $\sigma^{*,w}$ when $p=1/2$ is due to Goginava [7] (see also [3, 14]). In the endpoint case $p=1/2$ two positive results were showed. Weisz [20] proved that $\sigma^{*,w}$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$. Goginava [6] proved that the maximal operator $\widetilde{\sigma}^{*,w}$ defined by

$$\widetilde{\sigma}^{*,w}f:=\sup_{n\in\mathbb{N}}\frac{|\sigma_n^w f|}{\log^2(n+1)}$$

is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, where $\sigma_n^w$ is $n$-th Fejér means of Walsh-Fourier series. He also proved, that for any nondecreasing function $\varphi:\mathbb{N}\to[1,\ \infty)$, satisfying the condition

$$\overline{\lim_{n\to\infty}}\frac{\log^2(n+1)}{\varphi(n)}=+\infty,$$

the maximal operator

$$\sup_{n\in\mathbb{N}}\frac{|\sigma_n^w f|}{\varphi(n)}$$

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is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

For Walsh-Kaczmarz system analogical theorem was proved in [9] and for bounded Vilenkin system in [15].

The main aim of this paper is to prove that when $0<p<1/2$ the maximal operator

$$\widetilde{\sigma}_p^* f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{(n+1)^{1/p-2}} \tag{1}$$

is bounded from the Hardy space $H_p$ to the space $L_p$ (see Theorem 1), where $\sigma_n$ is Fejér means of bounded Vilenkin-Fourier series.

We also prove that for any nondecreasing function $\varphi:\mathbb{N}\to[1,\ \infty),$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-2}}{\varphi(n)} = +\infty, \tag{2}$$

the maximal operator

$$\sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\varphi(n)}$$

is not bounded from the Hardy space $H_p$ to the space $L_{p,\infty}$ when $0<p<1/2.$ Actually, we prove a stronger result (see Theorem 2) than the unboundedness of the maximal operator $\widetilde{\sigma}_p^*$ from the Hardy space $H_p$ to the spaces $L_{p,\infty}$. In particular, we prove that under condition (2) there exists a martingale $f\in H_p$ $(0<p<1/2)$ such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sigma_n f}{\varphi(n)} \right\|_{L_{p,\infty}} = \infty.$$

## 2. Definitions and Notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N}:=\mathbb{N}_+\cup\{0\}$.

Let $m:=(m_{0,\ldots,}\ m_n,\ldots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0,1,\ldots,\ m_k-1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$, as the complete direct product of the group $Z_{m_j}$, with the product of the discrete topologies of $Z_{m_j}$ 's.

The direct product $\mu$ of the measures

$$\mu_k(\{j\}) := 1/m_k, \qquad (j\in Z_{m_k}),$$

is the Haar measure on $G_m$, with $\mu(G_m)=1$.

If $\sup_n m_n<\infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded, then $G_m$ is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of $G_m$ represented by sequences

$$x := (x_0,x_1,\ldots,x_j,\ldots), \qquad (x_k\in Z_{m_k}).$$

It is easy to give a base, for the neighborhood of $x \in G_m$ :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}, \ (n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$.

Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m, \qquad (n \in \mathbb{N}).$$

Denote

$$I_N^{k,l} := \begin{cases} I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, 0, x_l \neq 0, x_{l+1}, \ldots, x_{N-1}, \ldots), \\ \text{where } x_i \in Z_{m_i},\ i \geqslant l+1, \text{ for } k < l < N, \\ I_N(0, \ldots, 0, x_k \neq 0, 0, \ldots, , x_{N-1} = 0,\ x_N, \ldots), \\ \text{where } x_i \in Z_{m_i},\ i \geqslant N, \text{ for } l = N. \end{cases}$$

and

$$I_N^{k,\alpha,l,\beta} := I_N(0, \ldots, 0, x_k = \alpha, 0, \ldots, 0, x_l = \beta, x_{l+1,\ldots,} x_{N-1}),\ \ k < l < N,$$

where $x_i \in Z_{m_i},\ i \geqslant l+1$.

It is evident

$$I_N^{k,l} = \bigcup_{\alpha=1}^{m_k-1} \bigcup_{\beta=1}^{m_l-1} I_N^{k,\alpha,l,\beta} \tag{3}$$

and

$$\overline{I_N} = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \bigcup \left( \bigcup_{k=1}^{N-1} I_N^{k,N} \right). \tag{4}$$

If we define the so-called generalized number system, based on $m$ in the following way :

$$M_0 := 1, \qquad M_{k+1} := m_k M_k, \qquad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j} \ \ (j \in \mathbb{N})$ and only a finite number of $n_j$'s differ from zero. Let $|n| := \max\ \{j \in \mathbb{N},\ n_j \neq 0\}$.

It is easy to show that

$$\sum_{A=0}^{l} M_A \leqslant cM_l. \tag{5}$$

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesque space.

Next, we introduce on $G_m$ an ortonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \to C,$ the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k/m_k), \qquad \left(i^2 = -1,\, x \in G_m,\, k \in \mathbb{N}\right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \qquad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh one if $m \equiv 2$.

The Vilenkin system is ortonormal and complete in $L_2(G_m)$ [1, 16].

Now we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels with respect to the Vilenkin system $\psi$ in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n d\mu, \qquad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\,\psi_k\,, \qquad (n \in \mathbb{N}_+,\, S_0 f := 0),$$

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} S_k f, \qquad (n \in \mathbb{N}_+),$$

$$D_n := \sum_{k=0}^{n-1} \psi_k\,, \qquad (n \in \mathbb{N}_+),$$

$$K_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k, \qquad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if} \quad x \in I_n, \\ 0\,, & \text{if} \quad x \notin I_n. \end{cases} \tag{6}$$

It is well-known that

$$\sup_n \int_{G_m} |K_n(x)|\, d\mu(x) \leqslant c < \infty, \tag{7}$$

and

$$n\,|K_n(x)| \leqslant c \sum_{A=0}^{|n|} M_A\,|K_{M_A}(x)|\,. \tag{8}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p\, d\mu(x)\right)^{1/p}, \qquad (0 < p < \infty).$$

The space $L_{p,\infty}(G_m)$ consists of all measurable functions $f$, for which

$$\|f\|_{L_{p,\infty}(G_m)} := \sup_{\lambda>0}\lambda^{p}\mu(f>\lambda) < +\infty.$$

The $\sigma-$algebra generated by the intervals $\{I_n(x): x\in G_m\}$ will be denoted by $\mathcal{F}_n$ $(n\in\mathbb{N})$. Denote by $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ a martingale with respect to $\mathcal{F}_n$ $(n\in\mathbb{N})$ (for details see e.g. [17]). The maximal function of a martingale $f$ is defend by

$$f^{*}=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|.$$

In case $f\in L_1$, the maximal functions are also be given by

$$f^{*}(x)=\sup_{n\in\mathbb{N}}\frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f(u)\mu(u)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p$ $(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^{*}\|_p < \infty.$$

If $f\in L_1$, then it is easy to show that the sequence $(S_{M_n}(f): n\in\mathbb{N})$ is a martingale. If $f=\left(f^{(n)}, n\in\mathbb{N}\right)$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty}\int_{G_m} f^{(k)}(x)\overline{\Psi}_i(x)\,d\mu(x).$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as martingale $(S_{M_n}(f): n \in \mathbb{N}$ obtained from $f$.

For the martingale $f$ we consider maximal operators

$$\sigma^{*}f := \sup_{n\in\mathbb{N}}|\sigma_n f|,$$

$$\widetilde{\sigma}^{*}f := \sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{\log^{2}(n+1)},$$

$$\widetilde{\sigma}_p^{*}f := \sup_{n\in\mathbb{N}}\frac{|\sigma_n f|}{(n+1)^{1/p-2}}.$$

A bounded measurable function $a$ is p-atom, if there exist interval $I$, such that

$$a)\ \int_I a\,d\mu = 0,$$

$$b)\ \|a\|_\infty \leqslant \mu(I)^{-1/p},$$

$$c)\ \operatorname{supp}(a)\subset I.$$

## 3. Formulation of main results

THEOREM 1. *Let* $0<p<1/2$. *Then the maximal operator* $\widetilde{\sigma}_p^*$ *is bounded from the Hardy martingale space* $H_p(G_m)$ *to the space* $L_p(G_m)$.

THEOREM 2. *Let* $\varphi:\mathbb{N}\to[1,\ \infty)$ *be a nondecreasing function, satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{(n+1)^{1/p-2}}{\varphi(n)}=+\infty. \tag{9}$$

Then there exists a martingale $f\in H_{1/2}$, such that

$$\sup_{n\in\mathbb{N}}\left\|\frac{\sigma_n f}{\varphi(n)}\right\|_{L_{p,\infty}}=\infty.$$

## 4. Auxiliary propositions

LEMMA 1. [19] *Suppose that an operator* $T$ *is sublinear and for some* $0<p\leqslant 1$

$$\int_{\overline{I}}|Ta|^p\,d\mu\leqslant c_p<\infty,$$

for every $p$-atom $a$, where $I$ denote the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$, then

$$\|Tf\|_{L_p(G_m)}\leqslant c_p\|f\|_{H_p(G_m)}.$$

LEMMA 2. [2, 8] *Let* $2<A\in\mathbb{N}_+,\ k\leqslant s<A$ *and* $q_A=M_{2A}+M_{2A-2}+\ldots+M_2+M_0$. *Then*

$$q_{A-1}\left|K_{q_{A-1}}(x)\right|\geqslant\frac{M_{2k}M_{2s}}{4},$$

*for*

$$x\in I_{2A}(0,\ldots,x_{2k}\neq 0,0,\ldots,0,x_{2s}\neq 0,x_{2s+1},\ldots x_{2A-1}),$$
$$k=0,\ 1,\ldots,\ A-3.\qquad s=k+2,\ k+3,\ldots,\ A-1.$$

LEMMA 3. [5] *Let* $A>t,\ t,A\in\mathbb{N},\ z\in I_t\setminus\ I_{t+1}$. *Then*

$$K_{M_A}(z)=\begin{cases}0, & \text{if } z-z_te_t\notin I_A,\\ \frac{M_t}{1-r_t(z)}, & \text{if } z-z_te_t\in I_A.\end{cases}$$

LEMMA 4. *Let* $x \in I_N^{k,l}$ , $k = 0,...,N-1,\ l = k+1,...,N$. *Then*

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leqslant \frac{cM_l M_k}{M_N^2}, \text{ when } n \geqslant M_N.$$

*Proof.* Let $x \in I_N^{k,\alpha,l,\beta}$. Then applying Lemma 3 we have

$$K_{M_A}(x) = 0, \text{ when } A > l.$$

Let $k < A \leqslant l$. Then we get

$$|K_{M_A}(x)| = \frac{M_k}{|1 - r_k(x)|} \leqslant \frac{m_k M_k}{2\pi\,\alpha}. \tag{10}$$

Let $x \in I_N^{k,l}$, for $0 \leqslant k < l \leqslant N-1$ and $t \in I_N$. Since $x - t \in I_N^{k,l}$ and $n \geqslant M_N$, combining (3), (5), (8) and (10) we obtain

$$n|K_n(x)| \leqslant c\sum_{A=0}^{l} M_A M_k \leqslant cM_k M_l$$

and

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leqslant \frac{cM_k M_l}{M_N^2}. \tag{11}$$

Let $x \in I_N^{k,N}$ , then applying (8) we have

$$\int_{I_N} n|K_n(x-t)|\, d\mu(t) \leqslant \sum_{A=0}^{|n|} M_A \int_{I_N} |K_{M_A}(x-t)|\, d\mu(t). \tag{12}$$

Let

$$\begin{cases} x = \left(0,...,0,x_k \neq 0,0,...0,x_N,x_{N+1},x_q,...,x_{|n|-1},...\right), \\ t = \left(0,...,0,x_N,...x_{q-1},t_q \neq x_q,t_{q+1},...,t_{|n|-1},...\right), \ q = N,...,|n|-1. \end{cases}$$

Using Lemma 3 in (12) it is easy to show that

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leqslant \frac{c}{n}\sum_{A=0}^{q-1} M_A \int_{I_N} M_k\, d\mu(t) \leqslant \frac{cM_k M_q}{nM_N} \leqslant \frac{cM_k}{M_N}. \tag{13}$$

Let

$$\begin{cases} x = \left(0,...,0,x_m \neq 0,0,...,0,x_N,x_{N+1},x_q,...,x_{|n|-1},...\right), \\ t = \left(0,0,...,x_N,...,x_{|n|-1},...\right). \end{cases}$$

If we apply Lemma 3 in (12) we obtain

$$\int_{I_N} |K_n(x-t)|\, d\mu(t) \leqslant \frac{c}{n}\sum_{A=0}^{|n|-1} M_A \int_{I_N} M_k\, d\mu(t) \leqslant \frac{cM_k}{M_N}. \tag{14}$$

Combining (11), (13) and (14) we complete the proof of Lemma 4. □

## 5. Proofs of the Theorems

*Proof of Theorem 1.* By Lemma 1, the proof of Theorem 1 will be complete, if we show that

$$\int_{\overline{I}_N}\left(\sup_{n\in\mathbb{N}}\frac{|\sigma_n a|}{(n+1)^{1/p-2}}\right)^p d\mu \leqslant c<\infty,$$

for every p-atom $a$, where $I$ denotes the support of the atom. The boundedness of $\sup_{n\in\mathbb{N}}|\sigma_n|/(n+1)^{1/p-2}$ from $L_\infty$ to $L_\infty$ follows from (7).

Let $a$ be an arbitrary p-atom, with support $I$ and $\mu(I)=M_N^{-1}$. We may assume that $I=I_N$. It is easy to see that $\sigma_n(a)=0$, when $n\leqslant M_N$. Therefore, we can suppose that $n>M_N$.

Since $\|a\|_\infty\leqslant cM_N^{1/p}$ we can write

$$\begin{aligned}\frac{|\sigma_n(a)|}{(n+1)^{1/p-2}} &\leqslant \frac{1}{(n+1)^{1/p-2}}\int_{I_N}|a(t)||K_n(x-t)|\,d\mu(t)\\ &\leqslant \frac{\|a\|_\infty}{(n+1)^{1/p-2}}\int_{I_N}|K_n(x-t)|\,d\mu(t)\\ &\leqslant \frac{cM_N^{1/p}}{(n+1)^{1/p-2}}\int_{I_N}|K_n(x-t)|\,d\mu(t).\end{aligned}$$

Let $x\in I_N^{k,l}$, $0\leqslant k<l\leqslant N$. From Lemma 4 we get

$$\frac{|\sigma_n(a)|}{(n+1)^{1/p-2}}\leqslant \frac{cM_N^{1/p}}{M_N^{1/p-2}}\frac{M_lM_k}{M_N^2}=cM_lM_k. \tag{15}$$

Combining (4) and (15) we obtain

$$\begin{aligned}&\int_{\overline{I_N}}|\sigma^* a(x)|^p\,d\mu(x)\\ &=\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\sum_{x_j=0,j\in\{l+1,\dots,N-1\}}^{m_{j-1}}\int_{I_N^{k,l}}|\sigma^* a(x)|^p\,d\mu(x)+\sum_{k=0}^{N-1}\int_{I_N^{k,N}}|\sigma^* a(x)|^p\,d\mu(x)\\ &\leqslant c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{m_{l+1}\dots m_{N-1}}{M_N}(M_lM_k)^p+c\sum_{k=0}^{N-1}\frac{1}{M_N}(M_NM_k)^p\\ &\leqslant c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{(M_lM_k)^p}{M_l}+c\sum_{k=0}^{N-1}\frac{M_k^p}{M_N^{1-p}}=I+II.\end{aligned}$$

Then

$$I = c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{1}{M_l^{1-2p}}\frac{(M_lM_k)^p}{M_l^{2p}}$$
$$\leqslant c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{1}{M_l^{1-2p}}$$
$$\leqslant c\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{1}{2^{l(1-2p)}}$$
$$\leqslant c\sum_{k=0}^{N-2}\frac{1}{2^{k(1-2p)}} < c < \infty.$$

It is evident

$$II \leqslant \frac{c}{M_N^{1-2p}} < c < \infty.$$

Which complete the proof of Theorem 1. □

*Proof of Theorem 2.* Let $0 < p < 1/2$ and $\{\lambda_k;\ k\in\mathbb{N}_+\}$ be an increasing sequence of the positive integers, such that

$$\lim_{k\to\infty}\frac{\lambda_k^{1/p-2}}{\varphi(\lambda_k)} = \infty.$$

It is evident that for every $\lambda_k$, there exists a positive integers $m_k^{'}$, such that $q_{m_k^{'}} < \lambda_k < cq_{m_k^{'}}$. Since $\varphi(n)$ is nondecreasing function, we have

$$\overline{\lim_{k\to\infty}}\frac{M_{2m_k^{'}}^{1/p-2}}{\varphi\left(q_{m_k^{'}}\right)} \geqslant c\overline{\lim_{k\to\infty}}\frac{q_{m_k^{'}}^{1/p-2}}{\varphi\left(q_{m_k^{'}}\right)}$$
$$\geqslant c\lim_{k\to\infty}\frac{\lambda_k^{1/p-2}}{\varphi(\lambda_k)} = \infty. \tag{16}$$

Let $\{n_k;\ k\in\mathbb{N}_+\}\subset\left\{m_k^{'};\ k\in\mathbb{N}_+\right\}$ such that

$$\lim_{k\to\infty}\frac{M_{2n_k}^{1/p-2}}{\varphi\left(q_{n_k}\right)} = \infty$$

and

$$f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x), \qquad n_k \geqslant 3.$$

It is evident

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, \text{ if } i = M_{2n_k},\dots,M_{2n_k+1}-1, \\ 0, \text{otherwise.}\end{cases}$$

Then we can write

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), \text{ if } i = M_{2n_k}, \dots, M_{2n_k+1} - 1, \\ f_{n_k}(x), \text{ if } i \geqslant M_{2n_k+1}, \\ 0, \qquad \text{otherwise.} \end{cases} \tag{17}$$

From (6) we get

$$\begin{aligned}
\left\|f_{n_k}\right\|_{H_p} &= \left\|\sup_{n\in\mathbb{N}} S_{M_n}\left(f_{n_k}\right)\right\|_{L_p} \\
&= \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_{L_p} \\
&= \left(\int_{I_{2n_k} \setminus I_{2n_k+1}} M_{2n_k}^p \, d\mu(x) + \int_{I_{2n_k+1}} \left(M_{2n_k+1} - M_{2n_k}\right)^p d\mu(x)\right)^{1/p} \\
&= \left(\frac{m_{2n_k} - 1}{M_{2n_k+1}} M_{2n_k}^p + \frac{\left(m_{2n_k} - 1\right)^p}{M_{2n_k+1}} M_{2n_k}^p\right)^{1/p} \\
&\leqslant M_{2n_k}^{1-1/p}.
\end{aligned}$$

By (17) we can write:

$$\begin{aligned}
\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} &= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k}-1} S_j f_{n_k}(x)\right| \\
&= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=M_{2n_k}}^{q_{n_k}-1} S_j f_{n_k}(x)\right| \\
&= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=M_{2n_k}}^{q_{n_k}-1} \left(D_j(x) - D_{M_{2n_k}}(x)\right)\right| \\
&= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k-1}-1} \left(D_{j+M_{2n_k}}(x) - D_{M_{2n_k}}(x)\right)\right|.
\end{aligned}$$

Since

$$D_{j+M_{2n_k}}(x) - D_{M_{2n_k}}(x) = \psi_{M_{2n_k}} D_j, \qquad j = 1, 2, .., M_{2n_k} - 1,$$

we obtain

$$\begin{aligned}
\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} &= \frac{1}{\varphi\left(q_{n_k}\right) q_{n_k}} \left|\sum_{j=0}^{q_{n_k-1}-1} D_j(x)\right| \\
&= \frac{1}{\varphi\left(q_{n_k}\right)} \frac{q_{n_k-1}}{q_{n_k}} \left|K_{q_{n_k-1}}(x)\right|.
\end{aligned}$$

Let $x \in I_{2n_k}^{2s,2l}$. Using Lemma 2 we obtain

$$\frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} \geqslant \frac{cM_{2s}M_{2l}}{M_{2n_k}\varphi\left(q_{n_k}\right)}.$$

Hence we can write:

$$\begin{aligned}
&\mu\left\{x \in G_m : \frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} \geqslant \frac{c}{M_{2n_k}\varphi\left(q_{n_k}\right)}\right\} \\
&\qquad \geqslant \mu\left\{x \in I_{2n_k}^{2,4} : \frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} \geqslant \frac{c}{M_{2n_k}\varphi\left(q_{n_k}\right)}\right\} \\
&\qquad \geqslant \mu\left(I_{2n_k}^{2,4}\right) > c > 0.
\end{aligned} \tag{18}$$

From (18) we have

$$\begin{aligned}
&\frac{\frac{c}{M_{2n_k}\varphi\left(q_{n_k}\right)}\left(\mu\left\{x \in G_m : \frac{\left|\sigma_{q_{n_k}} f_{n_k}(x)\right|}{\varphi\left(q_{n_k}\right)} \geqslant \frac{c}{M_{2n_k}\varphi\left(q_{n_k}\right)}\right\}\right)^{1/p}}{\left\|f_{n_k}(x)\right\|_{H_p}} \\
&\geqslant \frac{c}{M_{2n_k}\varphi\left(q_{n_k}\right)M_{2n_k}^{1-1/p}} \\
&= c\frac{M_{2n_k}^{1/p-2}}{\varphi\left(q_{n_k}\right)} \to \infty, \qquad \text{when } k \to \infty.
\end{aligned}$$

Theorem 2 is proved. □

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