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THE FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES WITH VARIABLE EXPONENT ON UNBOUNDED DOMAINS

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Abstract. The boundedness of fractional integral operator on Morrey spaces with variable exponent on unbounded domains is established.

1. Preliminaries and Definitions

The main results of this paper consist of the boundedness of the fractional integral operator and the boundedness of the fractional maximal operator on Morrey spaces with variable exponent on $\mathbb{R}^n$.

For any $0 < \alpha < n$, the fractional integral operator (Riesz potential) I_α is defined by

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The corresponding fractional maximal operator is defined by

$$(\mathrm{M}_\alpha f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy$$

where the supremum is taking over all cube containing x.

The boundedness of the fractional integral operator on Lebesgue spaces is also called as the Hardy-Littlewood-Sobolev theorem. It is classic and well known. The boundedness of I_α on Morrey spaces is given in [1, 30, 33, 34]. For any $1 \leqslant p < \infty$, denote the class of locally L^p-integrable functions by L^p_{loc}. For any $x \in \mathbb{R}^n$ and $r > 0$, write $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. In addition, define $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

Recall that the classical Morrey space is given by

$$M_{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc} : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} < \infty\}$$

where $0 \leqslant \lambda < n$ and

$$\|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{B \in \mathbb{B}} \left(\frac{1}{r^\lambda} \int_B |f(x)|^p dx \right)^{1/p}.$$

Peetre presented Spanne's result on the boundedness of the fractional integral operator on [34].

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THEOREM 1.1. *If* $0<\alpha<n$, $1<p<q<\infty$, $0\leqslant\lambda<\mu<n$, $-\frac{n}{p}+\alpha=-\frac{n}{q}$ *and* $\frac{\lambda}{p}=\frac{\mu}{q}$, *then* $I_\alpha : M_{p,\lambda}(\mathbb{R}^n)\to M_{q,\mu}(\mathbb{R}^n)$ *is bounded.*

An improvement on the above result is given by Adams [1].

THEOREM 1.2. *If* $0<\alpha<n$, $1<p<q<\infty$, $0\leqslant\lambda<n$, $-\frac{n}{p}+\frac{n\alpha}{n-\lambda}=-\frac{n}{q}$, *then* $I_\alpha : M_{p,\lambda}(\mathbb{R}^n)\to M_{q,\lambda}(\mathbb{R}^n)$ *is bounded.*

For the study of the fractional integral operators on Morrey spaces defined on quasi-metric spaces, the reader is referred to [19, 23, 24, 36]. In [23], they consider the boundedness of fractional integral operators on variable Morrey spaces defined on a space of homogeneous type (X,μ) with $\mu(X)<\infty$ and they extend the result to quasi-metric space with non-doubling measure μ satisfying $\mu(X)<\infty$ in [24]. For the study of the fractional integral operator on generalized Morrey spaces on bounded domains, the reader is referred to [20].

The reader may also consult the survey [31] for the boundedness results of the fractional integral operators on some other function spaces arising in analysis such as Orlicz spaces, Herz spaces and Hardy spaces. For the boundedness result of the fractional integral operator on Herz-Morrey spaces with variable exponent, the reader is referred to [22].

In [2, 14, 23, 24, 27, 28], the boundedness of the maximal operator on theirs Morrey space with variable exponent is established. More precisely, the results in [2] give the boundedness of the Hardy-Littlewood maximal operator on the Morrey spaces with variable exponent on bounded domains. This result is extended to $\mathbb{R}^n$ in [14, 27, 28].

In [17, 18], the boundedness of vector-valued singular integral operator and the Fefferman-Stein vector-valued maximal inequalities for $\mathscr{M}_{p(\cdot),u}$ are obtained (see Definition 1.5). Furthermore, their applications on the study of variable Triebel-Lizorkin-Morrey spaces are shown.

In this paper, we extend the Hardy-Littlewood-Sobolev theorem to Morrey spaces with variable exponent on $\mathbb{R}^n$. The technique used in [19, 23, 24, 36] to obtain the boundedness of the fractional integral operator on Morrey spaces with variable exponent is restricted to Morrey spaces defined on bounded domains only.

We present an alternative idea so that the boundedness of fractional integral operator on Morrey spaces defined on unbounded domains can be established. For brevity, we only consider the Morrey spaces defined on $\mathbb{R}^n$ and our result can be extended to Morrey spaces defined on any unbounded domain. The drawback of our technique is that we can only establish Spanne's type result.

In order to introduce our Morrey spaces with variable exponent, we recall the definition of Lebesgue space with variable exponent [25].

Let $\mathscr{P}$ denote the set of Lebesgue measurable functions $p:\mathbb{R}^n\to[1,\infty)$ satisfying

$$p_- = \operatorname{essinf}\{p(x) : x\in\mathbb{R}^n\} > 1 \quad \text{and} \quad p_+ = \operatorname{esssup}\{p(x) : x\in\mathbb{R}^n\} < \infty.$$

DEFINITION 1.1. Let $p\in\mathscr{P}$ be a Lebesgue measurable function. The variable Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\} < \infty.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}(\mathbb{R}^n)$.

THEOREM 1.3. *If* $1 < p(x) < \infty$, *then the associate space of* $L^{p(\cdot)}(\mathbb{R}^n)$ *is* $L^{p'(\cdot)}(\mathbb{R}^n)$ *where* p' *satisfies* $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

The function $p'(x)$ is called the conjugate function of $p(x)$.

Let $\mathscr{B}$ denote the set of all $p(\cdot)$ belonging to $\mathscr{P}$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. For the study of the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$, the reader is referred to [6, 9, 14, 32].

We recall the following characterization of $\mathscr{B}$ given by Diening in [9].

THEOREM 1.4. *Let* $p(\cdot) \in \mathscr{P}$. *Then the following conditions are equivalent:*

1. $p(\cdot) \in \mathscr{B}$.
2. $p'(\cdot) \in \mathscr{B}$.
3. $p(\cdot)/q \in \mathscr{B}$ *for some* $1 < q < p_-$.
4. $(p(\cdot)/q)' \in \mathscr{B}$ *for some* $1 < q < p_-$.

We recall some notations introduced in [17] for Banach function spaces.

DEFINITION 1.2. For any $p(\cdot) \in \mathscr{B}$, let $\kappa_{p(\cdot)}$ denote the supremum of those $q > 1$ such that $p(\cdot)/q \in \mathscr{B}$. Let $e_{p(\cdot)}$ be the conjugate of $\kappa_{p'(\cdot)}$.

We call $e_{p(\cdot)}$ the index of $p(\cdot)$. Theorem 1.4 ensures that $\kappa_{p(\cdot)}$ is well-defined and satisfies $1 < \kappa_{p(\cdot)} \leqslant p_-$. Moreover, $p_+ \geqslant e_{p(\cdot)}$. We present some basic properties satisfied by $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) \in \mathscr{B}$.

PROPOSITION 1.5. *Let* $p(\cdot) \in \mathscr{B}$, *then we have a constant* $C > 0$ *so that for any* $B \in \mathbb{B}$,

$$|B| \leqslant \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C|B|. \tag{1.1}$$

The preceding result is a direct consequence of the one-weight inequality in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces by taking constant weight in the Muckenhoupt-type weight condition in [7, 10].

The above inequalities can be considered as a generalization of the result for rearrangement-invariant Banach function spaces given in [3, Chapter 2, Theorem 5.2] to Lebesgue spaces with variable exponent. The above inequality was also obtained in [17, 18] and [21, Lemma 2].

DEFINITION 1.3. Let $p \in L^\infty$. For any $B \in \mathbb{B}$, define $\overline{p}_B$ by

$$\frac{1}{\overline{p}_B} = \frac{1}{|B|}\int_B \frac{1}{p(x)}dx.$$

Proposition 1.5 and [8, Lemma 3.4] generates the subsequent estimate of $\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

PROPOSITION 1.6. *Let $p \in \mathscr{B}$ and $1 < p_- \leqslant p_+ < \infty$. There exist $C_1, C_2 > 0$ so that for any $B \in \mathbb{B}$,*

$$C_1|B|^{\frac{1}{\overline{p}_B}} \leqslant \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C_2|B|^{\frac{1}{\overline{p}_B}}. \tag{1.2}$$

For the classical Morrey spaces, the weight function is given by $|B|^{\frac{1}{p}-\frac{1}{q}}$, $B \in \mathbb{B}$, with $0 < q \leqslant p < \infty$. On the other hand, we learn from [15, 30] that our result is also valid for Morrey spaces associated with weight functions defined in the following definition.

DEFINITION 1.4. Let $p \in L^\infty$ and $1 < p(x) < \infty$. A Lebesgue measurable function $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ is said to be a *Morrey weight function* for $L^{p(\cdot)}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x,2^{j+1}r) < Cu(x,r). \tag{1.3}$$

We denote the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Condition (1.3) is also used in [16] for the study of the Fefferman-Stein vector-valued inequalities in weighted Morrey spaces and the atomic decompositions of weighted Hardy-Morrey spaces.

In particular, when $p(x) \equiv p$, $1 < p < \infty$, is a constant function, we find that condition (1.3) can be rewritten as an integral condition. More precisely, assume that u satisfies

$$r \leqslant t \leqslant 2r \Rightarrow C^{-1}u(x,r) \leqslant u(x,t) \leqslant Cu(x,r), \quad \forall x \in \mathbb{R}^n$$

for some $C > 0$. Note that this is a well-known condition imposed on u, see [20, (3.2)] and [30, (1.1)]. We find that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^p}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^p}} u(x,2^{j+1}r) = \sum_{j=0}^{\infty} \frac{u(x,2^{j+1}r)}{2^{(j+1)\frac{n}{p}}}$$

and

$$\sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1}r} \frac{u(x,t)}{t^{\frac{n}{p}}} \frac{dt}{t} \sim \sum_{j=0}^{\infty} \frac{u(x,2^j r)}{(2^j r)^{\frac{n}{p}+1}} 2^j r.$$

Thus, condition (1.3) is equivalent to

$$\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{p}}} \frac{dt}{t} \leqslant C\frac{u(x,r)}{r^{\frac{n}{p}}}, \quad r > 0,\ \forall x \in \mathbb{R}^n.$$

Let $0<\alpha<n$. Recall that the condition imposed on the Morrey weight functions in [30] is

$$\int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha p+1}}dt \leqslant C\frac{u^p(x,r)}{r^{n-\alpha p}}, \quad r>0,\ \forall x\in\mathbb{R}^n \tag{1.4}$$

for some $C>0$. Note that we rewrite the condition stated in [30] to match with our Morrey space with variable exponent in Definition 1.5.

For any u satisfying (1.4), Hölder's inequality assures that

$$\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{p}}}\frac{dt}{t} \leqslant \left(\int_r^\infty \frac{u^p(x,t)}{t^{n-\alpha p}}\frac{dt}{t}\right)^{\frac{1}{p}}\left(\int_r^\infty \frac{1}{t^{\alpha p'}}\frac{dt}{t}\right)^{\frac{1}{p'}}$$

where p' is the conjugate of p.

Condition (1.4) yields

$$\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{p}}}\frac{dt}{t} \leqslant \frac{u(x,r)}{r^{\frac{n}{p}-\alpha}}r^{-\alpha} \leqslant \frac{u(x,t)}{r^{\frac{n}{p}}}.$$

Thus, condition (1.3) is weaker than (1.4) for constant exponent. Hence, our results also extend the boundedness of fractional integral operator on classical weighted Morrey spaces [30].

Furthermore, when $0<\alpha<n$, $p(x)=p$, $1<p<\infty$, is a constant function and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$, recall that the condition satisfied by the weight functions considered in [20] is given by

$$\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{q}}}\frac{dt}{t} \leqslant \frac{u(x,r)}{r^{\frac{n}{q}}}, \quad r>0,\ \forall x\in\mathbb{R}^n. \tag{1.5}$$

for some $C>0$. We obtain the preceding condition by inserting $\omega_1(x,t)=u(x,t)t^{-\frac{n}{p}}$, $\omega_2(x,t)=u(x,t)t^{-\frac{n}{q}}$ and $\alpha(x)=\frac{n}{p}-\frac{n}{q}$ in [20, (5.3)] (see Definition [20, Definition 3.1] for their definition of Morrey spaces with variable exponent). Moreover, we modify the condition given in [20, (5.3)] into our setting on unbounded domains by replacing l with ∞ on the upper limit of the above integral. For any u fulfilling (1.5), we have

$$\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{p}}}\frac{dt}{t} \leqslant \frac{1}{r^{\frac{n}{p}-\frac{n}{q}}}\int_r^\infty \frac{u(x,t)}{t^{\frac{n}{q}}}\frac{dt}{t} \leqslant C\frac{u(x,r)}{r^{\frac{n}{p}}}.$$

That is, (1.3) is in some sense weaker than (1.5) for constant p and (1.5) is equivalent with the condition $u\in\mathbb{W}_{q(\cdot)}$ when $q(x)=q$.

The definition for the Morrey weight functions (1.3) involves a summation including the terms $\frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$. Some properties satisfied by the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ can provide a pointwise estimate on these quotients.

PROPOSITION 1.7. *Let $p\in\mathscr{B}$. For any $1<q<\kappa_{p(\cdot)}$ and $1<s<\kappa_{p'(\cdot)}$, there exist constants $C_1,C_2>0$ such that for any $x_0\in\mathbb{R}^n$ and $r>0$, we have*

$$C_2 2^{jn(1-\frac{1}{s})} \leqslant \frac{\|\chi_{B(x_0,2^jr)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C_1 2^{\frac{jn}{q}}, \quad \forall j\in\mathbb{N}. \tag{1.6}$$

Proof. For any $B = B(x_0, r) \in \mathbb{B}$ and $j \in \mathbb{N}$, we have a constant $C > 0$ such that

$$C2^{-jn} \leqslant \mathrm{M}(\chi_B)(x)$$

when $x \in B(x_0, 2^j r)$, $j \in \mathbb{N}$. Thus, for any $q < \kappa_{p(\cdot)}$, there exists a $q < \tilde{q}$ so that $p(\cdot)/\tilde{q} \in \mathscr{B}$. Subsequently,

$$2^{-jn}\|\chi_{B(x_0,2^j r)}\|_{L^{p(\cdot)/\tilde{q}}} \leqslant C\|\mathrm{M}(\chi_B)\|_{L^{p(\cdot)/\tilde{q}}} \leqslant C\|\chi_B\|_{L^{p(\cdot)/\tilde{q}}}.$$

Since, for any $B \in \mathbb{B}$ and $q > 0$, $\|\chi_B\|_{L^{p(\cdot)/q}} = \|\chi_B\|^q_{L^{p(\cdot)}}$, we obtain the second inequality of (1.6).

According to Theorem 1.4, $p'(\cdot) \in \mathscr{B}$. Thus, for any $s < \kappa_{p'(\cdot)}$, we also have

$$\frac{\|\chi_{B(x_0,2^j r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant C_1 2^{\frac{jn}{s}}, \quad \forall j \in \mathbb{N}.$$

Therefore, Proposition 1.5 yields the first inequality in (1.6). □

A similar result is obtained in [21, Lemma 1]. The above result is a special case of the general result in [17] for Banach function spaces.

The notion of Boyd's indices gives us an estimate of the operator norm of the dilation operator $D_t f(x) = f(tx)$, $t > 0$, on rearrangement-invariant Banach function spaces [3, p.148-149]. Although the Boyd indices is not necessarily well defined on $L^{p(\cdot)}(\mathbb{R}^n)$, the preceding result provides some pivotal estimate of the action of the dilation operators on the characteristic function of $B(x_0, r)$.

Using Proposition 1.7, we find that for any $0 \leqslant \lambda < \frac{1}{e_{p(\cdot)}}$ whenever a weight function $u : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfies

$$\frac{u(x,2r)}{u(x,r)} \leqslant 2^{n\lambda}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, it belongs to $\mathbb{W}_{p(\cdot)}$. More precisely, for any $\lambda < \frac{1}{e_{p(\cdot)}}$, there exists a $s < \kappa_{p'(\cdot)}$ such that $\lambda < 1 - \frac{1}{s} < 1 - \frac{1}{\kappa_{p'(\cdot)}} = \frac{1}{e_{p(\cdot)}}$. Moreover, Proposition 1.7 is valid for any $1 < s < \kappa_{p'(\cdot)}$. By using the first inequality in (1.6), we have

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x,2^{j+1}r)}{u(x,r)} \leqslant C \sum_{j=0}^{\infty} 2^{-jn(1-\frac{1}{s})} 2^{jn\lambda} \leqslant C.$$

For instance, the weight function $u(x,r) = r^{\lambda(x)}$ where $0 \leqslant \lambda(x) \leqslant \lambda_+ < \frac{1}{e_{p(\cdot)}}$ is a member of $\mathbb{W}_{p(\cdot)}$.

DEFINITION 1.5. Let $p(x) \in \mathscr{B}$ and $u(x,r) \in \mathbb{W}_{p(\cdot)}$. The *Morrey space with variable exponent* $\mathscr{M}_{p(\cdot),u}$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathscr{M}_{p(\cdot),u}} = \sup_{z \in \mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|\chi_{B(z,R)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

We state a result concerning the boundedness of the fractional integral operator on Lebesgue spaces with variable exponent given in [4, 6].

PROPOSITION 1.8. *Let* $\alpha > 0$ *and* $p(x), q(x) \in \mathscr{P}$ *satisfy* $p_+ < \frac{n}{\alpha}$ *and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad a.e \text{ on } \quad \mathbb{R}^n.$$

If there exists q_0, $\frac{n}{n-\alpha} < q_0 < \infty$, *such that* $\frac{q(\cdot)}{q_0} \in \mathscr{B}$, *then*

$$\|I_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{1.7}$$

and

$$\|\mathrm{M}_\alpha f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

for some $C > 0$.

2. Main Result

The following is the Hardy-Littlewood-Sobolev theorem for Morrey spaces with variable exponent $\mathscr{M}_{p(\cdot),u}$.

THEOREM 2.1. *Let* $\alpha > 0$, $p(x), q(x) \in \mathscr{B}$. *Suppose that* $p(x)$, $q(x)$ *and* α *satisfy* $p_+ < \frac{n}{\alpha}$ *and*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad a.e \text{ on } \quad \mathbb{R}^n.$$

If $u \in \mathbb{W}_{q(\cdot)}$ *and there exists a* $q_0 < \infty$ *satisfying* $\frac{n}{n-\alpha} < q_0 < \infty$ *and* $\frac{q(\cdot)}{q_0} \in \mathscr{B}$, *then*

$$\|I_\alpha f\|_{\mathscr{M}_{q(\cdot),u}} \leqslant C\|f\|_{\mathscr{M}_{p(\cdot),u}} \tag{2.1}$$

and

$$\|\mathrm{M}_\alpha f\|_{\mathscr{M}_{q(\cdot),u}} \leqslant C\|f\|_{\mathscr{M}_{p(\cdot),u}}$$

for some $C > 0$.

Proof. We only consider the fractional integral operator as the proof for the fractional maximal operator follows from the boundedness of I_α and the fact that $\mathrm{M}_\alpha f \leqslant I_\alpha f$ for non-negative f.

Let $f \in \mathscr{M}_{p(\cdot),u}$. For any $z \in \mathbb{R}^n$ and $r > 0$, write $f(x) = f_0(x) + \sum_{j=1}^{\infty} f_j(x)$, where $f_0 = \chi_{B(z,2r)} f$ and $f_j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)} f$, $j \in \mathbb{N}\setminus\{0\}$. Proposition 1.8 shows that $\|I_\alpha f_0\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Thus, we find that

$$\begin{aligned}\frac{1}{u(z,r)}\|\chi_{B(z,r)}(I_\alpha f_0)\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C \sup_{\substack{y\in\mathbb{R}^n \\ r>0}} \frac{1}{u(y,r)}\|\chi_{B(y,r)} f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\end{aligned}$$

because inequality (1.3) and Proposition 1.7 imply that $u(z,2r) < Cu(z,r)$ for some constant $C>0$ independent of $z\in\mathbb{R}^n$ and $r>0$.

Furthermore, there is a constant $C>0$ such that, for any $j\geqslant 1$

$$\chi_{B(z,r)}(x)|(I_\alpha f_j)(x)| \leqslant C2^{-j(n-\alpha)}r^{-n+\alpha}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}|f(y)|dy. \tag{2.2}$$

The generalized Hölder inequality given in [25, Theorem 2.1] ensures that

$$\int_{B(z,2^{j+1}r)}|f(y)|dy \leqslant C\left\|\chi_{B(z,2^{j+1}r)}f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$

for some $C>0$.

Subsequently, applying the norm $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n)}$ on both sides of (2.2), we have

$$\begin{aligned}\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant & C2^{-j(n-\alpha)}r^{-n+\alpha}\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\times\left\|\chi_{B(z,2^{j+1}r)}f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{2.3}$$

Applying Proposition 1.5 with $B=B(z,2^{j+1}r)$, we have

$$\|\chi_{B(z,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C\frac{2^{(j+1)n}r^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Using the above inequality on (2.3), we obtain

$$\begin{aligned}&\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C2^{-j(n-\alpha)}r^{-n+\alpha}\frac{\|\chi_{B(z,r)}(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B(z,2^{j+1}r)}f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}2^{(j+1)n}r^n}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\\ &\leqslant C2^{j\alpha}r^\alpha\frac{\|\chi_{B(z,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left\|\chi_{B(z,2^{j+1}r)}f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

In view of the fact that for any $B\in\mathbb{B}$,

$$\frac{1}{|B|}\int_B\frac{1}{p(x)}dx-\frac{1}{|B|}\int_B\frac{1}{q(x)}dx=\frac{1}{\overline{p}_B}-\frac{1}{\overline{q}_B}=\frac{\alpha}{n},$$

Proposition 1.6 assures that

$$C_2|B|^{\frac{\alpha}{n}} \leqslant \frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \leqslant C_1|B|^{\frac{\alpha}{n}}, \quad \forall B\in\mathbb{B} \tag{2.4}$$

for some constants $C_1>C_2>0$ independent of $B\in\mathbb{B}$.

Hence, using (2.4) with $B=B(z,2^{j+1}r)$, we have

$$C_2\frac{2^{j\alpha}r^\alpha}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant \frac{1}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}.$$

Therefore,

$$\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus,

$$\begin{aligned}
&\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\frac{u(z,2^{j+1}r)}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\frac{\|\chi_{B(x,r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}u(z,2^{j+1}r)\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

As $u\in\mathbb{W}_{q(\cdot)}$ we obtain

$$\begin{aligned}
\frac{1}{u(z,r)}\|\chi_{B(z,r)}(I_\alpha f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant \frac{1}{u(z,r)}\sum_{j=0}^{\infty}\|\chi_{B(z,r)}(I_\alpha f_j)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\sup_{\substack{y\in\mathbb{R}^n\\ R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}
\end{aligned}$$

where the constant $C>0$ is independent of r and z. Taking supremum over $z\in\mathbb{R}^n$ and $r>0$ gives (2.1). □

We have the following generalization of the above result. We find that an integral condition on the Schwartz kernel of a linear operator is sufficient to assure the boundedness of the linear operator on Morrey spaces with variable exponent.

THEOREM 2.2. *Let $\alpha, p(x), q(x)$ and $u(x,r)$ satisfy the conditions in Theorem 2.1 and $T: L^{p(\cdot)}(\mathbb{R}^n)\to L^{q(\cdot)}(\mathbb{R}^n)$. If the Schwartz kernel of T, $K(x,y)$, satisfies*

$$\left(\frac{1}{|B(z,\mu)|}\int_{B(z,\mu)}|K(x,y)|^\varepsilon dy\right)^{\frac{1}{\varepsilon}}\leqslant C\mu^{\alpha-n}$$

for some $e_{p(\cdot)}<\varepsilon$, then T can be extended to be a bounded linear operator from $\mathscr{M}_{p(\cdot),u}$ to $\mathscr{M}_{q(\cdot),u}$.

Proof. We have the following modification of inequality (2.2)

$$\begin{aligned}
&\chi_{B(z,r)}(x)|(Tf_j)(x)|\\
&\leqslant C\chi_{B(z,r)}(x)\left(\int_{B(z,2^{j+1}r)}|f(y)|^{\varepsilon'}dy\right)^{\frac{1}{\varepsilon'}}\left(\int_{B(z,,2^{j+1}r)}|K(x,y)|^{\varepsilon}dy\right)^{\frac{1}{\varepsilon}}\\
&\leqslant C2^{j(\alpha-\frac{n}{\varepsilon'})}r^{\alpha-\frac{n}{\varepsilon'}}\chi_{B(z,r)}(x)\left(\int_{B(z,2^{j+1}r)}|f(y)|^{\varepsilon'}dy\right)^{\frac{1}{\varepsilon'}}
\end{aligned}$$

where ε' is the conjugate of ε. The identity $\||f|^{\varepsilon'}\|_{L^{p(\cdot)/\varepsilon'}(\mathbb{R}^n)} = \|f\|^{\varepsilon'}_{L^{p(\cdot)}(\mathbb{R}^n)}$ and the generalized Hölder inequality assert that

$$\int_{B(z,2^{j+1}r)} |f(y)|^{\varepsilon'} dy \leqslant C\|\chi_{B(z,2^{j+1}r)}|f|^{\varepsilon'}\|_{L^{p(\cdot)/\varepsilon'}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{(p(\cdot)/\varepsilon')'}(\mathbb{R}^n)}$$
$$\leqslant C\|\chi_{B(z,2^{j+1}r)}f\|^{\varepsilon'}_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(z,2^{j+1}r)}\|_{L^{(p(\cdot)/\varepsilon')'}(\mathbb{R}^n)}.$$

As $e_{p(\cdot)} < \varepsilon$, we have $\varepsilon' < \kappa_{p(\cdot)}$. That is, there exists a $\varepsilon' < q < \kappa_{p(\cdot)}$ such that $\frac{p(\cdot)}{q} \in \mathscr{B}$. Jensen's inequality guarantees that $\frac{p(\cdot)}{\varepsilon'} \in \mathscr{B}$. Theorem 1.4 assures that Proposition 1.5 is applicable to $L^{(p(\cdot)/\varepsilon')'}(\mathbb{R}^n)$. Subsequently, we have

$$\int_{B(z,2^{j+1}r)} |f(y)|^{\varepsilon'} dy \leqslant C2^{jn}r^n \frac{\|\chi_{B(z,2^{j+1}r)}f\|^{\varepsilon'}_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|^{\varepsilon'}_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Therefore, we obtain

$$\chi_{B(z,r)}(x)|(Tf_j)(x)| \leqslant C2^{j\alpha}r^{\alpha}\chi_{B(z,r)}(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(z,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$$

and the rest of the proof is the same as the proof of Theorem 2.1. For brevity, we leave the detail to the reader. □

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REFERENCES

[1] D. ADAMS, *A note on Riesz potentials*, Duke Math. J. **42** (1975), 765–778.
[2] A. ALMEIDA, J. HASANOV, AND S. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J. **15** (2008), 195–208.
[3] C. BENNETT, AND R. SHARPLEY, *Interpolation of Operators*, Academic Press, 1988.
[4] C. CAPONE, D. CRUZ-URIBE, AND A. FIORENZA, *The fractional maximal operator and fractional integrals on variable Lp spaces*, Rev. Mat. Iberoam. **23** (2007), 743–770.
[5] F. CHIARENZA, AND M. FRASCA, *Morrey Spaces and Hardy-Littlewood Maximal Function*, Rend. Mat. Appl. (7) **7** (1987), 273–279.
[6] D. CRUZ-URIBE, SFO, A. FIORENZA, J. MARTELL, AND C. PÉREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.
[7] D. CRUZ-URIBE, L. DIENING, AND P. HÄSTÖ, *The maximal operator on weighted variable Lebesgue spaces*, Frac. Calc. Appl. Anal. **14** (2011), 361–374.
[8] L. DIENING, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43.
[9] L. DIENING, *Maximal function on Orlicz-Musielak spaces and generalized Lebesgue space*, Bull. Sci. Math. **129** (2005), 657–700.
[10] L. DIENING, P. HARJULEHTO, P. HÄSTÖ, AND M. RUŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.
[11] D. EDMUNDS, AND J. RÁKOSNÍK, *Sobolev embeddings with variable exponent*, Studia Math. **143** (2000), 267–293.

[12] D. EDMUNDS, AND J. RÁKOSNÍK, *Sobolev embeddings with variable exponent II*, Math. Nachr. **246–247** (2002), 53–67.

[13] A. ERIDANI, V. KOKILASHVILI, AND A. MESKHI, *Morrey spaces and fractional integral operators*, Expo. Math. **27** (2009), 227–239.

[14] P. HÄSTÖ, *Local-to-golobal results in Variable Exponent spaces*, Math. Res. Lett. **16** (2009), 263–278.

[15] K.-P. HO, *Littlewood-Paley spaces*, Math. Scand. **108** (2011), 77–102.

[16] K.-P. HO, *Atomic decompositions of Weighted Hardy-Morrey Spaces*, accepted by Hokkaido Mathematical Journal.

[17] K.-P. HO, *Vector-valued singular integral operators on Morrey type spaces and variable Triebel-Lizorkin-Morrey spaces*, preprint.

[18] K.-P. HO, *Vector-valued maximal inequalities for Variable Morrey spaces and Variable Triebel-Lizorkin-Morrey spaces*, preprint.

[19] J. GARCÍA-CUERVA, AND A. GATTO, *Boundedness properties of fractional integral operator associated to non-doubling measure*, Studia. Math. **162** (2004), 245–261.

[20] V. GULIYEV, J. HASANOV, AND S. SAMKO, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand. **107** (2010), 285–304.

[21] M. IZUKI, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. Circ. Mat. Palermo (2) **59** (2010), 199–213.

[22] M. IZUKI, *Fractional integrals on Herz-Morrey spaces with variable exponent*, Hiroshima Math. J. **40** (2010), 343–355.

[23] V. KOKILASHVILI, AND A. MESKHI, *Boundedness of Maximal and Singular operators in Morrey Spaces with Variable Exponent*, Armen. J. Math. **1** (2008), 18–28.

[24] V. KOKILASHVILI, AND A. MESKHI, *Maximal functions and potentials in variable exponent Morrey spaces with non-doubling measure*, Complex Var. Elliptic Equ. **55** (2010), 923–936.

[25] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

[26] V. KOKILASHVILI, AND S. SAMKO, *On Sobolev Theorem for Riesz-type Potentials in Lebesgue Spaces with Variable Exponent*, Z. Anal. Anwend. **22** (2003), 899–910.

[27] Y. MIZUTA AND T. SHIMOMURA, *Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent*, J. Math. Soc. Japan **60** (2008), 583–602.

[28] Y. MIZUTA AND T. SHIMOMURA, *Continuity properties for Riesz potentials of functions in Morrey spaces of variable exponent*, Math. Inequ. Appl. **13** (2010), 99–122.

[29] Y. MIZUTA, E. NAKAI, T. OHNO, AND T. SHIMOMURA, *Boundedness of fractional integral operators on Morrey spaces and Sobolev embeddings for generalized Riesz potentials*, J. Math. Soc. Japan **62** (2010), 707–744.

[30] E. NAKAI, *Hardy-Littlewood Maximal Operator, Singular Integral Operators and the Riesz Potentials on Generalized Morrey Spaces*, Math. Nachr. **166** (1994), 95–104.

[31] E. NAKAI, *Recent Topics of Fractional Integrals*, Sugaku Expositions **20** (2007), 215–235.

[32] A. NEKVINDA, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$*, Math. Inequal. Appl. **7** (2004), 255–265.

[33] P. OLSEN, *Fractional integration, Morrey spaces and a Schrödinger equation*, Comm. Partial Differential Equations **115** (1995), 2005–2055.

[34] J. PEETRE, *On the Theory of $\mathscr{L}_{p,\lambda}$ Spaces*, J. Funct. Anal. **4** (1969), 71–87.

[35] S. SAMKO, *Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$*, Integral Transform. Spec. Funct. **7** (1998), 261–284.

[36] Y. SAWANO, AND H. TANAKA, *Morrey spaces for nondoubling measures*, Acta Math. Sinica **21** (2005), 1535–1544.

[37] H. TANAKA, *Morrey spaces and Fractional operators*, J. Aust. Math. Soc. **88** (2010), 247–259.

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