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NEW PERTURBATION BOUNDS FOR NONNEGATIVE AND POSITIVE POLAR FACTORS

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Abstract. The changes in the nonnegative and positive polar factors of generalized polar decomposition and polar decomposition are studied under the additive perturbation. Some new perturbation bounds are obtained. These bounds are different from precious ones in form and measure and, in some cases, may be smaller than the corresponding existing ones. Furthermore, the corresponding perturbation bounds for generalized nonnegative and positive polar factors of (M,N) weighted polar decomposition are also presented as the straightforward corollaries.

1. Introduction and preliminaries

Let $\mathbb{C}^{m\times n}$ and $\mathbb{C}_r^{m\times n}$ denote the set of $m\times n$ complex matrices and the subset of $\mathbb{C}^{m\times n}$ comprising matrices with rank r, respectively. Let I_r be the identity matrix of order r. Given $A\in\mathbb{C}^{m\times n}$, the symbols A^*, $A^\dagger$, $R(A)$, $\|A\|_2$, $\|A\|_F$, and $\|A\|$ stand for its conjugate transpose, Moore–Penrose inverse, range, spectral norm, Frobenius norm, and unitarily invariant norm, respectively.

For a matrix $A\in\mathbb{C}_r^{m\times n}$, there is a partial isometric matrix $Q\in\mathbb{C}^{m\times n}$ and a Hermitian positive semidefinite matrix $H\in\mathbb{C}^{n\times n}$ such that

$$A = QH. \tag{1.1}$$

The decomposition (1.1) is called the generalized polar decomposition of A (e.g., [2, 25]), and Q and H are called the subunitary polar factor and nonnegative polar factor of this decomposition, respectively. Usually, when $\operatorname{rank}(A)=n$, the decomposition (1.1) is called the polar decomposition and H is the positive polar factor.

In general, the generalized polar decomposition is not unique, while it has been proved that it is unique if the decomposition satisfies

$$R(Q^*) = R(H). \tag{1.2}$$

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The condition (1.2) can be found in [2]. Under this condition, the generalized polar decomposition (1.1) can be calculated from the singular value decomposition [2]

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* = U_1 \Sigma V_1^* \tag{1.3}$$

by

$$Q = U_1 V_1^* \text{ and } H = V_1 \Sigma V_1^*, \tag{1.4}$$

where $U = (U_1, U_2) \in \mathbb{C}_m^{m\times m}$ and $V = (V_1, V_2) \in \mathbb{C}_n^{n\times n}$ are unitary, $U_1 \in \mathbb{C}_r^{m\times r}, V_1 \in \mathbb{C}_r^{n\times r}$, $\Sigma = \operatorname{diag}(\sigma_1, \cdots, \sigma_r)$, and $\sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$ are the nonzero singular values of A. In this paper, we assume that the condition (1.2) is always satisfied. Note that the condition (1.2) is automatic when $\operatorname{rank}(A) = n$ and in this case

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^* = U_1 \Sigma V^*, Q = U_1 V^*, \text{ and } H = V \Sigma V^*. \tag{1.5}$$

The problem of estimating the perturbation bounds for the generalized polar decomposition and polar decomposition has been studied by many authors for the additive perturbation in various norms (e.g., [1, 3–10, 12–15, 17–22, 25, 30]). The additive perturbation refers to the situation when the perturbed matrix $\widetilde{A}$ is represented as $A+E$. In the past works, most attention was given to how the subunitary or unitary factor Q changed (e.g., [1, 3, 6–10, 12, 14, 15, 18–22, 25, 30]), but only some to how the nonnegative or positive polar factor H changed (e.g., [1, 3–6, 8, 10, 12, 13, 17, 22, 25, 30]). In the present paper, we focus our attention on the perturbation bounds for nonnegative and positive polar factors, and derive some new bounds in the unitary invariant norm, spectral norm, and Frobenius norm. These new bounds are different from precious ones in form and measure. As a result, they may be smaller than the corresponding existing bounds including the ones that are considered to be best. In addition, as pointed out in [28, 30], the (M,N) weighted polar decomposition, as a generalization of the generalized polar decomposition and polar decomposition, also has some important applications. Therefore, we also provide the corresponding perturbation bounds for generalized nonnegative and positive polar factors of this decomposition in Section 3.

The following are several existing perturbation bounds for nonnegative and positive polar factors, which will be used to compare with the results obtained in this paper.

Let $A = QH$ and $\widetilde{A} = \widetilde{Q}\widetilde{H}$ be the generalized polar decompositions or polar decompositions of A and $\widetilde{A} = A + E$, respectively. If $A, \widetilde{A} \in \mathbb{C}_n^{n\times n}$, the following result holds, which could be found in [3]:

$$\left\|\widetilde{H} - H\right\| \leqslant \left(1 + \frac{2\min\{\sigma_1, \widetilde{\sigma}_1\}}{\sigma_n + \widetilde{\sigma}_n}\right) \|E\|, \tag{1.6}$$

where σ_1, σ_n and $\widetilde{\sigma}_1, \widetilde{\sigma}_n$ are the biggest and smallest nonzero singular values of A and $\widetilde{A}$, respectively. If $A, \widetilde{A} \in \mathbb{C}_n^{m\times n}$ and $\|E\|_2 \leqslant \sigma_n$, Chen and Li [6] improved the bound (1.6) to some extent and obtained the following result:

$$\left\|\widetilde{H} - H\right\| \leqslant \left[(\kappa(A)+1)\omega(\left\|A^\dagger\right\|_2 \|E\|_2) - 1\right] \|E\| \tag{1.7}$$

$$\leqslant \frac{\kappa(A)}{1-\|A^{\dagger}\|_2\,\|E\|_2}\,\|E\|\,, \tag{1.8}$$

where $\omega(\varepsilon)=\frac{1}{\varepsilon}\ln\frac{1}{1-\varepsilon}$ and $\kappa(A)=\|A\|_2\left\|A^{\dagger}\right\|_2$.

As the special case of the result given in [30], we have the following perturbation bound for nonnegative polar factor, i.e, if $A,\widetilde{A}\in\mathbb{C}_r^{m\times n}$, then

$$\left\|\widetilde{H}-H\right\|\leqslant\left(2+\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\right)\|E\|\,, \tag{1.9}$$

where $\sigma_r,\widetilde{\sigma}_r$ are the smallest nonzero singular values of $A,\widetilde{A}$, respectively. Moreover, an alternative perturbation bound for positive polar factor can be also got from [30], i.e, if $A,\widetilde{A}\in\mathbb{C}_n^{m\times n}$ or $A,\widetilde{A}\in\mathbb{C}_n^{n\times n}$, then

$$\left\|\widetilde{H}-H\right\|\leqslant\left(\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_n+\widetilde{\sigma}_n}\right)\|E\|\,. \tag{1.10}$$

This bound was rediscovered by Chen [5] recently. Furthermore, in [5], Chen also presented the following perturbation bound for the positive polar factor:

$$\left\|\widetilde{H}-H\right\|\leqslant\left(\frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\right)\left(\left\|EA^{\dagger}\right\|+\left\|E\widetilde{A}^{\dagger}\right\|\right). \tag{1.11}$$

In the spectral norm, the following perturbation bound for positive polar factor can be found in [4], i.e., if $A,\widetilde{A}\in\mathbb{C}_n^{n\times n}$, then

$$\left\|\widetilde{H}-H\right\|_2\leqslant\left(1+\log(\max\{\sigma_1,\widetilde{\sigma}_1\}\max\{\frac{1}{\sigma_n},\frac{1}{\widetilde{\sigma}_n}\}\right)\|E\|_2\,. \tag{1.12}$$

Moreover, in [4], Bhatia also presented the following bound

$$\left\|\widetilde{H}-H\right\|_2\leqslant C(n)\,\|E\|_2\,, \tag{1.13}$$

where $C(n)=O(\log n)$ and is considered to be best possible.

In Frobenius norm, the following perturbation bound for nonnegative or positive polar factor can be found in [10, 13, 22, 25], i.e., if $A,\widetilde{A}\in\mathbb{C}_r^{m\times n}$, $\mathbb{C}_n^{m\times n}$, or $\mathbb{C}_n^{n\times n}$, then

$$\left\|\widetilde{H}-H\right\|_F\leqslant\sqrt{2}\,\|E\|_F\,. \tag{1.14}$$

In general, the factor $\sqrt{2}$ is considered to be best possible. Chen and Li [8] improved the bound (1.14) for positive polar factor, i.e., if $A,\widetilde{A}\in\mathbb{C}_n^{m\times n}$, then

$$\left\|\widetilde{H}-H\right\|_F\leqslant\sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_n^2}}{\sigma_1+\widetilde{\sigma}_n},\frac{\sqrt{\widetilde{\sigma}_1^2+\sigma_n^2}}{\widetilde{\sigma}_1+\sigma_n}\right\}\|E\|_F\,. \tag{1.15}$$

In order to obtain the results of this paper, two lemmas are needed, where Lemma 1.1 can be found in [11] and Lemma 1.2 can be found in [16].

LEMMA 1.1. *Let* $\Omega \in \mathbb{C}^{s\times s}$ *and* $\Gamma \in \mathbb{C}^{t\times t}$ *be two Hermitian matrices, and* $S \in \mathbb{C}^{s\times t}$, *and*

$$\Delta = [\alpha,\beta] \subset R,\ \Delta' = R\setminus[\alpha-\delta,\beta+\delta],\ \delta > 0.$$

Let $\lambda(\Omega)$ *and* $\lambda(\Gamma)$ *denote the eigenvalue sets of* Ω *and* Γ, *respectively. If*

$$\lambda(\Omega) \subset \Delta,\ \lambda(\Gamma) \subset \Delta',$$

then the equation $\Omega X - X\Gamma = S$ *has a unique solution* $X \in \mathbb{C}^{s\times t}$, *and moreover,* $\|X\| \leqslant \frac{\|S\|}{\delta}$ *for any unitarily invariant norm.*

LEMMA 1.2. *Let* $\Omega \in \mathbb{C}^{s\times s}$ *and* $\Gamma \in \mathbb{C}^{t\times t}$ *be two Hermitian matrices, and let* $\lambda(\Omega)$ *and* $\lambda(\Gamma)$ *denote the sets of eigenvalues of* Ω *and* Γ, *respectively. If* $\lambda(\Omega)\cap\lambda(\Gamma) = \emptyset$, *then for any* $E,F \in \mathbb{C}^{s\times t}$ *the equation* $\Omega X - X\Gamma = \Omega E + F\Gamma$ *has a unique solution* $X \in \mathbb{C}^{s\times t}$, *and moreover,*

$$\|X\|_F \leqslant \frac{1}{\eta}\sqrt{\|E\|_F^2 + \|F\|_F^2},$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{\sqrt{|\omega|^2+|\gamma|^2}}$. *If, in addition,* $F = 0$, *then we have a better bound*

$$\|X\|_F \leqslant \frac{1}{\eta}\|E\|_F,$$

where $\eta = \min_{\omega\in\lambda(\Omega),\gamma\in\lambda(\Gamma)} \frac{|\omega-\gamma|}{|\omega|}$.

2. New perturbation bounds for nonnegative and positive polar factors

Let the perturbed matrix be $\widetilde{A} \in \mathbb{C}_r^{m\times n}$. Similar to (1.1) and (1.3), we present the generalized polar decomposition and singular value decomposition of $\widetilde{A}$ as follows:

$$\widetilde{A} = \widetilde{Q}\widetilde{H}, \quad \widetilde{A} = \widetilde{U}\begin{pmatrix} \widetilde{\Sigma} & 0 \\ 0 & 0 \end{pmatrix}\widetilde{V}^* = \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}_1^*, \tag{2.1}$$

in which

$$\widetilde{Q} = \widetilde{U}_1\widetilde{V}_1^*, \quad \widetilde{H} = \widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*, \tag{2.2}$$

where $\widetilde{U} = (\widetilde{U}_1,\widetilde{U}_2) \in \mathbb{C}_m^{m\times m}$ and $\widetilde{V} = (\widetilde{V}_1,\widetilde{V}_2) \in \mathbb{C}_n^{n\times n}$ are unitary, $\widetilde{U}_1 \in \mathbb{C}_r^{m\times r}$, $\widetilde{V}_1 \in \mathbb{C}_r^{n\times r}$, $\widetilde{\Sigma} = \mathrm{diag}(\widetilde{\sigma}_1,\cdots,\widetilde{\sigma}_r)$, and $\widetilde{\sigma}_1 \geqslant \cdots \geqslant \widetilde{\sigma}_r > 0$ are the nonzero singular values of $\widetilde{A}$. When $\mathrm{rank}(\widetilde{A}) = n$,

$$\widetilde{A} = \widetilde{U}\begin{pmatrix} \widetilde{\Sigma} \\ 0 \end{pmatrix}\widetilde{V}^* = \widetilde{U}_1\widetilde{\Sigma}\widetilde{V}^*, \widetilde{Q} = \widetilde{U}_1\widetilde{V}^*, \text{ and } \widetilde{H} = \widetilde{V}\widetilde{\Sigma}\widetilde{V}^*. \tag{2.3}$$

In the following, we first consider the perturbation bounds for nonnegative polar factors.

THEOREM 2.1. *Let $A = QH, \widetilde{A} = \widetilde{Q}\widetilde{H}$ be the generalized polar decompositions of $A, \widetilde{A} = A + E \in \mathbb{C}_r^{m\times n}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\| \leqslant \left(\frac{\sigma_1^2}{\sigma_r + \widetilde{\sigma}_r} + \widetilde{\sigma}_1 + \sigma_1\right) \left\|(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger\right\|, \tag{2.4}$$

where σ_1, σ_r and $\widetilde{\sigma}_1, \widetilde{\sigma}_r$ are the biggest and smallest nonzero singular values of A and $\widetilde{A}$, respectively.

Proof. Using the singular value decomposition of A in (1.3), the Moore-Penrose inverse of A can be expressed as

$$A^\dagger = V \begin{pmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* = V_1 \Sigma^{-1} U_1^*. \tag{2.5}$$

Thus, considering (1.3) and (2.1), we have

$$A^*A = V_1\Sigma^2 V_1^*,\ (A^*A)^\dagger = V_1\Sigma^{-2}V_1^*,\ \widetilde{A}^*\widetilde{A} = \widetilde{V}_1\widetilde{\Sigma}^2\widetilde{V}_1^*.$$

From the above equations and some simple computations, it follows that

$$(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger = \widetilde{V}_1\widetilde{\Sigma}^2\widetilde{V}_1^* V_1 \Sigma^{-2} V_1^* - V_1V_1^*. \tag{2.6}$$

Premultiplying (2.6) by $\widetilde{V}_1^*$ and postmultiplying it by V_1, and noting $\widetilde{V}_1^*\widetilde{V}_1 = V_1^*V_1 = I_r$ implies

$$\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger V_1 = \widetilde{\Sigma}^2\widetilde{V}_1^*V_1\Sigma^{-2} - \widetilde{V}_1^*V_1. \tag{2.7}$$

Postmultiplying (2.7) by Σ^2 gives

$$\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger V_1\Sigma^2 = \widetilde{\Sigma}^2\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma^2,$$

which can be rewritten as

$$\widetilde{\Sigma}(\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma) + (\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma)\Sigma = \widetilde{V}_1^*(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger V_1\Sigma^2. \tag{2.8}$$

Applying Lemma 1.1 to (2.8) with $\Omega = \widetilde{\Sigma}, \Gamma = -\Sigma,\ X = \widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma$, and

$$S = \widetilde{V}_1^*(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger V_1\Sigma^2$$

leads to

$$\|X\| \leqslant \frac{1}{\eta}\|S\|, \tag{2.9}$$

where $\eta = \sigma_r + \widetilde{\sigma}_r$ and

$$\|S\| = \left\|\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger V_1\Sigma^2\right\| \leqslant \sigma_1^2 \left\|(\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^\dagger\right\|. \tag{2.10}$$

Note that

$$\widetilde{V}^*(\widetilde{H}-H)V=\widetilde{V}^*(\widetilde{V}_1\widetilde{\Sigma}\widetilde{V}_1^*-V_1\Sigma V_1^*)V=\begin{pmatrix}\widetilde{\Sigma}\widetilde{V}_1^*V_1-\widetilde{V}_1^*V_1\Sigma & \widetilde{\Sigma}\widetilde{V}_1^*V_2\\ -\widetilde{V}_2^*V_1\Sigma & 0\end{pmatrix}. \tag{2.11}$$

Then

$$\begin{aligned}\left\|\widetilde{H}-H\right\| &= \left\|\widetilde{V}^*(\widetilde{H}-H)V\right\| \leqslant \|X\|+\left\|\widetilde{\Sigma}\widetilde{V}_1^*V_2\right\|+\left\|\widetilde{V}_2^*V_1\Sigma\right\|\\ &\leqslant \|X\|+\widetilde{\sigma}_1\left\|\widetilde{V}_1^*V_2\right\|+\sigma_1\left\|\widetilde{V}_2^*V_1\right\|.\end{aligned} \tag{2.12}$$

Considering that $\widetilde{V}^*V=\begin{pmatrix}\widetilde{V}_1^*V_1 & \widetilde{V}_1^*V_2\\ \widetilde{V}_2^*V_1 & \widetilde{V}_2^*V_2\end{pmatrix}$ is unitary, we have

$$\left\|\widetilde{V}_1^*V_2\right\|=\left\|\widetilde{V}_2^*V_1\right\|, \tag{2.13}$$

which together with (2.12) gives

$$\left\|\widetilde{H}-H\right\| \leqslant \|X\|+(\widetilde{\sigma}_1+\sigma_1)\left\|\widetilde{V}_2^*V_1\right\|. \tag{2.14}$$

Moreover, premultiplying (2.6) by $\widetilde{V}_2^*$ and postmultiplying it by V_1, and noting $\widetilde{V}_2^*\widetilde{V}_1=0, V_1^*V_1=I_r$ yields

$$\widetilde{V}_2^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1=-\widetilde{V}_2^*V_1.$$

Thus,

$$\left\|\widetilde{V}_2^*V_1\right\|=\left\|\widetilde{V}_2^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\| \leqslant \left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|. \tag{2.15}$$

Then, it follows from (2.14), (2.9), (2.10), and (2.15) that the bound (2.4) holds. □

If the unitary invariant norm in Theorem 2.1 is replaced by the spectral norm, we have the following smaller perturbation bound.

COROLLARY 2.2. *Assume that the conditions of Theorem* 2.1 *hold. Then*

$$\left\|\widetilde{H}-H\right\|_2 \leqslant \left(\frac{\sigma_1^2}{\sigma_r+\widetilde{\sigma}_r}+\max\{\widetilde{\sigma}_1,\sigma_1\}\right)\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_2. \tag{2.16}$$

Proof. Note that (2.11) can be rewritten as

$$\widetilde{V}^*(\widetilde{H}-H)V=\begin{pmatrix}\widetilde{\Sigma}\widetilde{V}_1^*V_1-\widetilde{V}_1^*V_1\Sigma & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & \widetilde{\Sigma}\widetilde{V}_1^*V_2\\ -\widetilde{V}_2^*V_1\Sigma & 0\end{pmatrix}.$$

Then

$$\left\|\widetilde{H}-H\right\|_2 = \left\|\widetilde{V}^*(\widetilde{H}-H)V\right\|_2 \leqslant \left\|\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma\right\|_2 + \left\|\begin{pmatrix} 0 & \widetilde{\Sigma}\widetilde{V}_1^*V_2 \\ -\widetilde{V}_2^*V_1\Sigma & 0 \end{pmatrix}\right\|_2$$
$$\leqslant \|X\|_2 + \max\left\{\widetilde{\sigma}_1\left\|\widetilde{V}_1^*V_2\right\|_2, \sigma_1\left\|\widetilde{V}_2^*V_1\right\|_2\right\}.$$

Thus, the bound (2.16) follows from (2.9), (2.10), (2.13), and (2.15). □

If we replace the unitary invariant norm in Theorem 2.1 with the Frobenius norm, the following alternative perturbation bound can be obtained.

THEOREM 2.3. *Assume that the conditions of Theorem* 2.1 *hold. Then*

$$\left\|\widetilde{H}-H\right\|_F \leqslant \sqrt{\sigma_1^2+\widetilde{\sigma}_1^2}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_F. \tag{2.17}$$

Proof. Note that (2.8) can be rewritten as

$$\widetilde{\Sigma}(\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma) + (\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma)\Sigma = \widetilde{V}_1^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\Sigma\Sigma. \tag{2.18}$$

Then, applying Lemma 1.2 to (2.18) with $\Omega = \widetilde{\Sigma}$, $\Gamma = -\Sigma$, $X = \widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma$, $E = 0$, and $F = -\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\Sigma$, we have

$$\|X\|_F \leqslant \frac{1}{\eta}\|F\|_F, \tag{2.19}$$

where $\eta = \frac{\sigma_1+\widetilde{\sigma}_r}{\sigma_1}$ and

$$\|F\|_F \leqslant \sigma_1\left\|\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\|_F. \tag{2.20}$$

From (2.11), we have

$$\left\|\widetilde{H}-H\right\|_F^2 = \left\|\widetilde{V}^*(\widetilde{H}-H)V\right\|_F^2 = \left\|\widetilde{\Sigma}\widetilde{V}_1^*V_1 - \widetilde{V}_1^*V_1\Sigma\right\|_F^2 + \left\|\widetilde{\Sigma}\widetilde{V}_1^*V_2\right\|_F^2 + \left\|\widetilde{V}_2^*V_1\Sigma\right\|_F^2$$
$$\leqslant \|X\|_F^2 + \widetilde{\sigma}_1^2\left\|\widetilde{V}_1^*V_2\right\|_F^2 + \sigma_1^2\left\|\widetilde{V}_2^*V_1\right\|_F^2. \tag{2.21}$$

Substituting (2.19), (2.20), (2.13), and (2.15) into (2.21) and considering the property of Frobenius norm implies

$$\left\|\widetilde{H}-H\right\|_F^2 \leqslant \frac{\sigma_1^2}{\eta^2}\left\|\widetilde{V}_1^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\|_F^2$$
$$+ (\widetilde{\sigma}_1^2+\sigma_1^2)\left\|\widetilde{V}_2^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\|_F^2$$
$$= \frac{\sigma_1^2}{\eta^2}\left\|(\widetilde{V}_1,\widetilde{V}_2)^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\|_F^2$$
$$+ \left(\widetilde{\sigma}_1^2 + (1-\frac{1}{\eta^2})\sigma_1^2\right)\left\|\widetilde{V}_2^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger V_1\right\|_F^2$$

$$
\begin{aligned}
&\leqslant \frac{\sigma_1^2}{\eta^2}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_F^2 \\
&\quad+\left(\widetilde{\sigma}_1^2+(1-\frac{1}{\eta^2})\sigma_1^2\right)\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_F^2 \\
&=(\widetilde{\sigma}_1^2+\sigma_1^2)\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_F^2.
\end{aligned}
$$

Taking the square root on both sides gives the bound (2.17). □

Now, we consider the perturbation bounds for positive polar factors.

THEOREM 2.4. *Let* $A=QH,\widetilde{A}=\widetilde{Q}\widetilde{H}$ *be the polar decompositions of* $A,\widetilde{A}=A+E\in\mathbb{C}_n^{m\times n}$*, respectively. Then*

$$\left\|\widetilde{H}-H\right\|\leqslant\frac{\sigma_1^2}{\sigma_n+\widetilde{\sigma}_n}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}\right\|, \tag{2.22}$$

where σ_1,σ_n *are the biggest and smallest singular values of* A*, respectively, and* $\widetilde{\sigma}_n$ *is the smallest singular value of* $\widetilde{A}$*.*

Proof. Note that when $\operatorname{rank}(A)=n$, $V_1=V$ and $V_2=0$. In this case, (2.5) reduces to

$$A^\dagger=V(\Sigma^{-1},0)U^*=V\Sigma^{-1}U_1^*. \tag{2.23}$$

Thus, similar to the proof of Theorem 2.1 and observing (1.5) and (2.3), we can get

$$
\begin{aligned}
S&=\widetilde{V}^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}V\Sigma^2,\\
X&=\widetilde{V}^*(\widetilde{H}-H)V=\widetilde{\Sigma}\widetilde{V}^*V-\widetilde{V}^*V\Sigma.
\end{aligned} \tag{2.24}
$$

The remaining proof is similar to the corresponding proof of Theorem 2.1. □

THEOREM 2.5. *Assume that the conditions of Theorem* 2.4 *hold. Then*

$$\left\|\widetilde{H}-H\right\|_F\leqslant\frac{\sigma_1^2}{\sigma_1+\widetilde{\sigma}_n}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}\right\|_F. \tag{2.25}$$

Proof. Similar to the proof of Theorem 2.3, from (2.23), (1.5), and (2.3), it follows that

$$F=-\widetilde{V}^*(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}V\Sigma$$

and (2.24) holds. The remaining proof is similar to the corresponding proof of Theorem 2.3. □

REMARK 2.6. From the proofs of Theorems 2.4 and 2.5, and the forms of the bounds (2.22) and (2.25), it is easy to find that the bounds (2.22) and (2.25) also hold when $A, \widetilde{A} = A + E \in \mathbb{C}_n^{n\times n}$.

REMARK 2.7. In the theorems and corollary above, the expansion of $\widetilde{V}^*(\widetilde{H} - H)V$ is used to study the perturbation bounds for nonnegative and positive polar factors. Similarly, we can also use the expansion of $V^*(\widetilde{H} - H)\widetilde{V}$ to investigate these bounds. Whereas, the results obtained by the two ways are the same. In fact,

$$\left\| V^*(\widetilde{H} - H)\widetilde{V} \right\| = \left\| \left(V^*(\widetilde{H} - H)\widetilde{V} \right)^* \right\| = \left\| \widetilde{V}^*(\widetilde{H} - H)V \right\|.$$

REMARK 2.8. The perturbation bounds for nonnegative and positive polar factors obtained above are different from previous ones in form and measure (e.g., [1, 3–6, 8, 10, 12, 13, 17, 22, 25, 30]). None of these types of bounds are generally uniformly better than the others though some bounds such as (1.13) and (1.14) are deemed to be best. In some cases, the bounds derived in this paper may be smaller. Three examples are given below for which the bounds (2.16), (2.17), (2.22), and (2.25) are a little better than the corresponding ones listed in Section 1. In practical computation, the unitarily invariant norm is replaced by the spectral norm.

EXAMPLE 2.9. Let

$$A = \begin{pmatrix} 5.4699 & 0 \\ 0 & 4.1691 \end{pmatrix} \in \mathbb{C}_2^{2\times 2},\ \widetilde{A} = \begin{pmatrix} 5.4699 & -0.0017 \\ 0.0024 & 4.1692 \end{pmatrix} \in \mathbb{C}_2^{2\times 2}.$$

Then, we can get the bounds (1.6), (1.10)–(1.13), and (2.22):

$$\left(1 + \frac{2\min\{\sigma_1, \widetilde{\sigma}_1\}}{\sigma_n + \widetilde{\sigma}_n}\right) \|E\|_2 = 0.0056, \quad \left(\frac{\sigma_1 + \widetilde{\sigma}_1}{\sigma_n + \widetilde{\sigma}_n}\right) \|E\|_2 = 0.0032,$$
$$\left(\frac{\sigma_1 \widetilde{\sigma}_1}{\sigma_1 + \widetilde{\sigma}_1}\right) \left(\|EA^{-1}\|_2 + \left\|E\widetilde{A}^{-1}\right\|_2 \right) = 0.0024,$$
$$\left(1 + \log(\max\{\sigma_1, \widetilde{\sigma}_1\} \max\{\frac{1}{\sigma_n}, \frac{1}{\widetilde{\sigma}_n}\}\right) \|E\|_2 = 0.0056, \quad (\log n)\|E\|_2 = 7.2373 \times 10^{-4},$$
$$\frac{\sigma_1^2}{\sigma_n + \widetilde{\sigma}_n} \left\| (\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^{-1} \right\|_2 = 2.3589 \times 10^{-4}.$$

In addition, we can also get the bounds (1.15) and (2.25):

$$\sqrt{2}\max\left\{ \frac{\sqrt{\sigma_1^2 + \widetilde{\sigma}_n^2}}{\sigma_1 + \widetilde{\sigma}_n}, \frac{\sqrt{\widetilde{\sigma}_1^2 + \sigma_n^2}}{\widetilde{\sigma}_1 + \sigma_n} \right\} \|E\|_F = 0.0030,$$
$$\frac{\sigma_1^2}{\sigma_1 + \widetilde{\sigma}_n} \left\| (\widetilde{A}^*\widetilde{A} - A^*A)(A^*A)^{-1} \right\|_F = 2.0896 \times 10^{-4}.$$

EXAMPLE 2.10. Let

$$A = \begin{pmatrix} 2.176668561308338 & 0 \\ 0 & 8.294322803399506 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}_2^{3\times 2},$$

$$\widetilde{A} = \begin{pmatrix} 2.176677950165361 & 0.000034598006502 \\ -0.000003566450528 & 8.294298211401168 \\ 0.000294393953112 & 0.000318584423070 \end{pmatrix} \in \mathbb{C}_2^{3\times 2}.$$

Then, we can get $\|E\|_2 = 4.354273275372769 \times 10^{-4} < \sigma_{\rm n} = 2.1767$, and the bounds (1.7), (1.10), (1.11), and (2.22):

$$\begin{aligned}
&\left[(\kappa(A)+1)\omega(\|A^\dagger\|_2\|E\|_2)-1\right]\|E\|_2 = 0.001659430825011,\\
&\left(\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_n+\widetilde{\sigma}_n}\right)\|E\|_2 = 0.001659215242030,\\
&\left(\frac{\sigma_1\widetilde{\sigma}_1}{\sigma_1+\widetilde{\sigma}_1}\right)\left(\|EA^\dagger\|_2+\left\|E\widetilde{A}^\dagger\right\|_2\right) = 0.001167155001170,\\
&\frac{\sigma_1^2}{\sigma_n+\widetilde{\sigma}_n}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}\right\|_2 = 2.155014797359645\times 10^{-4}.
\end{aligned}$$

In addition, we can also get the bounds (1.15) and (2.25):

$$\begin{aligned}
&\sqrt{2}\max\left\{\frac{\sqrt{\sigma_1^2+\widetilde{\sigma}_n^2}}{\sigma_1+\widetilde{\sigma}_n}, \frac{\sqrt{\widetilde{\sigma}_1^2+\sigma_n^2}}{\widetilde{\sigma}_1+\sigma_n}\right\}\|E\|_F = 5.049204182305015\times 10^{-4},\\
&\frac{\sigma_1^2}{\sigma_1+\widetilde{\sigma}_n}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^{-1}\right\|_F = 9.380800465787683\times 10^{-5}.
\end{aligned}$$

EXAMPLE 2.11. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}, \quad \widetilde{A} = A + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}_2^{4\times 3}.$$

Then, we have the bounds (1.14) and (2.17):

$$\begin{aligned}
&\sqrt{2}\left\|\widetilde{A}-A\right\|_F = 0.003162277660168,\\
&\sqrt{\sigma_1^2+\widetilde{\sigma}_1^2}\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_F = 0.002834086966442.
\end{aligned}$$

Furthermore, we also have the bounds (1.9) and (2.16)

$$\begin{aligned}
&\left(2+\frac{\sigma_1+\widetilde{\sigma}_1}{\sigma_r+\widetilde{\sigma}_r}\right)\left\|\widetilde{A}-A\right\|_2 = 0.007827357444956,\\
&\left(\frac{\sigma_1^2}{\sigma_r+\widetilde{\sigma}_r}+\max\{\widetilde{\sigma}_1,\sigma_1\}\right)\left\|(\widetilde{A}^*\widetilde{A}-A^*A)(A^*A)^\dagger\right\|_2 = 0.003507170668442.
\end{aligned}$$

3. New perturbation bounds for generalized nonnegative and positive polar factors

In this section, let $A^{\#}_{MN}$ and $A^{\dagger}_{MN}$ denote the weighted conjugate transpose and the weighted Moore–Penrose inverse of A, respectively, whose definitions can be found in [23, 27]. In addition, without specification, we always assume that the weight matrices $M \in \mathbb{C}^{m\times m}_m$ and $N \in \mathbb{C}^{n\times n}_n$ are Hermitian positive definite.

For a matrix $A \in \mathbb{C}^{m\times n}_r$, there is an (M,N) weighted partial isometric matrix Q (e.g., [28, 29]) and a matrix H satisfying $NH \in \mathbb{C}^n_{\geqslant}$ such that

$$A = QH. \tag{3.1}$$

The above decomposition is a generalization of the generalized polar decomposition and polar decomposition and is called the (M,N) weighted polar decomposition [28] of A. The matrices Q and H are called the (M,N) weighted unitary polar factor and generalized nonnegative polar factor of this decomposition, respectively. When $\operatorname{rank}(A) = n$, H is the generalized positive polar factor.

Like the generalized polar decomposition, the (M,N) weighted polar decomposition is not unique generally. It will be unique if the decomposition satisfies

$$R(Q^{\#}_{MN}) = R(H). \tag{3.2}$$

The above condition was given by Yang and Li [31]. Under this condition, the (M,N) weighted polar decomposition can be calculated from the (M,N) singular value decomposition [23, 26]

$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^* = U_1\Sigma V_1^*$$

by

$$Q = U_1V_1^* \text{ and } H = N^{-1}V_1\Sigma V_1^*,$$

where $U = (U_1,U_2) \in \mathbb{C}^{m\times m}_m$ and $V = (V_1,V_2) \in \mathbb{C}^{n\times n}_n$ satisfy $U^*MU = I_m$ and $V^*N^{-1}V = I_n$, $U_1 \in \mathbb{C}^{m\times r}_r, V_1 \in \mathbb{C}^{n\times r}_r$, $\Sigma = \operatorname{diag}(\sigma_1,\cdots,\sigma_r)$, and $\sigma_1 \geqslant \cdots \geqslant \sigma_r > 0$ are the nonzero (M,N) singular values of A. In this section, we assume that the decomposition (3.1) always satisfies the condition (3.2).

Note that if $A = QH$ is the (M,N) weighted polar decomposition of A, then

$$M^{1/2}AN^{-1/2} = (M^{1/2}QN^{-1/2})(N^{1/2}HN^{-1/2})$$

is the generalized polar decomposition or polar decomposition of $M^{1/2}AN^{-1/2}$, and vice versa. Therefore, if the weighted norms are defined as in Definition 3.1 below, the perturbation bounds for generalized nonnegative and positive polar factors, as the counterparts of the corresponding bounds of nonnegative and positive polar factors, can be obtained as the straightforward corollaries.

DEFINITION 3.1. Let $A \in \mathbb{C}_r^{m\times n}$. We call the norms $\|A\|_{(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|$, $\|A\|_{2(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_2$, and $\|A\|_{F(MN)} = \left\|M^{1/2}AN^{-1/2}\right\|_F$ the weighted unitary invariant norm, the weighted spectral norm, and the weighted Frobenius norm of A, respectively.

It is worth pointing out that the weighted spectral norm of A is synonymous with the weighted norm of A defined as $\|A\|_{MN} = \left\|M^{1/2}AN^{-1/2}\right\|_2$ in [27], and the weighted unitary invariant norm is equivalent to the (M,N)-invariant norm defined by Rao and Rao [24] in essence.

In the following, we present the the perturbation bounds for generalized nonnegative and positive polar factors without proof.

THEOREM 3.2. *Let $A = QH, \widetilde{A} = \widetilde{Q}\widetilde{H}$ be the (M,N) weighted polar decompositions of $A, \widetilde{A} = A + E \in \mathbb{C}_r^{m\times n}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leqslant \left(\frac{\sigma_1^2}{\sigma_r + \widetilde{\sigma}_r} + \widetilde{\sigma}_1 + \sigma_1\right)\left\|(\widetilde{A}^{\#}_{MN}\widetilde{A} - A^{\#}_{MN}A)(A^{\#}_{MN}A)^{\dagger}_{NN}\right\|_{(NN)}, \tag{3.3}$$

where σ_1, σ_r and $\widetilde{\sigma}_1, \widetilde{\sigma}_r$ are the biggest and smallest nonzero (M,N) singular values of A and $\widetilde{A}$, respectively.

COROLLARY 3.3. *Assume that the conditions of Theorem 3.2 hold. Then*

$$\left\|\widetilde{H} - H\right\|_{2(NN)} \leqslant \left(\frac{\sigma_1^2}{\sigma_r + \widetilde{\sigma}_r} + \max\{\widetilde{\sigma}_1, \sigma_1\}\right)\left\|(\widetilde{A}^{\#}_{MN}\widetilde{A} - A^{\#}_{MN}A)(A^{\#}_{MN}A)^{\dagger}_{NN}\right\|_{2(NN)}.$$

THEOREM 3.4. *Assume that the conditions of Theorem 3.2 hold. Then*

$$\left\|\widetilde{H} - H\right\|_{F(NN)} \leqslant \sqrt{\sigma_1^2 + \widetilde{\sigma}_1^2}\left\|(\widetilde{A}^{\#}_{MN}\widetilde{A} - A^{\#}_{MN}A)(A^{\#}_{MN}A)^{\dagger}_{NN}\right\|_{F(NN)}. \tag{3.4}$$

THEOREM 3.5. *Let $A = QH, \widetilde{A} = \widetilde{Q}\widetilde{H}$ be the (M,N) weighted polar decompositions of $A, \widetilde{A} = A + E \in \mathbb{C}_n^{m\times n}$, respectively. Then*

$$\left\|\widetilde{H} - H\right\|_{(NN)} \leqslant \frac{\sigma_1^2}{\sigma_n + \widetilde{\sigma}_n}\left\|(\widetilde{A}^{\#}_{MN}\widetilde{A} - A^{\#}_{MN}A)(A^{\#}_{MN}A)^{\dagger}_{NN}\right\|_{(NN)}, \tag{3.5}$$

where σ_1 and σ_n are the biggest and smallest (M,N) weighted singular values of A, respectively, and $\widetilde{\sigma}_n$ is the smallest (M,N) singular value of $\widetilde{A}$.

THEOREM 3.6. *Assume that the conditions of Theorem 3.5 hold. Then*

$$\left\|\widetilde{H} - H\right\|_{F(NN)} \leqslant \frac{\sigma_1^2}{\sigma_1 + \widetilde{\sigma}_n}\left\|(\widetilde{A}^{\#}_{MN}\widetilde{A} - A^{\#}_{MN}A)(A^{\#}_{MN}A)^{\dagger}_{NN}\right\|_{F(NN)}. \tag{3.6}$$

4. Concluding remarks

In this paper, we mainly consider the perturbation bounds for nonnegative and positive polar factors of generalized polar decomposition and polar decomposition. The corresponding perturbation bounds for generalized nonnegative and positive polar factors of (M,N) weighted polar decomposition are also presented as the straightforward corollaries. It is worth pointing out that those bounds are confined to the condition that the ranks of the original matrix and perturbed matrix are the same. It is interesting to relax this restriction as did in [5] and [8]. We will consider this topic in the future. In addition, note that $H=(A^*A)^{1/2}$ if H is the nonnegative or positive polar factor and $H=(A^{\#}_{MN}A)^{1/2}$ if H is the generalized nonnegative or generalized positive polar factor. Then the results derived in this paper can also be regarded as the perturbation bounds for the square root of the positive semidefinite matrix or the generalized positive semidefinite matrix.

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