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NECESSARY AND SUFFICIENT CONDITIONS FOR SYMMETRIC HOMOGENEOUS POLYNOMIAL INEQUALITIES IN NONNEGATIVE REAL VARIABLES

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Abstract. Let $f_n(x,y,z)$ be a symmetric homogeneous polynomial of degree n. In this paper, we give the necessary and sufficient conditions to have $f_n(x,y,z) \geqslant 0$ for $n \leqslant 6$ and any nonnegative real numbers x,y,z. In addition, we extend some results to $n=7$ and $n=8$, and then apply the proposed method to prove several elegant symmetric homogeneous polynomial inequalities of degree n, $4 \leqslant n \leqslant 8$.

1. Introduction

The fourth degree Schur's inequality is a well-known symmetric homogeneous polynomial inequality which states that

$$\sum x^4 + xyz\sum x \geqslant \sum xy(x^2+y^2) \tag{1.1}$$

for any real numbers x,y,z, where $\sum$ denotes throughout the paper a cyclic sum over x, y, z. The following generalization of the fourth degree Schur's inequality is proved in [2] for any nonnegative real numbers x,y,z.

PROPOSITION 1.1. *Let α and β be real numbers. The inequality*

$$\sum x^4 + \alpha\sum x^2y^2 + (1-\alpha+2\beta)xyz\sum x \geqslant (1+\beta)\sum xy(x^2+y^2) \tag{1.2}$$

holds for any nonnegative real numbers x,y,z if and only if

$$\alpha \geqslant \begin{cases} 2\beta, & \text{if } \beta \leqslant 0 \\ \beta^2+2\beta, & \text{if } \beta > 0 \end{cases}, \tag{1.3}$$

with equality when $x=y=z$. For $\alpha = 2\beta$ and $\beta \leqslant 0$, equality holds again when $x=0$ and $y=z$, while for $\alpha = \beta^2+2\beta$ and $\beta > 0$ - when $\dfrac{x}{\beta} = y = z$.

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On the other hand, it is known that if a,b,c are the side-lengths of a non-degenerate or degenerate triangle, then we can use the following substitutions $a = y+z$, $b = z+x$ and $c = x+y$, where x,y,z are nonnegative real numbers. Since

$$\sum a^4 + \alpha\sum a^2b^2 + (1-\alpha+2\beta)abc\sum a - (1+\beta)\sum ab(a^2+b^2) =$$
$$= (\alpha-2\beta)\sum x^4 + (4-\alpha+2\beta)\sum x^2y^2 - 4(1-\beta)xyz\sum x - 2\beta\sum xy(x^2+y^2), \quad (1.4)$$

from Proposition 1.1, we can deduce the following two corollaries.

COROLLARY 1.1. *Let α and β be real numbers. The inequality*

$$\sum a^4 + \alpha\sum a^2b^2 + (1-\alpha+2\beta)abc\sum a \geqslant (1+\beta)\sum ab(a^2+b^2) \quad (1.5)$$

holds for any side-lengths a,b,c of a non-degenerate or degenerate triangle if and only if

$$\alpha \geqslant \begin{cases} 2\beta, & \text{if } \beta \leqslant 0 \\ \beta^2+2\beta, & \text{if } 0<\beta<2, \\ 6\beta-4, & \text{if } \beta \geqslant 2 \end{cases} \quad (1.6)$$

with equality when $a=b=c$. For $\alpha = 2\beta$ and $\beta \leqslant 0$, equality also holds when $a=0$ and $b=c$, for $\alpha=\beta^2+2\beta$ and $0<\beta<2$ - when $\frac{a}{\beta}=b=c$, and for $\alpha=6\beta-4$ and $\beta \geqslant 2$ - when $\frac{a}{2}=b=c$.

COROLLARY 1.2. *Let α and β be real numbers. The inequality*

$$\sum a^4 + \alpha\sum a^2b^2 + (1-\alpha+2\beta)abc\sum a \leqslant (1+\beta)\sum ab(a^2+b^2) \quad (1.7)$$

holds for any side-lengths a,b,c of a non-degenerate or degenerate triangle if and only if

$$\alpha \leqslant \begin{cases} 6\beta-4, & \text{if } \beta \leqslant 1 \\ 2\beta, & \text{if } \beta \geqslant 1 \end{cases}, \quad (1.8)$$

with equality when $a=b=c$. For $\alpha=6\beta-4$ and $\beta \leqslant 1$, equality also holds when $\frac{a}{2}=b=c$, while for $\alpha=2\beta$ and $\beta \geqslant 1$ - when $a=0$ and $b=c$.

A symmetric and homogeneous polynomial of degree six has the general form

$$f_6(x,y,z) = A_1\sum x^6 + A_2\sum xy(x^4+y^4) + A_3\sum x^2y^2(x^2+y^2)$$
$$+ A_4\sum x^3y^3 + A_5xyz\sum x^3 + A_6xyz\sum xy(x+y) + A_7x^2y^2z^2, \quad (1.9)$$

where $A_1,\ldots,A_7$ are real constants. Using the substitutions

$$p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz, \quad (1.10)$$

and the identities

$$\begin{cases} \sum x^6 = 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3, \\ \sum xy(x^4 + y^4) = -3r^2 + (7pq - p^3)r + p^4q - 4p^2q^2 + 2q^3, \\ \sum x^2y^2(x^2 + y^2) = -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3, \\ \sum x^3y^3 = 3r^2 - 3pqr + q^3, \\ \sum x^4 = 4pr + p^4 - 4p^2q + 2q^2, \quad \sum xy(x^2 + y^2) = -pr + p^2q - 2q^2, \\ \sum x^3 = 3r + p^3 - 3pq, \quad \sum xy(x + y) = pq - 3r, \end{cases} \tag{1.11}$$

we may express any polynomial $f_6(x,y,z)$ in terms of p,q,r as follows

$$f_6(x,y,z) = Ar^2 + g_1(p,q)r + g_2(p,q), \tag{1.12}$$

where

$$g_1(p,q) = Bp^3 + Cpq, \quad g_2(p,q) = Dp^6 + Ep^4q + Fp^2q^2 + Gq^3, \tag{1.13}$$

A, B, C, D, E, F, G being real constants. Throughout the paper, we call the coefficient A of r^2 in the expansion (1.12) the *highest coefficient* of $f_6(x,y,z)$.

Our method for proving any inequality $f_6(x,y,z) \geqslant 0$ by means of the necessary and sufficient conditions has as starting-point the form (1.12) of the polynomial $f_6(x,y,z)$. To bring a given sixth degree symmetric homogeneous polynomial to this form, the following identities can be also useful

$$\begin{aligned}(x-y)^2(y-z)^2(z-x)^2 = &\sum x^2y^2(x^2+y^2) - 2\sum x^3y^3 - 2xyz\sum x^3 \\ &+ 2xyz\sum xy(x+y) - 6x^2y^2z^2,\end{aligned} \tag{1.14}$$

$$(x-y)^2(y-z)^2(z-x)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3. \tag{1.15}$$

More general, for $n \in \{6,7,8\}$, the expansion of a symmetric and homogeneous polynomial $f_n(x,y,z)$ of degree n in terms of p, q and r has the form

$$f_n(x,y,z) = h_0(p,q)r^2 + h_1(p,q)r + h_2(p,q), \tag{1.16}$$

where $h_0(p,q)$, $h_1(p,q)$ and $h_2(p,q)$ are polynomial functions. The *highest polynomial* $h_0(p,q)$ has the form

$$h_0(p,q) = \begin{cases} A, & \text{if } n = 6 \\ Ap, & \text{if } n = 7 \\ Ap^2 + Bq, & \text{if } n = 8 \end{cases}, \tag{1.17}$$

where A and B are real constants. In the following section we will show that the proof of an inequality $f_n(x,y,z) \geqslant 0$ for $n \in \{6,7,8\}$ is much simpler in the case when $h_0(p,q) \leqslant 0$ for all $x,y,z \geqslant 0$.

The proposed necessary and sufficient conditions to have $f_n(x,y,z) \geqslant 0$ for any nonnegative real variables x,y,z are presented in section 2 and proved in section 3. In section 4, we apply the obtained results for proving some interesting symmetric homogeneous polynomial inequalities of degree five, six, seven and eight. Notice that almost all these inequalities were recently posted on the well-known website Art of Problem Solving, and no solution was given to some of them (see [3]...[9]).

2. Main Results

Our results rely on the following lemma.

LEMMA 2.1. *Let $x \leqslant y \leqslant z$ be nonnegative real numbers such that $x+y+z=p$ and $xy+yz+zx=q$, where p and q are given nonnegative real numbers satisfying $p^2 \geqslant 3q \geqslant 0$. Then, the product $r=xyz$ is maximal when $x=y$, and is minimal when $y=z$ or $x=0$; more exactly,*

$$r \in [r_{min}(p,q), r_{max}(p,q)], \tag{2.1}$$

where

$$r_{max}(p,q) = \frac{(p-\sqrt{p^2-3q})^2(p+2\sqrt{p^2-3q})}{27}, \tag{2.2}$$

and

$$r_{min}(p,q) = \begin{cases} \dfrac{(p-2\sqrt{p^2-3q})(p+\sqrt{p^2-3q})^2}{27}, & \text{if } 3q \leqslant p^2 < 4q \\ 0, & \text{if } p^2 \geqslant 4q. \end{cases} \tag{2.3}$$

Using Lemma 2.1, we can prove the following theorems.

THEOREM 2.1. *Let $f_n(x,y,z)$ be a symmetric homogeneous polynomial of degree $n \leqslant 5$. The inequality*

$$f_n(x,y,z) \geqslant 0$$

holds for all nonnegative real numbers x,y,z if and only if $f_n(x,1,1) \geqslant 0$ and $f_n(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$.

REMARK 2.1. Using Theorem 2.1, we can give short solutions for any fourth degree symmetric homogeneous polynomial inequality in nonnegative real numbers. For instant, with regard to Proposition 1.1, setting

$$f_4(x,y,z) = \sum x^4 + \alpha \sum x^2y^2 + (1-\alpha+2\beta)xyz\sum x - (1+\beta)\sum xy(x^2+y^2), \tag{2.4}$$

we have

$$\begin{aligned} f_4(x,1,1) =& (x-1)^2(x^2-2\beta x+\alpha-2\beta) \\ =& (x-1)^2[(x-\beta)^2+\alpha-\beta^2-2\beta] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} f_4(0,y,z) =&(y-z)^4+(3-\beta)yz(y-z)^2+(\alpha-2\beta)y^2z^2 \\ =&(y-z)^4+(3-\beta)yz(y-z)^2+\beta^2y^2z^2+(\alpha-\beta^2-2\beta)y^2z^2. \end{aligned} \tag{2.6}$$

Using Theorem 2.1, it is easy to reach the required conclusion.

THEOREM 2.2. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A \leqslant 0$. The inequality*

$$f_6(x,y,z) \geqslant 0$$

holds for all nonnegative real numbers x,y,z if and only if $f_6(x,1,1) \geqslant 0$ and $f_6(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$.

With regard to the polynomial $f_6(x,y,z)$ written in the form

$$f_6(x,y,z) = Ar^2 + g_1(p,q)r + g_2(p,q), \tag{2.7}$$

let us denote

$$h(t) = 2At + g_1(t+2, 2t+1) \tag{2.8}$$

and

$$d(p,q) = g_1^2(p,q) - 4Ag_2(p,q). \tag{2.9}$$

In addition, assume that

$$d(p,q) > 0 \iff \frac{p^2}{q} \in \mathbb{I} \cup \mathbb{J}, \tag{2.10}$$

where $\mathbb{I}$ is a union of intervals $\mathbb{I}_i \subseteq [3,4)$, and $\mathbb{J}$ is a union of intervals $\mathbb{J}_i \subseteq [4,\infty]$, and

$$\frac{(t+2)^2}{2t+1} \in \mathbb{I}_i,\ 0 < t \leqslant 1 \iff t \in \mathbb{K}_i, \tag{2.11}$$

$$\frac{(t+2)^2}{2t+1} \in \mathbb{I}_i,\ 1 \leqslant t < 4 \iff t \in \mathbb{L}_i, \tag{2.12}$$

$$\frac{(t+2)^2}{2t+1} \in \mathbb{J}_i,\ t \geqslant 4 \iff t \in \mathbb{M}_i. \tag{2.13}$$

THEOREM 2.3. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A > 0$. The inequality*

$$f_6(x,y,z) \geqslant 0$$

holds for all nonnegative real numbers x,y,z if and only if the following three conditions are fulfilled:

(a) $f_6(x,1,1) \geqslant 0$ and $f_6(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$;

(b) for each interval $\mathbb{I}_i$, we have $h(t) \geqslant 0$ for $t \in \mathbb{K}_i$ or $h(t) \leqslant 0$ for $t \in \mathbb{L}_i$;

(c) for each interval $\mathbb{J}_i$, we have $g_1(\sqrt{w},1) \geqslant 0$ for $w \in \mathbb{J}_i$ or $h(t) \leqslant 0$ for $t \in \mathbb{M}_i$.

The following corollary is an immediate consequence of Theorem 2.3.

COROLLARY 2.1. *Let $f_6(x,y,z)$ be a symmetric homogeneous polynomial of degree six having the highest coefficient $A>0$. The inequality*

$$f_6(x,y,z) \geqslant 0$$

holds for all $x,y,z \geqslant 0$ if the following three conditions are fulfilled:
(a) $f_6(x,1,1) \geqslant 0$ *and* $f_6(0,y,z) \geqslant 0$ *for all* $x,y,z \geqslant 0$;
(b) $h(t) \geqslant 0$ *for* $t \in (0,1]$ *or* $h(t) \leqslant 0$ *for* $t \in [1,4)$;
(c) $g_1(p,1) \geqslant 0$ *for* $p \geqslant 2$ *or* $h(t) \leqslant 0$ *for* $t \geqslant 4$.

In order to find similar results for the side-lengths of a non-degenerate or degenerate triangle, we need the following lemma.

LEMMA 2.2. *Let $a \geqslant b \geqslant c$ be the side-lengths of a non-degenerate or degenerate triangle such that $a+b+c=p$ and $ab+bc+ca=q$, where p and q are given positive real numbers satisfying $p^2 \geqslant 3q > 0$. The product $r=abc$ is minimal when $a=b$, and is maximal when $b=c$ or $a=b+c$.*

Using Lemma 2.2, we can prove the following theorems, which are useful to prove symmetric homogeneous polynomial inequalities of the form $g_n(a,b,c) \geqslant 0$, where a,b,c are triangle side-lengths.

THEOREM 2.4. *Let a,b,c be the side-lengths of a non-degenerate or degenerate triangle, and let $g_n(a,b,c)$ be a symmetric homogeneous polynomial of degree $n \leqslant 5$. The inequality*

$$g_n(a,b,c) \geqslant 0$$

holds for any triangle if and only if $g_n(x,1,1) \geqslant 0$ for all $0 \leqslant x \leqslant 2$, and $g_n(y+z,y,z) \geqslant 0$ for all $y,z \geqslant 0$.

THEOREM 2.5. *Let a,b,c be the side-lengths of a non-degenerate or degenerate triangle, and let $g_6(a,b,c)$ be a symmetric homogeneous polynomial of degree six whose highest coefficient is non-positive. The inequality*

$$g_6(a,b,c) \geqslant 0$$

holds for any triangle if and only if $g_6(x,1,1) \geqslant 0$ for all $0 \leqslant x \leqslant 2$, and $g_6(y+z,y,z) \geqslant 0$ for all $y,z \geqslant 0$.

We can extend Theorem 2.2 and Theorem 2.5 to symmetric homogeneous polynomial inequalities of degree $n=7$ and $n=8$.

THEOREM 2.6. *Let $f_7(x,y,z)$ be a symmetric homogeneous polynomial of degree seven whose highest polynomial is $h_0(p,q)=Ap$, where A is a non-positive real constant. The inequality*

$$f_7(x,y,z) \geqslant 0$$

holds for all nonnegative real numbers x,y,z *if and only if* $f_7(x,1,1) \geqslant 0$ *and* $f_7(0,y,z) \geqslant 0$ *for all* $x,y,z \geqslant 0$.

THEOREM 2.7. *Let* $f_8(x,y,z)$ *be a symmetric homogeneous polynomial of degree eight whose highest polynomial is* $h_0(p,q) = Ap^2 + Bq$, *where* A *and* B *are real constants satisfying* $A \leqslant 0$, $3A + B \leqslant 0$. *The inequality*

$$f_8(x,y,z) \geqslant 0$$

holds for all nonnegative real numbers x,y,z *if and only if* $f_8(x,1,1) \geqslant 0$ *and* $f_8(0,y,z) \geqslant 0$ *for all* $x,y,z \geqslant 0$.

THEOREM 2.8. *Let* a,b,c *be the side-lengths of a non-degenerate or degenerate triangle, and let* $g_7(a,b,c)$ *be a symmetric homogeneous polynomial of degree seven whose highest polynomial is* $h_0(p,q) = Ap$, *where* A *is a non-positive real constant. The inequality*

$$g_7(a,b,c) \geqslant 0$$

holds for any triangle if and only if $g_7(x,1,1) \geqslant 0$ *for all* $0 \leqslant x \leqslant 2$, *and* $g_7(y+z,y,z) \geqslant 0$ *for all* $y,z \geqslant 0$.

THEOREM 2.9. *Let* a,b,c *be the side-lengths of a non-degenerate or degenerate triangle, and let* $g_8(a,b,c)$ *be a symmetric homogeneous polynomial of degree eight whose highest polynomial is* $h_0(p,q) = Ap^2 + Bq$, *where* A *and* B *are real constants satisfying* $3A + B \leqslant 0$ *and* $4A + B \leqslant 0$. *The inequality*

$$g_8(a,b,c) \geqslant 0$$

holds for any triangle if and only if $g_8(x,1,1) \geqslant 0$ *for all* $0 \leqslant x \leqslant 2$, *and* $g_8(y+z,y,z) \geqslant 0$ *for all* $y,z \geqslant 0$.

3. Proof of lemmas and theorems

Proof of Lemma 2.1. Writing the inequality $(x-y)(x-z) \geqslant 0$ in the equivalent form $3x^2 - 2px + q \geqslant 0$, we get

$$x \leqslant x_{max} = \frac{p - \sqrt{p^2 - 3q}}{3},$$

with equality for $x = y$. Similarly, writing the inequality $(y+z)^2 \geqslant 4yz$ in the form $3x^2 - 2px + 4q - p^2 \leqslant 0$, we get

$$x \geqslant x_{min} = \begin{cases} \dfrac{p - 2\sqrt{p^2 - 3q}}{3}, & \text{if } 3q \leqslant p^2 < 4q \\ 0, & \text{if } p^2 \geqslant 4q \end{cases},$$

with equality for $y = z$ if $3q \leqslant p^2 < 4q$, and for $x = 0$ if $p^2 \geqslant 4q$.

On the other hand, the function $r(x)=x^3-px^2+qx$ is strictly increasing on $[x_{min},x_{max}]$ since $r'(x)=3x^2-2px+q=(x-y)(x-z)\geqslant 0$. Thus, $r(x)$ is maximal for $x=x_{max}$, when

$$x=y=\frac{p-\sqrt{p^2-3q}}{3},\quad z=\frac{p+2\sqrt{p^2-3q}}{3},$$

and is minimal for $x=x_{min}$, when

$$x=\frac{p-2\sqrt{p^2-3q}}{3},\quad y=z=\frac{p+\sqrt{p^2-3q}}{3}$$

if $3q\leqslant p^2<4q$, or

$$x=0$$

if $p^2\geqslant 4q$. □

Proof of Theorem 2.1. Let $p=x+y+z$, $q=xy+yz+zx$ and $r=xyz$. Any symmetric and homogeneous polynomial $f_n(x,y,z)$ of degree $n\leqslant 5$ can be written as

$$f_n(x,y,z)=A_n(p,q)r+B_n(p,q),$$

where $A_n(p,q)$ and $B_n(p,q)$ are polynomial functions. For fixed p and q, the linear function $g_n(r)=A_n(p,q)r+B_n(p,q)$ is minimal when r is minimal or maximal; that is, by Lemma 2.1, when two of x,y,z are equal or one of x,y,z is 0. Due to symmetry and homogeneity, the conclusion follows. □

Proof of Theorem 2.2. For fixed $p=x+y+z$ and $q=xy+yz+zx$, the inequality $f_6(x,y,z)\geqslant 0$ is equivalent to $g(r)\geqslant 0$, where $g(r)$ is a quadratic function having the form (2.7). Since $g(r)$ is concave for $A\leqslant 0$, $g(r)$ is minimal when r is minimal or maximal; that is, by Lemma 2.1, when two of x,y,z are equal or one of x,y,z is 0. □

Proof of Theorem 2.3. If $q=0$, then two of x,y,z are zero, and the desired inequality $f_6(x,y,z)\geqslant 0$ holds according to the condition $f_6(0,y,z)\geqslant 0$ in (a). On the other hand, since $d(p,q)$ is the discriminant of the quadratic function (defined for fixed p and q)

$$g(r)=Ar^2+g_1(p,q)r+g_2(p,q),\quad A>0,$$

the desired inequality $g(r)\geqslant 0$ holds if $d(p,q)\leqslant 0$. Therefore, we will consider further that $q>0$ and $d(p,q)>0$. By Lemma 2.1, for fixed p and q, r attains its extreme values r_{min} and r_{max} when two of x,y,z are equal or one of them is 0. Then, the necessary conditions $g(r_{min})\geqslant 0$ and $g(r_{max})\geqslant 0$ are satisfied if the necessary conditions in (a) are fulfilled. In addition, the inequality $g(r)\geqslant 0$ holds for all $x,y,z\geqslant 0$ if and only if $r_{min}(p,q)\geqslant \dfrac{-g_1(p,q)}{2A}$ or $r_{max}(p,q)\leqslant \dfrac{-g_1(p,q)}{2A}$; that is, either

$$2Ar_{min}(p,q)+g_1(p,q)\geqslant 0, \tag{3.1}$$

or

$$2Ar_{max}(p,q) + g_1(p,q) \leqslant 0. \tag{3.2}$$

We need to approach separately the cases $p^2/q \in \mathbb{I}_i \subseteq [3,4)$ and $p^2/q \in \mathbb{J}_i \subseteq [4,\infty)$, which generate the conditions in (b) and (c), respectively.

Case 1. $\dfrac{p^2}{q} \in \mathbb{I}_i \subseteq [3,4)$.

Let

$$a = \frac{p - 2\sqrt{p^2-3q}}{3} > 0, \;\; b = \frac{p + \sqrt{p^2-3q}}{3}, \;\; t = \frac{a}{b}.$$

From $3 \leqslant p^2/q < 4$, we get $0 < t \leqslant 1$. We have

$$a + 2b = p, \quad 2ab + b^2 = q, \quad \frac{p^2}{q} = \frac{(t+2)^2}{2t+1},$$

and hence the condition $p^2/q \in \mathbb{I}_i$ is equivalent to

$$\frac{(t+2)^2}{2t+1} \in \mathbb{I}_i, \;\; 0 < t \leqslant 1.$$

On the other hand, by Lemma 2.1, the condition (3.1) becomes

$$2Aab^2 + g_1(a+2b, 2ab+b^2) \geqslant 0,$$

which is equivalent to $h(t) \geqslant 0$.

Let now

$$a = \frac{p + 2\sqrt{p^2-3q}}{3}, \;\; b = \frac{p - \sqrt{p^2-3q}}{3} > 0, \;\; t = \frac{a}{b}.$$

From $3 \leqslant p^2/q < 4$, we get $1 \leqslant t < 4$. We have

$$a + 2b = p, \quad 2ab + b^2 = q, \quad \frac{p^2}{q} = \frac{(t+2)^2}{2t+1},$$

and hence the condition $p^2/q \in \mathbb{I}_i$ is equivalent to

$$\frac{(t+2)^2}{2t+1} \in \mathbb{I}_i, \;\; 1 \leqslant t < 4.$$

On the other hand, by Lemma 2.1, the condition (3.2) becomes

$$2Aab^2 + g_1(a+2b, 2ab+b^2) \leqslant 0,$$

which is equivalent to $h(t) \leqslant 0$.

Case 2. $\dfrac{p^2}{q} \in \mathbb{J}_i \subseteq [4,\infty)$.

By Lemma 2.1, the condition (3.1) becomes $g_1(p,q) \geqslant 0$. Setting $w = p^2/q \in \mathbb{J}_i$, this condition can be written as $g_1(\sqrt{w},1) \geqslant 0$.

Let now

$$a = \frac{p+2\sqrt{p^2-3q}}{3}, \quad b = \frac{p-\sqrt{p^2-3q}}{3} > 0, \quad t = \frac{a}{b}.$$

From $p^2/q \geqslant 4$, we get $t \geqslant 4$. It is easy to check that

$$a+2b = p, \quad 2ab+b^2 = q, \quad \frac{p^2}{q} = \frac{(t+2)^2}{2t+1},$$

and therefore the condition $p^2/q \in \mathbb{J}_i$ is equivalent to

$$\frac{(t+2)^2}{2t+1} \in \mathbb{J}_i, \quad t \geqslant 4.$$

On the other hand, according to Lemma 2.1, the condition (3.2) becomes

$$2Aab^2 + g_1(a+2b, 2ab+b^2) \leqslant 0.$$

which is equivalent to $h(t) \leqslant 0$.

Combining all results in the cases 1 and 2, the conclusion follows. □

Proof of Lemma 2.2. We use the substitutions $a = y+z$, $b = z+x$, $c = x+y$, where $x,y,z \geqslant 0$. If $a+b+c$ and $ab+bc+ca$ are fixed, then from

$$a+b+c = 2(x+y+z)$$

and

$$ab+bc+ca = (x+y+z)^2 + xy+yz+zx,$$

it follows that $x+y+z$ and $xy+yz+zx$ are fixed, and from

$$abc = (x+y+z)(xy+yz+zx) - xyz,$$

it follows that the product abc is minimal/maximal when the product xyz is maximal/minimal. So, by Lemma 2.1, the product abc is minimal when $x = y \leqslant z$, that is $a = b \geqslant c$, and is maximal when $y = z \geqslant x$ or $x = 0$, that is $a/2 \leqslant b = c \leqslant a$ or $a = b+c$. □

Proof of Theorem 2.4. Let $p = a+b+c$, $q = ab+bc+ca$, $r = abc$. Any symmetric and homogeneous polynomial $g_n(a,b,c)$ of degree $n \leqslant 5$ can be written as

$$g_n(a,b,c) = A_n(p,q)r + B_n(p,q),$$

where $A_n(p,q)$ and $B_n(p,q)$ are polynomial functions. For fixed p and q, the linear function $g(r) = A_n(p,q)r + B_n(p,q)$ is minimal when r is minimal or maximal; that is, by Lemma 2.2, when two of a,b,c are equal or one of a,b,c is the sum of the others. Due to symmetry and homogeneity, the conclusion follows. $\square$

Proof of Theorem 2.5. Let $p = a+b+c$, $q = ab+bc+ca$, $r = abc$. For fixed p and q, the inequality $g_6(a,b,c) \geqslant 0$ is equivalent to $g(r) \geqslant 0$, where $g(r)$ is a quadratic function having the form (2.7). Since $g(r)$ is concave for $A \leqslant 0$, $g(r)$ is minimal when r is minimal or maximal; that is, by Lemma 2.2, when two of a,b,c are equal or one of a,b,c is the sum of the others. Thus, the proof is completed. $\square$

Proof of Theorem 2.6. Since $A \leqslant 0$, the highest polynomial $h_0(p,q) = Ap$ is non-positive for all $x,y,z \geqslant 0$. Further, the proof is similar to the one of Theorem 2.2. $\square$

Proof of Theorem 2.7. Since $A \leqslant 0$, $3A+B \leqslant 0$ and $p^2 - 3q \geqslant 0$, we have

$$h_0(p,q) = Ap^2 + Bq = A(p^2 - 3q) + (3A+B)q \leqslant 0,$$

for all $x,y,z \geqslant 0$. Further, the proof is similar to the one of Theorem 2.2. $\square$

Proof of Theorem 2.8. Since $A \leqslant 0$, the highest polynomial $h_0(p,q) = Ap$ is non-positive. Further, the proof is similar to the one of Theorem 2.5. $\square$

Proof of Theorem 2.9. Since $3A+B \leqslant 0$, $4A+B \leqslant 0$, $p^2 - 3q \geqslant 0$ and

$$4q - p^2 = 2\sum ab - \sum a^2 = \sum a^2 - \sum (b-c)^2 = \sum (a+b-c)(a-b+c) \geqslant 0,$$

we have

$$h_0(p,q) = Ap^2 + Bq = (4A+B)(p^2 - 3q) + (3A+B)(4q - p^2) \leqslant 0.$$

Further, the proof is similar to the one of Theorem 2.5. $\square$

4. Applications

We will prove several inequalities of degree five, six, seven and eight, in nonnegative real variable and in triangle side-lengths. Notice that the coefficient of the product

$$(x-y)^2(y-z)^2(z-x)^2$$

in Propositions 4.5 ... 4.9 has the best possible values.

PROPOSITION 4.1. *Let x,y,z be nonnegative real numbers. If $k \leqslant 2$, then*

$$\sum x(x-y)(x-z)(x-ky)(x-kz) \geqslant 0$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$ ([3]).

Proof. Let $f_5(x,y,z)$ be the left side of the inequality. By Theorem 2.1, it suffices to prove that $f_5(x,1,1) \geqslant 0$ and $f_5(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$. Indeed, we have

$$f_5(x,1,1) = x(x-1)^2(x-k)^2 \geqslant 0,$$

$$f_5(0,y,z) = (y-z)^2(y+z)(y^2+z^2-kyz) \geqslant 0. \qquad \square$$

PROPOSITION 4.2. *If x,y,z are nonnegative real numbers, then*

$$\sum x(x-2y)(x-2z)(x-5y)(x-5z) \geqslant 0,$$

with equality for $x=0$ and $y^2+z^2-4yz=0$ (or any cyclic permutation) ([3]).

Proof. By Theorem 2.1, we need to show that $f_5(x,1,1) \geqslant 0$ and $f_5(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$, where $f_5(x,y,z)$ is the left side of the inequality. We have

$$f_5(x,1,1) = x^3(x-7)^2 + 20x^3 - 60x^2 + 44x + 8 \geqslant 20x^3 - 60x^2 + 44x + 8 > 0,$$

since

$$20x^3 - 60x^2 + 44x + 8 > 20x^2(x-3) \geqslant 0$$

for $x \geqslant 3$, and

$$20x^3 - 60x^2 + 44x + 8 = 5(2x-3)^2 + 8 - x \geqslant 8 - x \geqslant 0$$

for $x \leqslant 8$. Also,

$$f_6(0,y,z) = (y+z)(y^2+z^2-4yz)^2 \geqslant 0. \qquad \square$$

PROPOSITION 4.3. *Let x,y,z be nonnegative real numbers. If $k \in \mathbb{R}$, then*

$$\sum (y+z)(x-y)(x-z)(x-ky)(x-kz) \geqslant 0$$

with equality for $x=y=z$, for $y=z=0$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$.

Proof. Let

$$f_5(x,y,z) = \sum (y+z)(x-y)(x-z)(x-ky)(x-kz).$$

By Theorem 2.1, it suffices to prove that $f_5(x,1,1) \geqslant 0$ and $f_5(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$. Indeed,

$$f_5(x,1,1) = 2(x-1)^2(x-k)^2 \geqslant 0,$$

$$f_5(0,y,z) = k^2(y+z)y^2z^2 + yz(y+z)(y-z)^2 \geqslant 0. \qquad \square$$

PROPOSITION 4.4. *Let x,y,z be nonnegative real numbers, no two of which are zero. If $k>-2$, then*

$$\sum\frac{x(y+z)+(k-1)yz}{y^2+kyz+z^2}\geqslant\frac{3(k+1)}{k+2}$$

with equality for $x=y=z$, and for $x=0$ and $y=z$ (or any cyclic permutation) ([1], *pp.* 311).

Proof. Write the inequality as $f_6(x,y,z)\geqslant 0$, where

$$\begin{aligned}f_6(x,y,z)=&(k+2)\sum[x(y+z)+(k-1)yz](x^2+kxy+y^2)(x^2+kxz+z^2)\\&-3(k+1)\prod(y^2+kyz+z^2).\end{aligned}$$

Let $p=x+y+z$, $q=xy+yz+zx$, $r=xyz$. From

$$\begin{aligned}f_6(x,y,z)=&(k+2)\sum[q+(k-2)yz](p^2-2q+kxy-z^2)(p^2-2q+kxz-y^2)\\&-3(k+1)\prod(p^2-2q+kyz-x^2),\end{aligned}$$

it follows that $f_6(x,y,z)$ has the same highest coefficient as

$$\begin{aligned}f(x,y,z)&=(k+2)(k-2)\sum yz(kxy-z^2)(kxz-y^2)-3(k+1)\prod(kyz-x^2)\\&=3(-k^3-4k^2+k+1)r^2+k(k^2+3k+8)r\sum x^3-(2k^2+3k+4)\sum x^3y^3.\end{aligned}$$

Therefore,

$$A=3(-k^3-4k^2+k+1)+3k(k^2+3k+8)-3(2k^2+3k+4)=-9(k-1)^2.$$

Since $A\leqslant 0$, it suffices to show that the desired inequality holds for $y=z=1$ and for $x=0$ (Theorem 2.2). In these cases, the original inequality is equivalent to

$$(k+2)x(x-1)^2\geqslant 0$$

and

$$(y-z)^2[(k+2)(y^2+z^2)+(k^2+k+1)yz]\geqslant 0,$$

respectively, which are clearly true for $k>-2$. □

PROPOSITION 4.5. *If x,y,z are nonnegative real numbers, and*

$$\alpha_k=\begin{cases}4(k-2), & \text{if } 2\leqslant k\leqslant 6\\ \dfrac{(k+2)^2}{4}, & \text{if } k>6\end{cases},$$

then

$$\sum x(x-y)(x-z)(x-ky)(x-kz)+\frac{\alpha_k(x-y)^2(y-z)^2(z-x)^2}{x+y+z}\geqslant 0,$$

with equality for $x=y=z$, for $x=0$ and $y=z$ (or any cyclic permutation), for $x/k=y=z$ (or any cyclic permutation), and for $x=0$ and $y/z+z/y=(k-2)/2$ (or any cyclic permutation) if $k>6$ ([6]).

Proof. Write the inequality as $f_6(x,y,z) \geqslant 0$, where

$$f_6(x,y,z) = (x+y+z)\sum x(x-y)(x-z)(x-ky)(x-kz) + \alpha_k(x-y)^2(y-z)^2(z-x)^2.$$

According to (1.15), the highest coefficient of $f_6(x,y,z)$ is $A = -27\alpha_k$. Since $A \leqslant 0$ for $k \geqslant 2$, it suffices to show that $f_6(x,1,1) \geqslant 0$ and $f_6(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$ (Theorem 2.2). These conditions are true since

$$f_6(x,1,1) = x(x+2)(x-1)^2(x-k)^2,$$

$$f_6(0,y,z) = \begin{cases} \dfrac{(y-z)^4[(y-z)^2+(6-k)yz]}{y+z}, & \text{if } 2 \leqslant k \leqslant 6 \\ \dfrac{(y-z)^2[2(y-z)^2+(6-k)yz]^2}{4(y+z)}, & \text{if } k > 6 \end{cases}. \qquad \square$$

PROPOSITION 4.6. *If x,y,z are nonnegative real numbers, then*

$$\sum x(y+z)(x-y)(x-z)(x-2y)(x-2z) \geqslant (x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x=y=z$, for $x=0$ and $y=z$ (or any cyclic permutation), for $y=z=0$ (or any cyclic permutation), and for $x/2=y=z$ (or any cyclic permutation) ([9]).

Proof. Write the inequality as $f_6(x,y,z) \geqslant 0$, where

$$f_6(x,y,z) = \sum x(y+z)(x-y)(x-z)(x-2y)(x-2z) - (x-y)^2(y-z)^2(z-x)^2.$$

Since

$$f_6(x,y,z) = \sum(q-yz)(x^2+2yz-q)(x^2+6yz-2q) - (x-y)^2(y-z)^2(z-x)^2,$$

where $q = xy+yz+zx$, it follows that f_6 has the same highest coefficient as

$$\begin{aligned} f(x,y,z) &= -\sum yz(x^2+2yz)(x^2+6yz) + 27x^2y^2z^2 \\ &= -12\sum y^3z^3 - xyz\sum x^3 + 3x^2y^2z^2; \end{aligned}$$

that is, $A = -36-3+3 = -36$. Since $A < 0$, it suffices to show that $f_6(x,1,1) \geqslant 0$ and $f_6(0,y,z) \geqslant 0$ for all $x,y,z \geqslant 0$ (Theorem 2.2). We have

$$f_6(x,1,1) = 2x(x+2)(x-1)^2(x-2)^2 \geqslant 0,$$

$$f_6(0,y,z) = yz(y-z)^4 \geqslant 0. \qquad \square$$

PROPOSITION 4.7. *Let x,y,z be nonnegative real numbers. If $k > 0$, then*

$$\sum yz(x-y)(x-z)(x-ky)(x-kz) \geqslant 0,$$

with equality for $x=y=z$, for $y=z=0$ (or any cyclic permutation), and for $x/k = y=z$ (or any cyclic permutation) ([9]).

Proof. For $k=1$, the inequality has the obvious form

$$\sum yz(x-y)^2(x-z)^2 \geqslant 0.$$

We consider further that $k \neq 1$, denote the left side of the inequality by $f_6(x,y,z)$ and apply Theorem 2.3. Let $p=x+y+z$, $q=xy+yz+zx$, $r=xyz$. From

$$f_6(x,y,z) = k^2\sum x^3y^3 + r\sum x^3 - (k^2+k+1)r\sum xy(x+y) + 3(k+1)^2r^2,$$

using (1.11), we can write $f_6(x,y,z)$ in the form (2.7), where

$$A = 9(k^2+k+1) > 0,$$

$$g_1(p,q) = p^3 - (4k^2+k+4)pq, \quad g_2(p,q) = k^2q^3.$$

We have

$$\begin{aligned} h(t) &= 18(k^2+k+1)t + (t+2)^3 - (4k^2+k+4)(t+2)(2t+1) \\ &= t[t^2 - 2(4k^2+k+1)t - 2k^2 + 13k + 10], \end{aligned}$$

$$g_1(\sqrt{w},1) = \sqrt{w}(w - 4k^2 - k - 4)$$

and

$$d(p,q) = g_1^2 - 4Ag_2 = (p^2 - w_1q)(p^2 - w_2q)(p^2 - w_3q),$$

where

$$\begin{aligned} w_1 &= 4(k^2+k+1), \\ w_2 &= 2k^2 - k + 2 + 2(k-1)\sqrt{k^2+k+1}, \\ w_3 &= 2k^2 - k + 2 - 2(k-1)\sqrt{k^2+k+1}. \end{aligned}$$

For $t>0$, $h(t)$ has the same sign as

$$h_0(t) = t^2 - 2(4k^2+k+1)t - 2k^2 + 13k + 10,$$

whose derivative is

$$h_0'(t) = 2(t - 4k^2 - k - 1).$$

The condition (a) in Theorem 2.3 is fulfilled since

$$f_6(x,1,1) = (x-1)^2(x-k)^2 \geqslant 0, \quad f_6(0,y,z) = k^2y^3z^3 \geqslant 0.$$

We need to consider further the following three cases.

Case 1. $k \geqslant \frac{8}{7}$. Since $w_1 > w_2 \geqslant 4 > 3 > w_3$, we have

$$\mathbb{I}_1 = [3,4), \quad \mathbb{J}_1 = [4, w_2), \quad \mathbb{J}_2 = (w_1, \infty).$$

With regard to $\mathbb{I}_1$, since $h_0'(t) < 0$ for $t \in (0,4)$, $h_0(t)$ is decreasing. If $h_0(1) \geqslant 0$, then $h_0(t) \geqslant 0$ for $t \in (0,1]$, and hence for $t \in \mathbb{K}_1$. If $h_0(1) \leqslant 0$, then $h_0(t) \leqslant 0$ for $t \in [1,4)$, and hence for $t \in \mathbb{L}_1$.

With regard to $\mathbb{J}_1$, we will show that $h_0(t) \leqslant 0$ for $t \in \mathbb{M}_1$. Since

$$w_2 < 2k^2 - k + 2 + 2(k-1)(k+1) = 4k^2 - k < 4k^2,$$

we have

$$t \in \mathbb{M}_1 \iff 4 \leqslant \frac{(t+2)^2}{2t+1} < w_2,\ t \geqslant 4 \implies 4 \leqslant \frac{(t+2)^2}{2t+1} < 4k^2,\ t \geqslant 4$$
$$\iff 4 \leqslant t < 2(2k^2 - 1 + k\sqrt{4k^2-3}) \implies 4 \leqslant t < 8k^2.$$

Then, it suffices to show that $h_0(t) \leqslant 0$ for $4 \leqslant t < 8k^2$. Indeed,

$$h_0(t) \leqslant h_0(t) + (t-4)(8k^2 - t) = -2(k-1)t - 34k^2 + 13k + 10$$
$$< -34k^2 + 13k + 10 < 0.$$

With regard to $\mathbb{J}_2$, we will show that $g_1(\sqrt{w},1) \geqslant 0$ for $w \in \mathbb{J}_2$. This is true if $w_1 \geqslant 4k^2 + k + 4$, which is trivial.

Case 2. $1 < k \leqslant \frac{8}{7}$. Since $w_1 > 4 \geqslant w_2 > 3 > w_3$, we have

$$\mathbb{I}_1 = [3, w_2), \quad \mathbb{J}_1 = (w_1, \infty).$$

With regard to $\mathbb{I}_1$, we will show that $h(t) \geqslant 0$ for $t \in (0,1]$, and hence for $t \in \mathbb{K}_1$. Since $h_0'(t) < 0$ for $t \in (0,1]$, $h_0(t)$ is decreasing, and hence $h_0(t) \geqslant h_0(1) = -10k^2 + 11k + 9 > 0$.

With regard to $\mathbb{J}_1$, we will show that $g_1(\sqrt{w},1) \geqslant 0$ for $w \in \mathbb{J}_1$. This is true if $w_1 \geqslant 4k^2 + k + 4$, which is clearly true.

Case 3. $0 < k < 1$. Since $w_1 > 4 > w_3 > 3 > w_2$, we have

$$\mathbb{I}_1 = [3, w_3), \quad \mathbb{J}_1 = (w_1, \infty).$$

The proof is the same as the one of the case 2. □

PROPOSITION 4.8. *Let x, y, z be nonnegative real numbers. If $k \leqslant 4$, then*

$$\sum x^2(x-y)(x-z)(x-ky)(x-kz) \geqslant (5-3k)(x-y)^2(y-z)^2(z-x)^2,$$

with equality for $x = y = z$, for $x = 0$ and $y = z$ (or any cyclic permutation), and for $x/k = y = z$ (or any cyclic permutation) if $k > 0$ ([4]).

Proof. Denote the left-hand side of the inequality by $f(x,y,z)$ and write the desired inequality as $f_6(x,y,z) \geqslant 0$. Using (1.11), we have

$$\begin{aligned} f(x,y,z) =& \sum x^6 - (k+1)\sum xy(x^4+y^4) + k\sum x^2y^2(x^2+y^2) \\ &+ (k+1)^2 xyz\sum x^3 - k(k+1)xyz\sum xy(x+y) + 3k^2x^2y^2z^2 \\ =& 9(k^2+k+1)r^2 + [(k^2+k+8)p^3 - 2(2k^2+5k+11)pq]r + p^6 \\ &- (k+7)p^4q + (5k+13)p^2q^2 - 4(k+1)q^3. \end{aligned}$$

Using then (1.15), we can write $f_6(x,y,z)$ in the form (2.7), where

$$A = 9(k^2+k+1)+27(5-3k) = 9(4-k)^2,$$

$$g_1(p,q) = (4-k)(7-k)p(p^2-4q), \quad g_2(p,q) = (p^2-4q)^2[p^2-(k-1)q].$$

We have

$$h(t) = 2At + g_1(t+2,2t+1) = (4-k)t[(7-k)t^2-2(7-k)t+2(8-5k)]$$

and

$$d(p,q) = g_1^2-4Ag_2 = (k-1)(k-4)^2(p^2-4q)^2[36q-(13-k)p^2].$$

We see that $h(t)$ has for $t>0$ the same sign as

$$h_0(t) = (7-k)t^2-2(7-k)t+2(8-5k).$$

For $k=1$, the desired inequality $f_6(x,y,z) \geqslant 0$ is true since $A>0$ and $d(p,q)=0$. Also, for $k=4$, the inequality is true because

$$f_6(x,y,z) = (p^2-3q)(p^2-4q)^2 \geqslant 0.$$

Thus, it remains to consider the cases $1<k<4$ and $k<1$.

The condition (a) in Theorem 2.3 is fulfilled since

$$f_6(x,1,1) = x^2(x-1)^2(x-k)^2 \geqslant 0$$

and

$$f_6(0,y,z) = (y-z)^4[y^2+z^2+(3-k)yz] \geqslant 0.$$

Case 1. $1<k<4$. From $d(p,q)>0$, we get

$$\mathbb{I}_1 = \left[3, \frac{36}{13-k}\right).$$

We will show that $h_0(t) \leqslant 0$ for $t \in \mathbb{L}_1$. Therefore, we need to prove that

$$1-3\sqrt{\frac{k-1}{7-k}} \leqslant t \leqslant 1+3\sqrt{\frac{k-1}{7-k}}$$

for

$$t \in \mathbb{L}_1 \iff 3 \leqslant \frac{(t+2)^2}{2t+1} < \frac{36}{13-k}, \ 1 \leqslant t < 4 \iff 1 \leqslant t < \frac{2k+10+6\sqrt{3(k-1)}}{13-k}.$$

The left inequality is clearly true for $t \geqslant 1$. To prove the right inequality, it suffices to show that

$$\frac{2k+10+6\sqrt{3(k-1)}}{13-k} \leqslant 1+3\sqrt{\frac{k-1}{7-k}},$$

which holds if and only if

$$\frac{13-k}{\sqrt{7-k}} \geqslant \sqrt{k-1}+2\sqrt{3}.$$

Since

$$\sqrt{k-1}+2\sqrt{3} = \frac{13-k}{2\sqrt{3}-\sqrt{k-1}},$$

we need to show that

$$\sqrt{k-1}+\sqrt{7-k} \leqslant 2\sqrt{3}.$$

Indeed,

$$\sqrt{k-1}+\sqrt{7-k} < 2\sqrt{\frac{(k-1)+(7-k)}{2}} = 2\sqrt{3}.$$

Case 2. $k<1$. Since the condition $d(p,q)>0$ holds for $p^2 \geqslant 3q$, we have

$$\mathbb{I}_1 = [3,4), \quad \mathbb{J}_1 = [4,\infty).$$

With regard to $\mathbb{I}_1$, we have

$$h_0(t) = (7-k)(t-1)^2 + 9(1-k) > 0$$

for any $t>0$, and hence for $t \in \mathbb{K}_1$.

With regard to $\mathbb{J}_1$, we have

$$g_1(\sqrt{w},1) = (4-k)(7-k)\sqrt{w}(w-4) \geqslant 0$$

for $w \in \mathbb{J}_1$. □

PROPOSITION 4.9. *Let x,y,z be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\sum (x-y)(x-z)(x-ky)(x-kz) \geqslant \frac{(k+3)(x-y)^2(y-z)^2(z-x)^2}{xy+yz+zx}$$

with equality for $x=y=z$, for $x/k=y=z$ (or any cyclic permutation), and for $x=0$ and $y/z+z/y=k+2$ (or any cyclic permutation) ([7]).

Proof. Write the inequality as $f_6(x,y,z) \geqslant 0$, where

$$\begin{aligned} f_6(x,y,z) =& (xy+yz+zx)\sum (x-y)(x-z)(x-ky)(x-kz) \\ &- (k+3)(x-y)^2(y-z)^2(z-x)^2, \end{aligned}$$

and then apply Theorem 2.3. We have

$$\begin{aligned} &\sum (x-y)(x-z)(x-ky)(x-kz) = \\ &= \sum x^4 - (k+1)\sum xy(x^2+y^2) - (k^2-1)xyz\sum x + (k^2+2k)\sum x^2y^2 \\ &= 3(2-k-k^2)pr + p^4 - (k+5)p^2q + (k+2)^2q^2. \end{aligned}$$

Using (1.15), we can write $f_6(x,y,z)$ in the form (2.7), where

$$A = 27(k+3),$$

$$g_1(p,q) = 4(k+3)p^3 - 3(k^2+7k+16)pq, \quad g_2(p,q) = q(p^2-(k+4)q)^2.$$

We have

$$\begin{aligned} h(t) &= 54(k+3)t + 4(k+3)(t+2)^3 - 3(k^2+7k+16)(t+2)(2t+1) \\ &= t[4(k+3)t^2 - 6(k^2+3k+4)t - 3(5k^2+k-22)], \end{aligned}$$

$$g_1(\sqrt{w},1) = \sqrt{w}[4(k+3)w - 3(k^2+7k+16)]$$

and

$$d(p,q) = g_1^2 - 4Ag_2 = (p^2-w_1q)(p^2-w_2q)(p^2-w_3q),$$

where

$$w_1 = \frac{12}{k+3}, \quad w_2 = \frac{3(k+3)}{4}, \quad w_3 = \frac{3(k+4)^2}{4(k+3)}.$$

For $t>0$, the sign of $h(t)$ is the same with the sign of

$$h_0(t) = 4(k+3)t^2 - 6(k^2+3k+4)t - 3(5k^2+k-22).$$

The condition (a) in Theorem 2.3 is fulfilled since

$$f_6(x,1,1) = (2x+1)(x-1)^2(x-k)^2 \geqslant 0$$

and

$$f_6(0,y,z) = yz[y^2+z^2-(k+2)yz]^2 \geqslant 0.$$

We need to consider the following four cases.

Case 1. $k \geqslant \frac{7}{3}$. Since $w_1 < 3 < 4 \leqslant w_2 < w_3$, we have

$$\mathbb{I}_1 = [3,4), \quad \mathbb{J}_1 = [4,w_2), \quad \mathbb{J}_2 = (w_3,\infty).$$

With regard to $\mathbb{I}_1$, we will show that $h_0(t) \leqslant 0$ for $t \in [1,4)$, and hence for $t \in \mathbb{L}_1$. Since $h_0(t)$ is convex, it suffices to show that $h_0(1) \leqslant 0$ and $h_0(4) \leqslant 0$. Indeed,

$$h_0(1) = -21k^2 - 17k + 54 < 0, \quad h_0(4) = -39k^2 - 11k + 162 < 0.$$

With regard to $\mathbb{J}_1$, we will show that $h_0(t) \leqslant 0$ for $t \in \mathbb{M}_1$. We have

$$t \in \mathbb{M}_1 \iff 4 \leqslant \frac{(t+2)^2}{2t+1} < w_2,\ t \geqslant 4 \iff 4 \leqslant t < \frac{3k+1+3\sqrt{k^2+2k-3}}{4}.$$

Since

$$\frac{3k+1+3\sqrt{k^2+2k-3}}{4} < \frac{3k+1+3(k+1)}{4} = \frac{3k+2}{2},$$

it suffices to show that $h_0(t) \leqslant 0$ for $4 \leqslant t < \dfrac{3k+2}{2}$. Since $h_0(t)$ is convex, it suffices to show that $h_0(4) \leqslant 0$ and $h_0\left(\dfrac{3k+2}{2}\right) \leqslant 0$. Indeed,

$$h_0(4) = -39k^2 - 11k + 162 < 0, \quad h_0\left(\frac{3k+2}{2}\right) = -9k^2 - 17k + 54 < 0.$$

With regard to $\mathbb{J}_2$, we will show that $g_1(\sqrt{w}, 1) \geqslant 0$ for $w \in \mathbb{J}_2$. This is true if $4(k+3)w_3 - 3(k^2+7k+16) \geqslant 0$, which is true for any $k \geqslant 0$.

Case 2. $\dfrac{-17+5\sqrt{193}}{42} \leqslant k \leqslant \dfrac{7}{3}$. Since $w_1 < 3 < w_2 \leqslant 4 < w_3$, we have

$$\mathbb{I}_1 = [3, w_2), \quad \mathbb{J}_1 = (w_3, \infty).$$

With regard to $\mathbb{I}_1$, we will show that $h_0(t) \leqslant 0$ for $t \in \mathbb{L}_1$. We have

$$t \in \mathbb{L}_1 \iff 3 \leqslant \frac{(t+2)^2}{2t+1} < w_2, \ 1 \leqslant t < 4$$
$$\iff 4t^2 - 2(3k+1)t - 3k + 7 < 0, \ 1 \leqslant t < 4 \iff 1 \leqslant t < t_1,$$

where

$$t_1 = \frac{3k+1+3\sqrt{k^2+2k-3}}{4}.$$

Since $h_0(t)$ is convex, it suffices to show that $h_0(1) \leqslant 0$ and $h_0(t_1) \leqslant 0$. Since $k \geqslant (-17+5\sqrt{193})/42$, we have

$$h_0(1) = -21k^2 - 17k + 54 \leqslant 0.$$

From $4t_1^2 = 2(3k+1)t_1 + 3k - 7$, we have

$$\begin{aligned} h_0(t_1) &= (k+3)[2(3k+1)t_1 + 3k - 7] - 6(k^2+3k+4)t_1 - 3(5k^2+k-22) \\ &= 2(k-9)t_1 - 12k^2 - k + 45. \end{aligned}$$

Then, we need to show that $2(k-9)t_1 - 12k^2 - k + 45 \leqslant 0$. Since

$$t_1 > \frac{3k+1+(3k-1)}{4} = \frac{3k}{2},$$

we have

$$2(k-9)t_1 - 12k^2 - k + 45 < 3k(k-9) - 12k^2 - k + 45 = -9k^2 - 28k + 45 < 0.$$

With regard to $\mathbb{J}_1$, we will show that $g_1(\sqrt{w}, 1) \geqslant 0$ for $w \in \mathbb{J}_1$. This is true if $4(k+3)w_3 - 3(k^2+7k+16) \geqslant 0$, which is true for any $k \geqslant 0$.

Case 3. $1 \leqslant k \leqslant \dfrac{-17+5\sqrt{193}}{42}$. Since $w_1 \leqslant 3 \leqslant w_2 < 4 < w_3$, we have

$$\mathbb{I}_1 = [3, w_2), \quad \mathbb{J}_1 = (w_3, \infty).$$

With regard to $\mathbb{I}_1$, we will show that $h_0(t) \geqslant 0$ for $t \in (0,1]$, and hence for $t \in \mathbb{K}_1$. Indeed,

$$h_0(t) \geqslant h_0(t) - 4(k+3)(t-1)^2 = -2k(3k+5)t - 15k^2 - 7k + 54$$
$$\geqslant -2k(3k+5) - 15k^2 - 7k + 54 = -21k^2 - 17k + 54 \geqslant 0.$$

With regard to $\mathbb{J}_1$, we will show that $g_1(\sqrt{w},1) \geqslant 0$ for $w \in \mathbb{J}_1$. This is true if $4(k+3)w_3 - 3(k^2+7k+16) \geqslant 0$, which is true for any $k \geqslant 0$.

Case 4. $0 \leqslant k \leqslant 1$. Since $w_2 \leqslant 3 \leqslant w_1 \leqslant 4 \leqslant w_3$, we have

$$\mathbb{I}_1 = [3, w_1), \quad \mathbb{J}_1 = [w_3, \infty),$$

and the proof is the same as the one of the case 3. $\square$

PROPOSITION 4.10. *Let a,b,c be the side-lengths of a triangle. If $a+b+c=3$, then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{41}{6} \geqslant 3(a^2+b^2+c^2),$$

with equality for a degenerate triangle having $a = 3/2$, $b = 1$, $c = 1/2$ (or any permutation) ([8]).

Proof. Write the desired inequality in the form

$$\frac{1}{abc} + 6 \geqslant \frac{121}{6(ab+bc+ca)}.$$

By Lemma 2.2, it suffices to prove this inequality for $a/2 \leqslant b = c \leqslant a$ and for $a = b+c$.

Case 1. $a/2 \leqslant b = c \leqslant a$. From $a+b+c=3$, we have $b = c = (3-a)/2$, $a \in [1, 3/2]$. The desired inequality is equivalent to

$$(3-a)(27a^4 - 135a^3 + 202a^2 - 102a + 18) \geqslant 0,$$

which is true since

$$27a^4 - 135a^3 + 202a^2 - 102a + 18 =$$
$$= 27\left(a - \frac{3}{2}\right)^4 + 27a\left(a - \frac{3}{2}\right)^2 + \frac{163}{2}(a-1)\left(\frac{3}{2} - a\right) - 2a + \frac{57}{16} > 0.$$

Case 2. $a = b+c$. From $a+b+c=3$, we have $a = 3/2$ and $b+c = 3/2$. The desired inequality reduces to $(2bc-1)^2 \geqslant 0$. $\square$

PROPOSITION 4.11. *If a,b,c are the side-lengths of a triangle, then*

$$\frac{a^2}{4a^2+5bc} + \frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5bc} \geqslant \frac{1}{3},$$

with equality for an equilateral triangle, and for a degenerate triangle having $a/2 = b = c$ (or any cyclic permutation).

Proof. Write the inequality as $g_6(a,b,c) \geqslant 0$, where

$$g_6(a,b,c) = 3\sum a^2(4b^2+5ca)(4c^2+5ab) - \prod(4a^2+5bc)$$
$$= -45a^2b^2c^2 - 25abc\sum a^3 + 40\sum a^3b^3.$$

Thus, $g_6(a,b,c)$ has the highest coefficient $A = -45-75+120 = 0$. By Theorem 2.5, it suffices to prove the original inequality for $b=c=1$ and $0 \leqslant a \leqslant 2$, and for $a=b+c$.

For $b=c=1$, the desired inequality is equivalent to the obvious inequality $(2-a)(a-1)^2 \geqslant 0$.

For $a=b+c$, using the Cauchy-Schwarz inequality,

$$\frac{b^2}{4b^2+5ca} + \frac{c^2}{4c^2+5ab} \geqslant \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)},$$

it suffices to show that

$$\frac{a^2}{4a^2+5bc} + \frac{(b+c)^2}{4(b^2+c^2)+5a(b+c)} \geqslant \frac{1}{3}.$$

This is equivalent to the obvious inequality

$$(b-c)^2(3b^2+3c^2-4bc) \geqslant 0. \qquad \square$$

PROPOSITION 4.12. *Let a,b,c be the side-lengths of a triangle. If $k > -2$, then*

$$\sum \frac{a(b+c)+(k+1)bc}{b^2+kbc+c^2} \leqslant \frac{3(k+3)}{k+2},$$

with equality for an equilateral triangle, and for a degenerate triangle with $a/2=b=c$ (or any cyclic permutation) ([5]).

Proof. Write the inequality as $g_6(a,b,c) \geqslant 0$, where

$$g_6(a,b,c) = 3(k+3)\prod(b^2+c^2+kbc)$$
$$-(k+2)\sum[a(b+c)+(k+1)bc](a^2+c^2+kac)(a^2+b^2+kab).$$

Let $p=a+b+c$ and $q=ab+bc+ca$. From

$$g_6(a,b,c) = 3(k+3)\prod(p^2-2q-a^2+kbc)$$
$$-(k+2)\sum(q+kbc)(p^2-2q-b^2+kac)(p^2-2q-c^2+kab),$$

it follows that g_6 has the same highest coefficient as

$$g(a,b,c) = 3(k+3)\prod(-a^2+kbc) - k(k+2)\sum bc(-b^2+kac)(-c^2+kab)$$
$$= 3(k+3)[(k^3-1)a^2b^2c^2 - k^2abc\sum a^3 + k\sum a^3b^3]$$
$$-k(k+2)(3k^2a^2b^2c^2 - 2kabc\sum a^3 + \sum a^3b^3).$$

Therefore,

$$A = 3(k+3)(k^3-1-3k^2+3k)-k(k+2)(3k^2-6k+3) = -9(k-1)^2.$$

Since $A \leqslant 0$, by Theorem 2.5, it suffices to prove the original inequality for $b=c=1$ and $0 \leqslant a \leqslant 2$, and for $a=b+c$.

For $b=c=1$, the desired inequality is equivalent to the obvious inequality $(2-a)(a-1)^2 \geqslant 0$.

For $a=b+c$, the inequality becomes

$$\frac{3bc}{b^2+c^2+kbc}+\frac{bc-c^2}{b^2+(k+2)(c^2+bc)}+\frac{bc-b^2}{c^2+(k+2)(b^2+bc)} \leqslant \frac{3}{k+2},$$

and is true since

$$\frac{3bc}{b^2+c^2+kbc} \leqslant \frac{3}{k+2}$$

and

$$\frac{bc-c^2}{b^2+(k+2)(c^2+bc)}+\frac{bc-b^2}{c^2+(k+2)(b^2+bc)} \leqslant 0,$$

the last inequality being equivalent to $(b-c)^2(b^2+bc+c^2) \geqslant 0$. □

PROPOSITION 4.13. *Let a,b,c be the side-lengths of a triangle. If $k>-2$, then*

$$\sum \frac{2a^2+(4k+9)bc}{b^2+kbc+c^2} \leqslant \frac{3(4k+11)}{k+2},$$

with equality for an equilateral triangle, and for a degenerate triangle with $a/2=b=c$ (or any cyclic permutation) ([5]).

Proof. Write the inequality as $g_6(a,b,c) \geqslant 0$, where

$$g_6(a,b,c) = 3(4k+11)\prod(b^2+c^2+kbc)$$
$$-(k+2)\sum[2a^2+(4k+9)bc](a^2+c^2+kac)(a^2+b^2+kab).$$

Let $p=a+b+c$ and $q=ab+bc+ca$. From

$$g_6(a,b,c) = 3(4k+11)\prod(p^2-2q-a^2+kbc)$$
$$-(k+2)\sum[2a^2+(4k+9)bc](p^2-2q-b^2+kac)(p^2-2q-c^2+kab),$$

it follows that g_6 has the same highest coefficient as

$$\begin{aligned} g(a,b,c) =& 3(4k+11)\prod(-a^2+kbc)-(k+2)\sum[2a^2+(4k+9)bc](-b^2+kac)(-c^2+kab) \\ =& 3(4k+11)[(k^3-1)a^2b^2c^2-k^2abc\sum a^3+k\sum a^3b^3] \\ & -(k+2)[3(4k^3+9k^2+2)a^2b^2c^2-6k(k+3)abc\sum a^3+9\sum a^3b^3]. \end{aligned}$$

Therefore,

$$\begin{aligned}A =&3(4k+11)(k^3-1-3k^2+3k)-(k+2)[3(4k^3+9k^2+2)-18k(k+3)+27]\\ =&-9(4k+11)(k-1)^2.\end{aligned}$$

Since $A \leqslant 0$, by Theorem 2.5, it suffices to prove the original inequality for $b=c=1$ and $0 \leqslant a \leqslant 2$, and for $a=b+c$.

For $b=c=1$, the desired inequality is equivalent to the obvious inequality $(2-a)(a-1)^2 \geqslant 0$.

For $a=b+c$, the inequality becomes

$$\frac{(2k+13)bc}{b^2+c^2+kbc}+\frac{(2k+5)(b+c)c}{b^2+(k+2)(c^2+bc)}+\frac{(2k+5)(b+c)b}{c^2+(k+2)(b^2+bc)} \leqslant \frac{3(2k+7)}{k+2}.$$

Setting $x=b/c+c/b$, $x \geqslant 2$, we can write this inequality as follows

$$\frac{2k+13}{x+k}+\frac{(2k+5)(x+2)(x+2k+3)}{(k+2)x^2+(k+2)(k+3)x+2k^2+6k+5} \leqslant \frac{3(2k+7)}{k+2},$$

$$(x-2)[4(k+2)(k+4)x^2+2(k+2)Ax+B] \geqslant 0,$$

where

$$A=2k^2+13k+22, \quad B=8k^3+51k^2+98k+65,$$

This is true since

$$A=2(k+2)^2+5(k+2)+4>0, \ \ B=8(k+2)^3+2k^2+(k+1)^2>0. \qquad \square$$

PROPOSITION 4.14. *Let x,y,z be positive real numbers. If $k \geqslant \sqrt{2}-1$, then*

$$\frac{x^2(y+z)}{kx^2+y^2+z^2}+\frac{y^2(x+x)}{ky^2+z^2+x^2}+\frac{z^2(x+y)}{kz^2+x^2+y^2} \leqslant \frac{2(x+y+z)}{k+2},$$

with equality for $x=y=z$. If $k=\sqrt{2}-1$, then equality also holds for $x/\sqrt{2}=y=z$ (or any cyclic permutation).

Proof. Write the inequality as $f_7(x,y,z) \geqslant 0$, where

$$\begin{aligned}f_7(x,y,z) =&2(x+y+z)\prod(kx^2+y^2+z^2)\\ &-(k+2)\sum x^2(y+z)(ky^2+z^2+x^2)(kz^2+x^2+y^2).\end{aligned}$$

Let $p=x+y+z$, $q=xy+yz+zx$, $r=xyz$. From

$$\begin{aligned}f_7(x,y,z) =&2p\prod[p^2-2q+(k-1)x^2]\\ &-(k+2)\sum x^2(p-x)[p^2-2q+(k-1)y^2][p^2-2q+(k-1)z^2],\end{aligned}$$

it follows that the highest polynomial of $f_7(x,y,z)$ is

$$h_0(p,q) = 2(k-1)^3p - (k+2)(k-1)^2\sum(p-x) = -6(k-1)^2p.$$

By Theorem 2.6, since $h_0(p,q) \leqslant 0$, it suffices to show that the desired inequality holds for $y = z = 1$ and for $x = 0$. For $y = z = 1$, the original inequality is equivalent to

$$(x-1)^2(x+1)[kx^2 + 2(k-1)x + 2k] \geqslant 0,$$

which is true since

$$kx^2 + 2(k-1)x + 2k = k(x-1+\frac{1}{k})^2 + \frac{k^2+2k-1}{k} \geqslant 0.$$

For $x = 0$, the original inequality becomes

$$2kt^2 - (k+2)t + k^2 - 5k + 4 \geqslant 0,$$

where $t = y/z + z/y \geqslant 2$. This is also true since

$$2kt^2 - (k+2)t + k^2 - 5k + 4 = 2k(t-2)^2 + (7k-2)(t-2) + k(k+1) > 0. \qquad \square$$

PROPOSITION 4.15. *If x,y,z are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{2x^2+yz} + \frac{1}{2y^2+zx} + \frac{1}{2z^2+xy} \geqslant \frac{1}{xy+yz+zx} + \frac{2}{x^2+y^2+z^2},$$

with equality for $x = y = z$, and for $x = 0$ and $y = z$ (or any cyclic permutation).

Proof. Let $p = x+y+z$, $q = xy+yz+zx$, $r = xyz$. Write the inequality as $f_8(x,y,z) \geqslant 0$, where

$$f_8(x,y,z) = q(p^2-2q)\sum(2y^2+zx)(2z^2+xy) - p^2\prod(2x^2+yz).$$

Clearly, $f_8(x,y,z)$ has the same highest polynomial as

$$f(x,y,z) = -p^2\prod(2x^2+yz) = -p^2(9x^2y^2z^2 + 2xyz\sum x^3 + 4\sum y^3z^3);$$

that is, $h_0(p,q) = -(9+6+12)p^2 = -27p^2$. By Theorem 2.7, since $h_0(p,q) < 0$, it suffices to show that the desired inequality holds for $y = z = 1$ and for $x = 0$. In these cases, the original inequality reduces to the obvious forms

$$x(x-1)^2(x^2+x+1) \geqslant 0$$

and

$$(y^2-z^2)^2 \geqslant 0,$$

respectively. $\square$

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