Mathematical
Inequalities
& Applications
Volume 16, Number 2 (2013), 401–412
doi:10.7153/mia-16-30

MIXED RADIAL BLASCHKE–MINKOWSKI HOMOMORPHISMS AND COMPARISON OF VOLUMES

LIJUAN LIU

(*Communicated by Y. Burago*)

Abstract. In this paper we consider a Busemann-Petty type problem for mixed radial Blaschke-Minkowski homomorphisms.

1. Introduction

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n\,(n>2)$. Let $\mathscr{K}^n$ denote the space of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$ with the Hausdorff topology. Let B denote the unit ball in $\mathbb{R}^n$, the surface of B is S^{n-1}. A compact, convex set K is uniquely determined by its support function $h(K,\cdot)$ on the unit sphere S^{n-1}, defined by $h(K,u)=\max\{u\cdot x:\ x\in K\}$.

Associated with a compact subset $K\in\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot):S^{n-1}\to\mathbb{R}$, defined for $u\in S^{n-1}$, by $\rho(K,u)=\max\{\lambda\geqslant 0:\lambda u\in K\}$. If $\rho(K,\cdot)$ is positive and continuous, we call K a star body. Let $\mathscr{S}^n$ denote the set of star bodies in $\mathbb{R}^n$, and $\mathscr{S}^n_e$ denote the subset of $\mathscr{S}^n$ that contains the origin-symmetric star bodies.

The intersection body IK of a star body K is defined by

$$\rho(IK,u)=vol_{n-1}(K\cap u^{\perp}),\ \ u\in S^{n-1}.$$

Intersection bodies have attracted increased interest in recent years. They appear already in a paper by Busemann [2] but were first explicitly defined and named by Lutwak [20]. Intersection bodies turned out to be critical for the solution of the Busemann-Petty problem (see [3–5, 10, 13–15, 29]).

Recently, Schuster [23] introduced a class of operators, called radial Blaschke-Minkowski homomorphisms which generalize the well known intersection operator.

DEFINITION 1.1. A map $\Phi:\mathscr{S}^n\to\mathscr{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies (a), (b) and (c).

(a) Φ is continuous with respect to radial Hausdorff metric.

(b) $\Phi(K\,\widetilde{\#}L)=\Phi K\,\widetilde{+}\,\Phi L$ for all $K,L\in\mathscr{S}^n$.

Mathematics subject classification (2010): 52A40, 52A20.
Keywords and phrases: Mixed radial Blaschke-Minkowski homomorphisms, mixed dual volume.
Supported in part by the National Natural Science Foundation of China (Grant No. 11071156).

(c) Φ is $SO(n)$ equivariant, i.e., $\Phi(\vartheta K) = \vartheta\Phi K$ for all $\vartheta \in SO(n)$, where $SO(n)$ is the group of rotations in $\mathbb{R}^n$.

Here $\Phi K \tilde{+} \Phi L$ is the radial Minkowski sum of ΦK and ΦL, $K\tilde{\#}L$ is the radial Blaschke sum of K and L, i.e.,

$$\rho(K\tilde{\#}L,u)^{n-1} = \rho(K,u)^{n-1} + \rho(L,u)^{n-1}.$$

Schuster [25] studied the following Busemann-Petty type problem for radial Blaschke-Minkowski homomorphisms and obtained answers analogous to the famous Busemann-Petty problem.

THEOREM A. [25] *Let $\Phi : \mathscr{S}^n \to \mathscr{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $K \in \Phi\mathscr{S}^n$ and $L \in \mathscr{S}^n$, then*

$$\Phi K \subseteq \Phi L \Rightarrow V(K) \leqslant V(L),$$

and $V(K) = V(L)$ if and only if $K = L$.

THEOREM B. [25] *Let $\Phi : \mathscr{S}^n \to \mathscr{S}^n$ be a radial Blaschke-Minkowski homomorphism. If $\mathscr{S}^n(\Phi)$ does not coincide with $\mathscr{S}^n$, then there exist star bodies $K, L \in \mathscr{S}^n$, such that*

$$\Phi K \subseteq \Phi L,$$

but

$$V(K) > V(L).$$

Here $\mathscr{S}^n(\Phi)$ denotes the injectivity set of Φ (see section 3 for a precise definition).

Schuster first introduced the concept of mixed radial Blaschke-Minkowski homomorphisms. Let $\Phi : \mathscr{S}^n \to \mathscr{S}^n$ be a radial Blaschke-Minkowski homomorphism. There is a continuous operator

$$\Phi : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-1} \to \mathscr{S}^n,$$

symmetric in its arguments such that, for $K_1, \ldots, K_m$ and $\lambda_1, \ldots, \lambda_m \geqslant 0$,

$$\Phi(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \tilde{\sum_{i_1,\cdots,i_{n-1}}} \Phi(K_{i_1}, \ldots, K_{i_{n-1}})\lambda_{i_1} \cdots \lambda_{i_{n-1}}.$$

Clearly, it generalizes the notion of radial Blaschke-Minkowski homomorphisms. We call

$$\Phi : \mathscr{S}^n \times \cdots \times \mathscr{S}^n \to \mathscr{S}^n$$

the mixed radial Blaschke-Minkowski homomorphism induced by Φ. If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$, we write $\Phi_i K$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$. For $i = 0, \cdots, n-1$, we write $\Phi_i(K,L)$ for $\Phi(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{L, \ldots, L}_{i})$. In particular, if $i = 0$, then $\Phi_0(K,L) = \Phi K$

In this paper, we focus on the study of the Busemann-Petty type problem for mixed radial Blaschke-Minkowski homomorphisms. We generalize Schuster's results as follows:

THEOREM 1.1. *For* $i = 0, \cdots, n-1$, *let* $\Phi_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $K \in \Phi_i \mathscr{S}^n$ *and* $L \in \mathscr{S}^n$, *then*

$$\Phi_i K \subseteq \Phi_i L \Rightarrow \widetilde{W}_i(K) \leqslant \widetilde{W}_i(L),$$

and $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ *if and only if* $K = L$.

THEOREM 1.2. *For* $i = 0, \cdots, n-1$, *let* $\Phi_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$ *be a mixed radial Blaschke-Minkowski homomorphism. If* $\mathscr{S}^n(\Phi_i)$ *does not coincide with* $\mathscr{S}^n$, *then there exist star bodies* $K, L \in \mathscr{S}^n$, *such that*

$$\Phi_i K \subseteq \Phi_i L,$$

but

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

Here $\mathscr{S}^n(\Phi_i)$ *denotes the injectivity set of* Φ_i *(see section 3 for a precise definition).*

2. Notation and background material

2.1. Dual mixed volumes and mixed radial Blaschke-Minkowski homomorphisms

For $K_1, K_2 \in \mathscr{S}^n$ and $\lambda_1, \lambda_2 \geqslant 0$, the radial Minkowski linear combination $\lambda_1 K_1 \widetilde{+} \lambda_2 K_2$ is the star body defined by

$$\rho(\lambda_1 K_1 \widetilde{+} \lambda_2 K_2, \cdot) = \lambda_1 \rho(K_1, \cdot) + \lambda_2 \rho(K_2, \cdot). \tag{2.1}$$

If $K_i \in \mathscr{S}^n (i = 1, 2, \ldots, m)$ and $\lambda_i (i = 1, 2, \ldots, m)$ are nonnegative real numbers, then the volume of $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m$ is a homogeneous polynomial of degree n in λ_i given by

$$V(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_m K_m) = \sum_{i_1, \ldots, i_n} \widetilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{2.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding m. The coefficient $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by the above identity, it is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$. More explicitly, the dual mixed volume $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$ has the following integral representation [19]:

$$\widetilde{V}(K_{i_1}, \ldots, K_{i_n}) = \frac{1}{n} \int_{S^{n-1}} \rho(K_{i_1}, u) \cdots \rho(K_{i_n}, u) du, \tag{2.3}$$

where du is the spherical Lebesgue measure on S^{n-1}.

The coefficients $\widetilde{V}(K_{i_1}, \ldots, K_{i_n})$ are nonnegative, symmetric and monotone (with respect to set inclusion). They are also multilinear with respect to radial Minkowski addition and $\widetilde{V}(K, \ldots, K) = V(K)$. Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the dual mixed volume $\widetilde{V}(K_1, \ldots, K_n)$ is usually written as $\widetilde{V}_i(K, L)$. If $L = B$, then $\widetilde{V}_i(K,B)$ is the dual Quermassintegral of K and is written as $\widetilde{W}_i(K)$. For $0 \leqslant i \leqslant n$, then we write $\widetilde{W}_i(K,L)$ for the dual mixed volume $\widetilde{V}_i(\underbrace{K,\ldots,K}_{n-i-1},\underbrace{B,\ldots,B}_{i},L)$.

From (2.3), if $K,L \in \mathscr{S}^n$, $i \in \mathbb{R}$, we have

$$\widetilde{W}_i(K,L) = \int_{S^{n-1}} \rho(K,u)^{n-i-1}\rho(L,u)du. \tag{2.4}$$

LEMMA 2.1. [32] *If $K,L \in \mathscr{S}^n$, and $0 \leqslant i \leqslant n-1$, then*

$$\widetilde{W}_i(K,L)^{n-i} \leqslant \widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L), \tag{2.5}$$

equality holds if and only if K and L are dilatations of each other.

Schuster characterized completely all mixed radial Blaschke Minkowski homomorphisms.

LEMMA 2.2. [23] *A map $\Phi : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-1} \to \mathscr{S}^n$ is a mixed radial Blaschke-Minkowski homomorphism if and only if there is a nonnegative measure $\mu \in \mathscr{M}_+(S^{n-1},)$ such that*

$$\rho(\Phi(K_1,\ldots,K_{n-1}),\cdot) = (\rho(K_1,\cdot)\cdots\rho(K_{n-1},\cdot)) * \mu. \tag{2.6}$$

In particular, if $K_1 = \cdots = K_{n-1} = K$, then

$$\rho(\Phi K,\cdot) = \rho(K,\cdot)^{n-1} * \mu. \tag{2.7}$$

For the definition of the convolution, see the next section.

Additional information on convex body valued valuations or star body valued valuations can be found in references [1, 7–9, 11, 12, 16–18, 21–27, 30, 31].

2.2. Spherical harmonics

Some basic notions on spherical harmonics will be required. The article by Grinberg and Zhang [6] and the article by Schuster [25] are excellent general references on spherical harmonics. As usual, $SO(n)$ and S^{n-1} will be equipped with the invariant probability measures. Let $\mathscr{C}(SO(n)),\mathscr{C}(S^{n-1})$ be the spaces of continuous functions on $SO(n)$ and S^{n-1} with uniform topology and let $\mathscr{M}(SO(n)),\mathscr{M}(S^{n-1})$ denote their dual spaces of signed finite Borel measures with weak* topology. The group $SO(n)$ acts on these spaces by left translation, i.e., for $f \in \mathscr{C}(S^{n-1})$ and $\mu \in \mathscr{M}(S^{n-1})$, we have $\vartheta f(u) = f(\vartheta^{-1}u), \vartheta \in SO(n)$, and $\vartheta\mu$ is the image measure of μ under the rotation ϑ. If $\mu,\sigma \in \mathscr{M}(SO(n))$, the convolution $\mu * \sigma$ is defined by:

$$\int_{SO(n)} f(\vartheta)d(\mu*\sigma)(\vartheta) = \int_{SO(n)}\int_{SO(n)} f(\eta\tau)d\mu(\eta)d\sigma(\tau), \tag{2.8}$$

for every $f \in \mathscr{C}(SO(n))$.

The sphere S^{n-1} is identified with the homogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the subgroup of rotations leaving the pole $\widehat{e}$ of S^{n-1} fixed.

The projection from $SO(n)$ onto S^{n-1} is $\vartheta \mapsto \widehat{\vartheta} := \vartheta\widehat{e}$. Right $SO(n-1)$-invariant functions on $SO(n)$ are defined by $\check{f}(\vartheta) = f(\widehat{\vartheta})$, for $f \in \mathscr{C}(S^{n-1})$. In fact, $\mathscr{C}(S^{n-1})$ is isomorphic to the subspace of right $SO(n-1)$-invariant functions in $\mathscr{C}(SO(n))$ and this correspondence carries over to an identification of the space $\mathscr{M}(S^{n-1})$ with right $SO(n-1)$-invariant measures in $\mathscr{M}(SO(n))$. It is easy to check that the Dirac measure $\delta_{\widehat{e}}$ is the unique rightneutral element for the convolution on $\mathscr{M}(S^{n-1})$.

The convolution $\mu * f \in \mathscr{C}(S^{n-1})$ of a measure $\mu \in \mathscr{M}(SO(n))$ and a function $f \in \mathscr{C}(S^{n-1})$ is defined by:

$$(\mu * f)(u) = \int_{SO(n)} \vartheta f(u) d\mu(\vartheta). \tag{2.9}$$

The canonical pairing of $f \in \mathscr{C}(S^{n-1})$ and $\mu \in \mathscr{M}(S^{n-1})$ is defined by:

$$\langle \mu, f \rangle = \langle f, \mu \rangle = \int_{S^{n-1}} f(u) d\mu(u). \tag{2.10}$$

If $\mu, \nu \in \mathscr{M}(S^{n-1})$ and $f \in \mathscr{C}(S^{n-1})$, then

$$\langle \mu * \nu, f \rangle = \langle \mu, f * \nu \rangle. \tag{2.11}$$

A function $f \in \mathscr{C}(S^{n-1})$ is called zonal, if $\vartheta f = f$ for every $\vartheta \in SO(n-1)$. Zonal functions depend only on the value $u \cdot \widehat{e}$. The set of continuous zonal functions on S^{n-1} will be denoted by $\mathscr{C}(S^{n-1}, \widehat{e})$ and the definition of $\mathscr{M}(S^{n-1}, \widehat{e})$ is analogous. A map $\Lambda : \mathscr{C}[-1,1] \to \mathscr{C}(S^{n-1}, \widehat{e})$ defined by:

$$\Lambda f(u) = f(u \cdot \widehat{e}), \quad u \in S^{n-1}. \tag{2.12}$$

The map Λ is also an isomorphism between functions on $[-1,1]$ and zonal functions on S^{n-1}.

If $f \in \mathscr{C}(S^{n-1}), \mu \in \mathscr{M}(S^{n-1}, \widehat{e})$ and $\eta \in SO(n)$, then

$$(f * \mu)(\widehat{\eta}) = \int_{S^{n-1}} f(\eta u) d\mu(u). \tag{2.13}$$

If $\mu \in \mathscr{M}(S^{n-1}, \widehat{e})$, for each $f \in \mathscr{C}(S^{n-1})$ and every $\vartheta \in SO(n)$, then

$$(\vartheta f) * \mu = \vartheta (f * \mu). \tag{2.14}$$

We use $\mathscr{H}_k^n$ to denote the finite dimensional vector space of spherical harmonics of dimension n and order k. Let $N(n,k)$ denote the dimension of $\mathscr{H}_k^n$. The space of all finite sums of spherical harmonics of dimension n is denoted by $\mathscr{H}^n$. The spaces $\mathscr{H}_k^n$ are pairwise orthogonal with respect to the usual inner product on $\mathscr{C}(S^{n-1})$. Clearly, $\mathscr{H}_k^n$ is invariant with respect to rotations.

Let $P_k^n \in \mathscr{C}[-1,1]$ denote the Legendre polynomial of dimension n and order k. The zonal function ΛP_k^n is up to a multiplicative constant the unique zonal spherical harmonic in $\mathscr{H}_k^n$. In each space $\mathscr{H}_k^n$ we choose an orthonormal basis $H_{k1}, \cdots, H_{kN(n,k)}$.

The collection $\{H_{k1},\cdots,H_{kN(n,k)}:\ k\in\mathbb{N}\}$ forms a complete orthogonal system in $\mathscr{L}^2(S^{n-1})$. In particular, for every $f\in\mathscr{L}^2(S^{n-1})$, the series

$$f\sim\sum_{k=0}^{\infty}\pi_k f$$

converges to f in the $\mathscr{L}^2(S^{n-1})$-norm, where $\pi_k f\in\mathscr{H}_k^n$ is the orthogonal projection of f on the space $\mathscr{H}_k^n$. Using well-known properties of the Legendre polynomials, it is not hard to show that

$$\pi_k f = N(n,k)(f*\Lambda P_k^n). \tag{2.15}$$

This motivates the spherical expansion of a measure $\mu\in\mathscr{M}(S^{n-1})$,

$$\mu\sim\sum_{k=0}^{\infty}\pi_k\mu,$$

where $\pi_k\mu\in\mathscr{H}_k^n$ is defined by:

$$\pi_k\mu = N(n,k)(\mu*\Lambda P_k^n). \tag{2.16}$$

From $P_0^n(t)=1$, $N(n,0)=1$ and $P_1^n(t)=t$, $N(n,1)=n$, we obtain, for $\mu\in\mathscr{M}(S^{n-1})$, the following special cases of (2.16):

$$\pi_0\mu=\mu(S^{n-1})\quad and\quad (\pi_1\mu)(u)=n\int_{S^{n-1}}u\cdot v\,d\mu(v). \tag{2.17}$$

Let κ_n denote the volume of the Euclidean unit ball B. By (2.4) and (2.17), for every star body $K\in\mathscr{S}^n$, it follows that

$$\kappa_n\pi_0\rho(K,\cdot)=\widetilde{W}_i(B,K)\quad and\quad \kappa_n\pi_0\rho(K,\cdot)^{n-i-1}=\widetilde{W}_i(K,B). \tag{2.18}$$

A measure $\mu\in\mathscr{M}(S^{n-1})$ is uniquely determined by its series expansion. Using the fact that ΛP_k^n is (essentially) the unique zonal function in $\mathscr{H}_k^n$, a simple calculation shows that for $\mu\in\mathscr{M}(S^{n-1},\widehat{e})$, formula (2.16) becomes

$$\pi_k\mu = N(n,k)\langle\mu,\Lambda P_k^n\rangle\Lambda P_k^n. \tag{2.19}$$

Thus, a zonal measure $\mu\in\mathscr{M}(S^{n-1},\widehat{e})$ is defined by its so-called Legendre coefficients $\mu_k:=\langle\mu,\Lambda P_k^n\rangle$. Using $\pi_k H=H$ for every $H\in\mathscr{H}_k^n$ and the fact that spherical convolution of zonal measures is commutative, we have the Funk-Hecke Theorem: If $\mu\in\mathscr{M}(S^{n-1},\widehat{e})$ and $H\in\mathscr{H}_k^n$, then $H*\mu=\mu_k H$.

A map $\Phi:\mathscr{D}\subseteq\mathscr{M}(S^{n-1})\to\mathscr{M}(S^{n-1})$ is called a multiplier transformation if there exist real numbers c_k, the multipliers of Φ, such that, for every $k\in\mathbb{N}$,

$$\pi_k\Phi\mu = c_k\pi_k\mu,\quad \forall\mu\in\mathscr{D}. \tag{2.20}$$

3. Main results

THEOREM 3.1. *If* $\Phi_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$ *is a mixed radial Blaschke-Minkowski homomorphism, then, for* $K, L \in \mathscr{S}^n$,

$$\widetilde{W}_i(K, \Phi_i L) = \widetilde{W}_i(L, \Phi_i K). \tag{3.1}$$

Proof. Let $\mu \in \mathscr{M}(S^{n-1}, \widehat{e})$ be the generating measure of Φ_i. Using (2.4), Lemma 2.2 and (2.11), it follows that

$$\begin{aligned}
\widetilde{W}_i(K, \Phi_i L) &= \kappa_n \langle \rho(\Phi_i L, \cdot), \rho(K, \cdot)^{n-i-1} \rangle \\
&= \kappa_n \langle \rho(L, \cdot)^{n-i-1} * \mu, \rho(K, \cdot)^{n-i-1} \rangle \\
&= \kappa_n \langle \rho(L, \cdot)^{n-i-1}, \rho(K, \cdot)^{n-i-1} * \mu \rangle \\
&= \kappa_n \langle \rho(L, \cdot)^{n-i-1}, \rho(\Phi_i K, \cdot) \rangle \\
&= \widetilde{W}_i(L, \Phi_i K). \quad \square
\end{aligned}$$

Using Lemma 2.2 and the fact that spherical convolution operators are multiplier transformations, one obtains that

LEMMA 3.2. *If* Φ_i *is a mixed radial Blaschke-Minkowski homomorphism which is generated by the zonal measure* μ, *then, for every star body* $K \in \mathscr{S}^n$,

$$\pi_k \rho(\Phi_i K, \cdot) = \mu_k \pi_k \rho(K, \cdot)^{n-i-1}, \tag{3.2}$$

where the numbers μ_k *are the Legendre coefficients of* μ.

DEFINITION 3.1. If Φ_i is a mixed radial Blaschke-Minkowski homomorphism, generated by the zonal measure μ, then we call the subset $\mathscr{S}^n(\Phi_i)$ of $\mathscr{S}^n$, defined by

$$\mathscr{S}^n(\Phi_i) = \{K \in \mathscr{S}^n : \pi_k \rho(K, \cdot)^{n-i-1} = o \ \ if \ \mu_k = 0\},$$

the injectivity set of Φ_i.

It is easy to verify that for every mixed radial Blaschke-Minkowski homomorphism, the set $\mathscr{S}^n(\Phi_i)$ is a non-empty rotation and dilatation invariant subset of $\mathscr{S}^n$ which is closed under L_{n-i-1} radial sum. By Lemma 3.2, a star body $K \in \mathscr{S}^n(\Phi_i)$ is uniquely determined by its image $\Phi_i K$.

DEFINITION 3.2. A star body $K \in \mathscr{S}^n$ is called polynomial if $\rho(K, \cdot) \in \mathscr{H}^n$.

Clearly, the set of polynomial star bodies is dense in $\mathscr{S}^n$ and the set of all origin-symmetric polynomial star bodies is dense in $\mathscr{S}^n_e$.

THEOREM 3.3. *If* $\Phi_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$ *is a mixed radial Blaschke-Minkowski homomorphism such that* $\mathscr{S}^n_e \in \mathscr{S}^n(\Phi_i)$, *then, for every polynomial body* $L \in \mathscr{S}^n_e$, *there exist symmetric star bodies* $K_1, K_2 \in \mathscr{S}^n_e$ *such that*

$$L \widetilde{+} \Phi_i K_1 = \Phi_i K_2.$$

Proof. Let $L \in \mathscr{S}^n_e$ be a polynomial star body. From definition 3.2 we have

$$\rho(L,\cdot) = \sum_{k=0}^{m} \pi_k \rho(L,\cdot). \tag{3.3}$$

Since $L \in \mathscr{S}^n_e$ and by the properties of the orthogonal projection of f on the space $\mathscr{H}^n_k$, we have $\pi_k \rho(L,\cdot) = 0$ for all odd $k \in \mathbb{N}$.

Let $\mu \in \mathscr{M}(S^{n-1}, \widehat{e})$ be the generating measure of Φ_i and let μ_k denote the Legendre coefficients of μ. From $\mathscr{S}^n_e \subseteq \mathscr{S}^n(\Phi_i)$ and definition 3.1, it follows that $\mu_k \neq 0$ for every even $k \in \mathbb{N}$. We define

$$f := \sum_{k=0}^{m} c_k \pi_k \rho(L,\cdot), \tag{3.4}$$

where $c_k = 0$ for odd and $c_k = \mu_k^{-1}$ if k is even. Clearly, f is an even continuous function on S^{n-1} and since spherical convolution operators are multiplier transformations, one can obtain

$$f * \mu = \sum_{k=0}^{m} c_k \mu_k \pi_k \rho(L,\cdot) = \sum_{k=0}^{m} \pi_k \rho(L,\cdot) = \rho(L,\cdot). \tag{3.5}$$

Denote by f^+ and f^- the positive and negative parts of f, and let K_1 and K_2 be the star bodies such that $\rho(K_1,\cdot)^{n-i-1} = f^-$ and $\rho(K_2,\cdot)^{n-i-1} = f^+$. Hence, (3.5) can be rewritten as

$$\rho(K_2,\cdot)^{n-i-1} * \mu = \rho(K_1,\cdot)^{n-i-1} * \mu + \rho(L,\cdot).$$

By Lemma 2.2, it follows that

$$L \widetilde{+} \Phi_i K_1 = \Phi_i K_2. \quad \square$$

In this paper, we study the following Busemann-Petty type problem for mixed radial Blaschke-Minkowski homomorphisms.

PROBLEM 3.1. Suppose $\Phi_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$ is a mixed radial Blaschke-Minkowski homomorphism. If

$$\Phi_i K \subseteq \Phi_i L,$$

does it follow that

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L)?$$

Proof of Theorem 1.1. For $K \in \Phi_i \mathscr{S}^n$, there exists a star body K_0 such that $K = \Phi_i K_0$. Using Theorem 3.1 and the fact that if $0 \leqslant i \leqslant n-1$, the dual mixed volume $\widetilde{W}_i$ is monotone with respect to set inclusion, we can conclude

$$\widetilde{W}_i(L,K) = \widetilde{W}_i(L,\Phi_i K_0) = \widetilde{W}_i(K_0,\Phi_i L) \geqslant \widetilde{W}_i(K_0,\Phi_i K) = \widetilde{W}_i(K,\Phi_i K_0) = \widetilde{W}_i(K).$$

Applying the Minkowski inequality (2.5), we obtain

$$\widetilde{W}_i(K) \leqslant \widetilde{W}_i(L).$$

Equality holds if and only if K and L are dilatations of each other. Clearly, star bodies of equal volume which are dilatations of each other must be equal. $\square$

Proof of Theorem 1.2. Let $\mu \in \mathscr{M}(S^{n-1},\widehat{e})$ be the generating measure of Φ_i and μ_k denote its Legendre coefficients. Since $\mathscr{S}^n(\Phi_i) \neq \mathscr{S}^n$ and Φ_i is non-trivial, by definition 3.1 there exists an integer $k \in \mathbb{N}$ and $k \geqslant 1$ such that $\mu_k = 0$. We can choose $\alpha > 0$ such that the function $f(u) = 1 + \alpha P_k^n(u \cdot \widehat{e}), u \in S^{n-1}$, is positive. Let $K \in \mathscr{S}^n$ be the star body with $\rho(K,\cdot)^{n-i-1} = f$. Since $\pi_k \rho(K,\cdot)^{n-i-1} = \pi_k(1 + \alpha P_k^n(u \cdot \widehat{e})) \neq 0$, from definition 3.1 we have $K \notin \mathscr{S}^n(\Phi_i)$.

From (2.18) and the properties of the orthogonal projection on the space $\mathscr{H}_k^n$, we have

$$\widetilde{W}_i(K,B) = \kappa_n \pi_0 \rho(K,\cdot)^{n-i-1} = \kappa_n = \widetilde{W}_i(B). \tag{3.6}$$

Using the fact that a star body $K \in \mathscr{S}^n(\Phi_i)$ is uniquely determined by its image $\Phi_i K$, we see that $\Phi_i B = \Phi_i K$.

Noting that K is just a perturbation of B, we use (3.6) and the Minkowski inequality (2.5) to get

$$\widetilde{W}_i(B) = \widetilde{W}_i(K,B) < \widetilde{W}_i(B)^{\frac{1}{n-i}} \widetilde{W}_i(K)^{\frac{n-i-1}{n-i}}.$$

Hence

$$\widetilde{W}_i(B) < \widetilde{W}_i(K). \quad \square$$

THEOREM 3.4. *Suppose $\mathscr{S}_e^n \subseteq \mathscr{S}^n(\Phi_i)$. If $L \in \mathscr{S}_e^n$ is a polynomial star body whose radial function is positive, then, if $L \notin \Phi_i \mathscr{S}^n$, there exists a star body $K \in \mathscr{S}_e^n$, such that*

$$\Phi_i K \subseteq \Phi_i L,$$

but

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

Proof. Let $\mu \in \mathscr{M}(S^{n-1},\widehat{e})$ be the generating measure of Φ_i. Since $L \in \mathscr{S}_e^n$ is a polynomial star body, it follows from the proof of Theorem 3.3 that there exists an even function $f \in \mathscr{H}^n$,such that

$$\rho(L,\cdot) = f * \mu. \tag{3.7}$$

The function f must assume negative values, otherwise, by Lemma 2.2 we have $L = \Phi_i L_0$, where L_0 is the star body with $\rho(L_0,\cdot)^{n-i-1} = f$. Let $F \in \mathscr{C}(S^{n-1})$ be a non-constant even function, such that $F(u) \geqslant 0$ if $f(u) < 0$, and $F(u) = 0$ if $f(u) \geqslant 0$.

By suitable approximation of the function F with spherical harmonics, we can find a non-negative, even function $G \in \mathscr{H}^n$ and an even function $H \in \mathscr{H}^n$ such that

$$\langle f, G\rangle < 0, \quad \textit{and} \quad G = H * \mu. \tag{3.8}$$

Since the radial function $\rho(L,\cdot)$ is positive, there exists a $\beta > 0$ and an origin-symmetric star body K such that

$$\rho(K,\cdot)^{n-i-1} = \rho(L,\cdot)^{n-i-1} - \beta H. \tag{3.9}$$

From (3.7) and Lemma 2.2, we see that $\rho(\Phi_i K,\cdot) = \rho(\Phi_i L,\cdot) - \beta G$. Since $G \geqslant 0$, it follows that

$$\rho(\Phi_i K,\cdot) \leqslant \rho(\Phi_i L,\cdot), \tag{3.10}$$

or equivalently

$$\Phi_i K \subseteq \Phi_i L.$$

On the other hand, applying (2.4) (2.11) (3.8) and (3.9), we obtain

$$\begin{aligned}\widetilde{W}_i(L) - \widetilde{W}_i(K,L) &= \frac{1}{n}\int_{S^{n-1}} \rho(L,u)(\rho(L,u)^{n-i-1} - \rho(K,u)^{n-i-1})dS(u)\\ &= \kappa_n \beta \langle f * \mu, H\rangle\\ &= \kappa_n \beta \langle f, H * \mu\rangle\\ &= \kappa_n \beta \langle f, G\rangle\\ &< 0.\end{aligned}$$

To complete the proof, we can use (2.5) to conclude

$$\widetilde{W}_i(K) > \widetilde{W}_i(L). \quad \square$$

REMARK. If $i = 0$, Theorem 1.1 and Theorem 1.2 are just Theorem A and Theorem B, respectively.

An immediate consequence of Theorem 1.1 is:

COROLLARY 3.5. *Suppose* $K, L \in \Phi_i \mathscr{S}^n$, *then* $\Phi_i K = \Phi_i L$ *if and only if* $K = L$. *A map* $\Phi : \mathscr{S}^n \to \mathscr{S}^n$ *is called even if* $\Phi K = \Phi(-K)$ *for every* $K \in \mathscr{S}^n$.

Based on the intersection operator, Zhang [28] introduced the mixed intersection operator $I_i : \underbrace{\mathscr{S}^n \times \cdots \times \mathscr{S}^n}_{n-i-1} \to \mathscr{S}^n$. If we restrict to origin-symmetric star bodies and Φ_i changes to be the mixed intersection operator I_i, Problem 3.1 is just a generalization of the famous Busemann-Petty problem. The mixed intersection operator I_i is an even mixed radial Blaschke-Minkowski homomorphism. Its generating measure is the (suitably normalized) invariant measure $\mu_{S_0^{n-2}}$ which is concentrated on $S_0^{n-2} = S^{n-1} \cap \widehat{e}^{\perp}$, i.e., $\rho(I_i K,\cdot)$ is the spherical Radon transform of $\rho(K,\cdot)^{n-i-1}$:

$$\rho(I_i K,\cdot) = \rho(K,\cdot)^{n-i-1} * \mu_{S_0^{n-2}}.$$

COROLLARY 3.6. *If* $K \in I_i\mathscr{S}^n$ *and* $L \in \mathscr{S}_e^n$*, then*

$$I_iK \subseteq I_iL \Rightarrow \widetilde{W}_i(K) \leqslant \widetilde{W}_i(L),$$

and $\widetilde{W}_i(K) = \widetilde{W}_i(L)$ *if and only if* $K = L$.

COROLLARY 3.7. *If* $L \notin I_i\mathscr{S}^n$*, then there exists a star body* K *such that*

$$I_iK \subseteq I_iL,$$

but

$$\widetilde{W}_i(K) > \widetilde{W}_i(L).$$

The special case $i = 0$ of Corollary 3.6 and Corollary 3.7 are the following results which are the answer to the famous Busemann-Petty problem.

COROLLARY 3.8. [20] *If* K *is an intersection body and* $L \in \mathscr{S}_e^n$*, then*

$$IK \subseteq IL \Rightarrow V(K) \leqslant V(L),$$

and $V(K) = V(L)$ *if and only if* $K = L$.

COROLLARY 3.9. [20] *If* $L \notin I\mathscr{S}^n$*, then there exists a star body* K *such that*

$$IK \subseteq IL,$$

but

$$V(K) > V(L).$$

Acknowledgement. We are grateful to the referee for the suggested improvements.

REFERENCES

[1] S. ALESKER, A. BERNIG, F. E. SCHUSTER, *Harmonic Analysis of Translation Invariant Valuations*, Geom. Funct. Anal. **21** (2011), 751–773.

[2] K. BALL, *Volume in terms of concurrent cross-sections*, Pacific J. Math. **3** (1953), 1–12.

[3] R. J. GARDNER, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. math. Soc. **342** (1994), 435–445.

[4] R. J. GARDNER, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. Math. **140** (1994), 435–447.

[5] R. J. GARDNER, A. KOLDOBSKY, AND T. SCHLUMPRECHT, *An analytic solution of the Busemann-Petty problem on sections of convex bodies*, Ann. Math. **149** (1999), 691–703.

[6] E. GRINBERG, G. ZHANG, *Convolutions, transforms and convex bodies*, Proc. London Math. Soc. **78** (1999), 77–115.

[7] C. HABERL, *L_p intersection bodies*, Adv. Math. **217** (2008), 2599–2624.

[8] C. HABERL, *Star body valued valuations*, Indiana Univ. Math. J. **58** (2009), 2253–2276.

[9] C. HABERL, F. E. SCHUSTER, *Genneral L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.

[10] N. J. KALTON, A. KOLDOBSKY, *Intersection bodies and L_p-spaces*, Adv. Math. **196** (2005), 257–275.

[11] M. KIDERLEN, *Blaschke- and Minkowski-endomorphisms of convex bodies*, Trans. Amer. Math. Soc. **358** (2006), 5539–5564.

[12] D. A. KLAIN, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80–101.
[13] A. KOLDOBSKY, *Intersection bodies, positive definite distributions, and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), 827–840.
[14] A. KOLDOBSKY, *A functional analytic approach to intersection bodies*, Geom. Funct. Anal. **10** (2000), 1507–1526.
[15] A. KOLDOBSKY, *Fourier Analysis in Convex Geometry*, Math. Surveys Monogr., vol. **116**, Amer. Math. Soc., 2005.
[16] M. LUDWIG, *Projection bodies and valuations*, Adv. Math. **172**, 2 (2002), 158–168.
[17] M. LUDWIG, *Minkowski valuations*, Trans. Amer. Math. Soc. **357**, 10 (2005), 4191–4213.
[18] M. LUDWIG, *Intersection bodies and valuations*, Amer. J. Math. **128**, 6 (2006), 1409–1428.
[19] E. LUTWAK, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.
[20] E. LUTWAK, *Intersection bodies and dual mixed volumes*, Adv. Math. **71** (1988), 232–261.
[21] P. MCMULLEN, R. SCHNEIDER, *Valuations on convex Bodies*, in: P. M. Gruber, J. M. Wills (Eds.), Convexity and Its Applications, Birkh*ä*user, Basel, 1983, pp. 170–247.
[22] R. SCHNEIDER, F. E. SCHUSTER, *Rotation equivariant Minkowski valuations*, Int. Math. Res. Not. **2006**, Art. ID 72894, 20 pp.
[23] F. E. SCHUSTER, *Volume inequalities and additive maps of convex bodies*, Mathematika **53**, 2 (2006), 211–234 (2007).
[24] F. E. SCHUSTER, *Convolutions and multiplier transformations of convex bodies*, Trans. Amer. Math. Soc. **359**, 11 (2007), 5567–5591.
[25] F. E. SCHUSTER, *Valuations and Busemann-Petty type problems*, Adv. Math. **219**, 1 (2008), 344–368.
[26] F. E. SCHUSTER, *Crofton measures and Minkowski valuations*, Duke Math. J. **154**, 1 (2010), 1–30.
[27] W. WANG, L. J. LIU AND B. W. HE, *L_p radial Minkowski homomorphisms*, Taiwan J. Math **15**, 3 (2011), 1183–1199.
[28] G. ZHANG, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. **345** (1994), 777–801.
[29] G. ZHANG, *A positive answer to the Busemann-Petty problem in four dimensions*, Ann. Math. **149** (1999), 535–543.
[30] C. J. ZHAO, *On Blaschke-Minkowski homomorphisms*, Geom. Dedicata **149** (2010), 373–378.
[31] C. J. ZHAO AND W. S. CHEUNG, *Radial Blaschke-Minkowski homomorphisms and volume differences*, Geom. Dedicata **154** (2011), 81–91.
[32] C. J. ZHAO, G. S. LENG, *Brunn-Minkowski inequality for mixed projection bodies*, J. Math. Anal. Appl. **301** (2005), 115–123.

(Received July 13, 2011)

Lijuan Liu
Department of Mathematics
Shanghai University
Shanghai, 200444
P. R. China
e-mail: ljliu@shu.edu.cn