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EMBEDDINGS PROPERTIES ON HERZ–TYPE BESOV AND TRIEBEL–LIZORKIN SPACES

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Abstract. We study the embeddings problems on Herz-type Besov-Triebel-Lizorkin spaces. In particular we will give a proof of the Sobolev-type embedding for these function spaces. All these results generalize the classical results on Besov and Triebel-Lizorkin spaces.

1. Introduction

Many researchers are interested in the analysis of the class of function spaces, because they are well-established tools in the analysis of partial differential equation. Some example of these spaces can be mentioned such as: Hölder spaces, Bessel potential spaces, Besov spaces and Triebel-Lizorkin spaces. For more details one can refer to Triebel's books [17], [18], [19] and other literatures. In recent years many researchers have modified the classical spaces and have generalized the classical results to these modified ones. For example. Besov-Morrey spaces which have been studied in [9]. Besov-Morrey spaces are modified Besov spaces where the base norm is of Morrey type, instead of L^p. For a general theory on these spaces see [9], [10], [11], [15] and [20]. Another example can be mentioned, that is the Herz-type Besov and Triebel-Lizorkin spaces. They are modelled on Besov spaces and Triebel-Lizorkin spaces, but the underlying norm is of $\dot{K}_q^{\alpha,p}$ type rather than L^p.

It is well known that Herz spaces play an important role in Harmonic Analysis. After they have been introduced in [2], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [1], in the summability of Fourier transforms [5] and in regularity theory for elliptic equations in divergence form [16].

The purpose of this paper is then to consider Herz-type Besov spaces $\dot{K}_q^{\alpha,p}B_\beta^s$ and Herz-type Triebel-Lizorkin spaces $\dot{K}_q^{\alpha,p}F_\beta^s$. These function spaces introduced earlier in the papers of J. Xu and D. Yang [21] , [22] and [23].

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. In [14], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations.

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The main aim of this paper is to study the embeddings problems of $\dot{K}_q^{\alpha,p}B_\beta^s$ and $\dot{K}_q^{\alpha,p}F_\beta^s$ spaces. All these results generalize the existing classical results on Besov and Triebel-Lizorkin spaces by taking $\alpha=0$ and $p=q$.

Let us now present the contents of this paper. Section 2 collects fundamental notation and concepts. Some necessary tools are given in Section 3. In particular we generalize the classical Plancherel-Polya-Nikolskij inequality on $\dot{K}_q^{\alpha,p}$ spaces instead of L^p spaces. In Section 4 we recall the definitions of the Herz-type Besov and Triebel-Lizorkin spaces and present a few aspects of their properties. Finally, in Section 5 we present some embeddings properties. In particular we will prove the Sobolev embedding theorem for these function spaces and present some consequences.

2. Preliminaries

As usual, $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers.

For any $u>0, k\in\mathbb{Z}$ we set $C(u)=\{x\in\mathbb{R}^n : u/2<|x|\leqslant u\}$ and $C_k=C(2^k)$. For $x\in\mathbb{R}^n$ and $r>0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r and $\overline{B}(x,r)$ is the closure of the open ball $B(x,r)$. Let χ_k, for $k\in\mathbb{Z}$, denote the characteristic function of the set C_k. The Euclidean scalar product of $x=(x_1,...,x_n)$ and $y=(y_1,...,y_n)$ is given by $x\cdot y=x_1y_1+...+x_ny_n$.

As usual $L^p(\mathbb{R}^n)$ for $0<p\leqslant\infty$ stands for the Lebesgue spaces on $\mathbb{R}^n$ normed by (quasi-normed for $p<1$)

$$\|f\mid L^p(\mathbb{R}^n)\|=\|f\|_p=\left(\int_{\mathbb{R}^n}|f(x)|^p\,dx\right)^{1/p}<\infty,\quad 0<p<\infty$$

and

$$\|f\mid L^\infty(\mathbb{R}^n)\|=\|f\|_\infty=\operatorname*{ess-sup}_{x\in\mathbb{R}^n}|f(x)|<\infty.$$

We define for any $x\in\mathbb{R}^n$ and $N,R>0$

$$w_{R,N}(x)=R^n\left(1+R|x|\right)^{-N}.$$

By $\mathscr{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. The topology in the complete locally convex space $\mathscr{S}(\mathbb{R}^n)$ is generated by the norms

$$p_N(\varphi)=\sup_{x\in\mathbb{R}^n}(1+|x|)^N\sum_{|\alpha|\leqslant N}|D^\alpha\varphi(x)|,\quad N=1,2,3,....$$

By $\mathscr{S}'(\mathbb{R}^n)$ we denote the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f\in\mathscr{S}(\mathbb{R}^n)$ by

$$\mathscr{F}(f)(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)dx.$$

Its inverse is denoted by $\mathscr{F}^{-1}f$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are extended to the dual Schwartz space $\mathscr{S}'(\mathbb{R}^n)$ in the usual way.

By ℓ_q, $0 < q \leqslant \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm

$$\big\|\{a_k\}_{k\in\mathbb{Z}} \mid \ell_q\big\| = \Big(\sum_{k=-\infty}^{\infty} |a_k|^q\Big)^{1/q}$$

(with the usual modification if $q=\infty$).

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

3. Basic tools

In this section we derive several technical lemmas that were used in the other sections. We start by recalling the definition and some of the properties of the homogeneous Herz spaces $\dot{K}_q^{\alpha,p}$.

DEFINITION 3.1. Let $\alpha \in \mathbb{R}, 0 < p,q \leqslant \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by

$$\dot{K}_q^{\alpha,p} := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \big\|f \mid \dot{K}_q^{\alpha,p}\big\| < \infty\},$$

where

$$\big\|f \mid \dot{K}_q^{\alpha,p}\big\| = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\, \|f\chi_k\|_q^p\right)^{1/p} < \infty,$$

with the usual modifications made when $p=\infty$ and/or $q=\infty$.

The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p,q) \geqslant 1$ then $\dot{K}_q^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leqslant \infty$ then $\dot{K}_p^{0,p}$ coincides with the Lebesque spaces $L^p(\mathbb{R}^n)$. If $0 < p_1 \leqslant p_2 \leqslant \infty$, then we may derive the embedding

$$\dot{K}_q^{\alpha,p_1} \hookrightarrow \dot{K}_q^{\alpha,p_2}.$$

A detailed discussion of the properties of these spaces may be found in the papers [6], [7], [12], [13], and references therein.

LEMMA 3.2. *Let $r,R > 0$ and $m > n$. Then there exists $c = c(r,m,n) > 0$ such that for all $g \in \mathscr{S}'(\mathbb{R}^n)$ with* supp $\mathscr{F}g \subset \overline{B}(0,R)$, *we have*

$$|g(x)| \leqslant c\,(w_{R,m} * |g|^r(x)))^{1/r}, \quad x \in \mathbb{R}^n.$$

This Lemma is from [4, Lemma A.7].

The classical Plancherel-Polya-Nikolskij inequality (cf. [17, 1.3.2/5, Rem. 1.4.1/4]), says that $\|f\|_q$ can be estimated by

$$c\,R^{n(1/p-1/q)}\,\|f\|_p$$

for any $0<p\leqslant q\leqslant\infty$, $R>0$ and any $f\in L^p(\mathbb{R}^n)\cap\mathscr{S}'(\mathbb{R}^n)$ with supp $\mathscr{F}f\subset\overline{B}(0,R)$. The constant $c>0$ is independent of R. This inequality plays an important role in theory of function spaces and PDE's. Our aim is to extend this result to Herz spaces. Let us start with the following lemma.

LEMMA 3.3. *Let $\alpha\in\mathbb{R}, 0<p,q\leqslant\infty$ and $R\geqslant H>0$. Then there exists a constant $c>0$ independent of R and H such that for all $f\in\dot{K}_q^{\alpha,p}\cap\mathscr{S}'(\mathbb{R}^n)$ with* supp *$\mathscr{F}f\subset\overline{B}(0,R)$, we have*

$$\sup_{x\in B(0,1/H)}|f(x)|\leqslant c\left(\frac{R}{H}\right)^{n/d}H^{n/q+\alpha}\left\|f\mid\dot{K}_q^{\alpha,p}\right\|$$

for any $0<d<\min(q,1/(1/q+\alpha/n))$.

Proof. By Lemma 3.2 we have for any $d,R>0,N>n/d$ and any $x\in B(0,1/H)$

$$\begin{aligned}|f(x)|^\rho &\leqslant c\left(\int_{\mathbb{R}^n}|f(y)|^d\,w_{R,dN}(x-y)dy\right)^{\rho/d}\\ &\leqslant c\left(\int_{\overline{B}(0,2^2/H)}(\cdots)dy\right)^{\rho/d}+c\left(\int_{\mathbb{R}^n\setminus\overline{B}(0,2^2/H)}(\cdots)dy\right)^{\rho/d},\end{aligned}$$

where $\rho=\min(1,d)$. Using the following decomposition

$$\int_{\overline{B}(0,2^2/H)}(\cdots)dy=\sum_{j=0}^{\infty}\int_{C(2^{2-j}/H)}(\cdots)dy,$$

$$\int_{\mathbb{R}^n\setminus\overline{B}(0,2^2/H)}(\cdots)dy=\sum_{j=0}^{\infty}\int_{C(2^{j+3}/H)}(\cdots)dy$$

and the well-known inequality

$$\left(\sum_{j=0}^{\infty}|a_j|\right)^{\sigma}\leqslant\sum_{j=0}^{\infty}|a_j|^{\sigma},\quad\{a_j\}_j\subset\mathbb{C},\ \sigma\in[0,1]\tag{3.4}$$

we obtain that $|f(x)|^\rho$ can be estimated by

$$c\sum_{j=0}^{\infty}\left(V_{j,R,H}^1(x)+V_{j,R,H}^2(x)\right),\tag{3.5}$$

where

$$V^1_{j,R,H}(x) = \left(w_{R,dN} * |f\chi_{C(2^{2-j}/H)}|^d(x)\right)^{\rho/d}, \quad V^2_{j,R,H}(x) = V^1_{-j-1,R,H}(x).$$

Here N is chosen large enough such that $N > \max(n/d, n/d - n/q - \alpha)$. Let us give the estimation of the first term in (3.5). A simple change of variables and the Hölder inequality (with $\frac{1}{d} = \frac{1}{q} + \frac{1}{d} - \frac{1}{q}$), yield for any $d, R > 0$ and any $x \in B(0, 1/H)$

$$\begin{aligned}\sum_{j=0}^{\infty} V^1_{j,R,H}(x) &\leqslant R^{n\rho/d} \sum_{j=0}^{\infty} \left\| f\chi_{C(2^{2-j}/H)} \right\|_d^{\rho} \\ &\leqslant R^{n\rho/d} \sum_{k=-\infty}^{2-[\log_2 H]} \left\| f\chi_{\widetilde{C}_k} \right\|_d^{\rho} \\ &\leqslant c\, R^{n\rho/d} \sum_{k=-\infty}^{2-[\log_2 H]} 2^{n\rho k(1/d-1/q)} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{\rho}, \end{aligned} \tag{3.6}$$

where $\widetilde{C}_k = \{x \in \mathbb{R}^n : 2^{k-2} < |x| \leqslant 2^k\}$ and $[a]$ is the integer part of the real number a. Now we can distinguish two cases as follows:

If $0 < p \leqslant \rho$, then by the embedding $\ell_{p/\rho} \hookrightarrow \ell_1$, we obtain that the right-hand side of (3.6) is bounded by

$$\begin{aligned} &c\, R^{n\rho/d} \left(\sum_{k=-\infty}^{2-[\log_2 H]} 2^{nk(1/d-1/q)p} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{p} \right)^{\rho/p} \\ &\leqslant c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{p} \right)^{\rho/p} \\ &\leqslant c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \left\| f \mid \dot{K}_q^{\alpha,p} \right\|^{\rho}, \end{aligned} \tag{3.7}$$

where we have used the fact that $1/d > 1/q + \alpha/n$ and $2^{k-3}H < 1$.

If $\rho < p$, then the Hölder inequality in ℓ_ρ implies that

$$\begin{aligned} &\sum_{j=0}^{\infty} V^1_{j,R,H}(x) \\ &\leqslant c\, R^{n\rho/d} \sum_{k=-\infty}^{2-[\log_2 H]} 2^{k\rho\alpha} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{\rho} 2^{k\rho(n/d-n/q-\alpha)} \\ &\leqslant c\, R^{n\rho/d} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{p} \right)^{\rho/p} \left(\sum_{k=-\infty}^{2-[\log_2 H]} 2^{k\rho(n/d-n/q-\alpha)p/(p-\rho)} \right)^{(p-\rho)/p} \\ &\leqslant c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \left\| f \mid \dot{K}_q^{\alpha,p} \right\|^{\rho}, \end{aligned} \tag{3.8}$$

where in the last inequality we have used (since, again $1/d > 1/q + \alpha/n$ and $2^{k-3}H < 1$)

$$\left(\sum_{k=-\infty}^{2-[\log_2 H]} \left(2^k H\right)^{\rho(n/d-n/q-\alpha)p/(p-\rho)} \right)^{(p-\rho)/p} \leqslant c.$$

Now we estimate the second term in (3.5). We see that for any $y \in C(2^{3+j}/H)$ and any $x \in B(0,1/H)$, we have $|x-y| > 2^j/H$, so that for any $d > 0, N \in \mathbb{N}$ and $j \in \mathbb{N}_0$

$$w_{R,N}(x-y) \leqslant R^n \left(\frac{2^j R}{H}\right)^{-N} \leqslant 2^{-jN} R^n.$$

Hence by a simple change of variables and the Hölder inequality (with $\frac{1}{d} = \frac{1}{q} + \frac{1}{d} - \frac{1}{q}$),

$$\begin{aligned}
&\sum_{j=0}^{\infty} V_{j,R,H}^2(x) \\
&\leqslant R^{n\rho/d} \sum_{j=0}^{\infty} 2^{-jN\rho} \left\| f\chi_{C(2^{j+3}/H)} \right\|_d^{\rho} \\
&\leqslant c\, R^{n\rho/d} H^{-N\rho} \sum_{k=2-[\log_2 H]}^{\infty} 2^{-kN\rho} \left\| f\chi_{\widetilde{C}_k} \right\|_d^{\rho} \\
&\leqslant c\, R^{n\rho/d} H^{-N\rho} \sum_{k=2-[\log_2 H]}^{\infty} 2^{k\rho(n/d-n/q-N)} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{\rho} \\
&= c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \sum_{k=2-[\log_2 H]}^{\infty} \left(2^k H\right)^{(n/d-n/q-N-\alpha)\rho} 2^{k\rho\alpha} \left\| f\chi_{\widetilde{C}_k} \right\|_q^{\rho}.
\end{aligned}$$

Since $N > n/d - n/q - \alpha$ and $2^k H > 1$, for any $k \geqslant 2 - [\log_2 H]$, the right-hand side of the last expression is bounded by

$$\begin{aligned}
&c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \left(\sup_{k \geqslant 2-[\log_2 H]} 2^{k\alpha} \left\| f\chi_{\widetilde{C}_k} \right\|_q \right)^{\rho} \\
&\leqslant c \left(\frac{R}{H}\right)^{n\rho/d} H^{(n/q+\alpha)\rho} \left\| f \mid \dot{K}_q^{\alpha,p} \right\|^{\rho}. \qquad (3.9)
\end{aligned}$$

Finally, the desired estimate follows from (3.7), (3.8) and (3.9) taking into account the decomposition (3.5). $\square$

The following lemma is the $\dot{K}_q^{\alpha,p}$-version of the Plancherel-Polya-Nikolskij inequality.

LEMMA 3.10. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s, p, q, r \leqslant \infty$. We suppose that $\alpha_1 + n/s > 0$, $0 < q \leqslant s \leqslant \infty$ and $\alpha_2 \geqslant \alpha_1$. Then there exist a positive constant $c > 0$ independent of R such that for all $f \in \dot{K}_q^{\alpha_2,p} \cap \mathscr{S}'(\mathbb{R}^n)$ with* supp $\mathscr{F} f \subset \overline{B}(0,R)$, *we have*

$$\left\| f \mid \dot{K}_s^{\alpha_1,r} \right\| \leqslant c\, R^{n/q-n/s+\alpha_2-\alpha_1} \left\| f \mid \dot{K}_q^{\alpha_2,\theta} \right\|,$$

where

$$\theta = \begin{cases} r \text{ if } \alpha_2 = \alpha_1 \\ p \text{ if } \alpha_2 > \alpha_1. \end{cases}$$

Proof. We write

$$\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \|f\chi_k\|_s^r = \sum_{k=-\infty}^{-[\log_2 R]} (\cdots) + \sum_{k=1-[\log_2 R]}^{\infty} (\cdots) = I_R + II_R. \tag{3.11}$$

Estimate of I_R. Lemma 3.3 gives for any $R > 0$

$$I_R \leqslant \sup_{x \in B(0,2/R)} |f(x)|^r \sum_{k=-\infty}^{-[\log_2 R]} 2^{k(\alpha_1 + n/s)r} \leqslant c\, R^{(n/q - n/s + \alpha_2 - \alpha_1)r} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^r,$$

because of $\alpha_1 + n/s > 0$ and $2^{k-1}R < 1$.

Estimate of II_R. First we consider the case $q = s = \infty$. It is easy to see that

$$\begin{aligned} II_R &= R^{(\alpha_2-\alpha_1)r} \sum_{k=1-[\log_2 R]}^{\infty} \left(2^k R\right)^{(\alpha_1-\alpha_2)r} 2^{k\alpha_2 r} \|f\chi_k\|_\infty^r \\ &\leqslant R^{(\alpha_2-\alpha_1)r} \sup_{k\in\mathbb{Z}} (2^{k\alpha_2} \|f\chi_k\|_\infty)^r \sum_{k=1-[\log_2 R]}^{\infty} \left(2^k R\right)^{(\alpha_1-\alpha_2)r} \\ &\leqslant c\, R^{(\alpha_2-\alpha_1)r} \left\| f \mid \dot{K}_\infty^{\alpha_2,p} \right\|^r, \end{aligned}$$

if $\alpha_2 > \alpha_1$. If $\alpha_2 = \alpha_1$ then it is clear that

$$II_R = \sum_{k=1-[\log_2 R]}^{\infty} 2^{k\alpha_2 r} \|f\chi_k\|_\infty^r \leqslant \left\| f \mid \dot{K}_\infty^{\alpha_2,r} \right\|^r.$$

Now we consider the case $q < \infty$. By Lemma 3.2 we have for any $R > 0, N > n/d$ and any $x \in C_k$

$$\begin{aligned} |f(x)| &\leqslant c \left(\int_{\mathbb{R}^n} |f(y)|^q w_{R,qN}(x-y)dy \right)^{1/q} \\ &\leqslant c \left(\int_{B(0,2^{k-2})} (\cdots)dy \right)^{1/q} + c \left(\int_{\widetilde{C}_k} (\cdots)dy \right)^{1/q} \\ &\quad + c \left(\int_{\mathbb{R}^n \setminus B(0,2^{k+2})} (\cdots)dy \right)^{1/q} \\ &= V_{R,k}^1(x) + V_{R,k}^2(x) + V_{R,k}^3(x). \end{aligned}$$

Here $\widetilde{C}_k := \left\{ x \in \mathbb{R}^n : 2^{k-2} \leqslant |x| \leqslant 2^{k+2} \right\}$. We choose N such that

$$N > \max\left(n/s, n/d, n/s - \alpha_2 + \alpha_1 + n/d, n/d - \alpha_2\right), \tag{3.12}$$

with d as in Lemma 3.3. It is easy to verify that if $x \in C_k$ and $y \in B(0,2^{k-2})$, then $|x-y| > 2^{k-2}$. This estimate and Lemma 3.3, yield for any $x \in C_k$ and any $k > -[\log_2 R]$

$$\begin{aligned} V^1_{R,k}(x) &\leqslant c \sup_{y\in B(0,2^{k-2})} |f(y)| \left(\int_{2^{k-2}<|x-y|<2^{k+1}} w_{R,qN}(x-y)dy \right)^{1/q} \\ &\leqslant c\, R^{(n/q-N)} \left(2^k R\right)^{n/d} 2^{-(\alpha_2+N)k} \left\| f \mid \dot{K}^{\alpha_2,p}_q \right\|. \end{aligned}$$

From this and (3.12), we get

$$\begin{aligned} &\sum_{k=1-[\log_2 R]}^{\infty} 2^{k\alpha_1 r} \left\| V^1_{R,k}\chi_k \right\|_s^r \\ &\leqslant c\, R^{(n/q-N+n/d)r} \left\| f \mid \dot{K}^{\alpha_2,p}_q \right\|^r \sum_{k=1-[\log_2 R]}^{\infty} 2^{k(n/s+n/d+\alpha_1-\alpha_2-N)r} \\ &\leqslant c\, R^{(n/q-n/s+\alpha_2-\alpha_1)r} \left\| f \mid \dot{K}^{\alpha_2,p}_q \right\|^r. \end{aligned}$$

Applying the Minkowski inequality, we obtain

$$\left\| V^2_{R,k}\chi_k \right\|_s \leqslant c\, R^{n/q} \left\| (1+R|\cdot|)^{-N} \right\|_s \left\| f\chi_{\widetilde{C}_k} \right\|_q \leqslant c\, R^{n/q-n/s} \left\| f\chi_{\widetilde{C}_k} \right\|_q,$$

where we have used the fact that $N > n/s$. Hence

$$\begin{aligned} &\left(\sum_{k=1-[\log_2 R]}^{\infty} 2^{k\alpha_1 r} \left\| V^2_{R,k}\chi_k \right\|_s^r \right)^{1/r} \\ &\leqslant c\, R^{n/q-n/s} \left(\sum_{k=1-[\log_2 R]}^{\infty} 2^{k(\alpha_1-\alpha_2)r} 2^{k\alpha_2 r} \left\| f\chi_{\widetilde{C}_k} \right\|_q^r \right)^{1/r} \\ &\leqslant c\, R^{n/q-n/s+\alpha_2-\alpha_1} \sup_{k\in\mathbb{Z}} \left(2^{k\alpha_2} \left\| f\chi_{\widetilde{C}_k} \right\|_q \right) \left(\sum_{k=1-[\log_2 R]}^{\infty} \left(2^k R\right)^{(\alpha_1-\alpha_2)r} \right)^{1/r} \\ &\leqslant c\, R^{n/q-n/s+\alpha_2-\alpha_1} \left\| f \mid \dot{K}^{\alpha_2,p}_q \right\|, \end{aligned}$$

if $\alpha_2 > \alpha_1$. The case $\alpha_2 = \alpha_1$ can be easily solved. For $V^3_{R,k}$, we see that $\int_{\mathbb{R}^n \setminus B(0,2^{k+2})} (\cdots)dy$ can be rewritten as

$$\sum_{i=0}^{\infty} \int_{C_{k+i+3}} (\cdots)dy.$$

Then, using (3.4), we get for any $x \in C_k$

$$\left(V^3_{R,k}(x)\right)^{\rho} \leqslant c \sum_{i=0}^{\infty} \left(w_{R,qN} * |f\chi_{C_{k+i+3}}|^q(x) \right)^{\rho/q},$$

with $\rho = \min(1,q)$. Since $|x-y| > 3\cdot 2^{k+i}$ for any $x \in C_k$ and any $y \in C_{k+i+3}$, the right-hand side of the last inequality is bounded by

$$\begin{aligned}
&c\, R^{\rho(n/q-N)} \sum_{i=0}^{\infty} 2^{-(k+i)\rho N} \left\| f\chi_{C_{k+i+3}} \right\|_q^{\rho} \\
&= c\, R^{\rho(n/q-N)} \sum_{j=k+3}^{\infty} 2^{-j\rho N} \left\| f\chi_{C_j} \right\|_q^{\rho} \\
&\leqslant c\, R^{\rho(n/q-N)} \sum_{j=k+3}^{\infty} 2^{j\rho(n/q-N)} \sup_{x\in B(0,2^j)} |f(x)|^{\rho} \\
&\leqslant c\, R^{\rho(n/q-N+n/d)} \sum_{j=k+3}^{\infty} 2^{j\rho(n/d-N-\alpha_2)} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^{\rho} \\
&\leqslant c\, R^{\rho(n/q-N+n/d)} 2^{k\rho(n/d-N-\alpha_2)} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^{\rho},
\end{aligned}$$

where we have used Lemma 3.3 (since $j > k > -[\log_2 R]$) and (3.12). Therefore,

$$\begin{aligned}
&\sum_{k=1-[\log_2 R]}^{\infty} 2^{k\alpha_1 r} \left\| V_{R,k}^3 \chi_k \right\|_s^r \\
&\leqslant c\, R^{(n/q-N+n/d)r} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^r \sum_{k=1-[\log_2 R]}^{\infty} 2^{k(n/s-\alpha_2+\alpha_1-N+n/d)r} \\
&\leqslant c\, R^{(n/q-n/s+\alpha_2-\alpha_1)r} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^r \sum_{k=1-[\log_2 R]}^{\infty} \left(2^k R\right)^{(n/s-\alpha_2+\alpha_1-N+n/d)r} \\
&\leqslant c\, R^{(n/q-n/s+\alpha_2-\alpha_1)r} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|^r,
\end{aligned}$$

where we have used again (3.12). The proof is completed. □

REMARK 3.13. We would like to mention that Lemma 3.10 generalizes the classical Plancherel-Polya-Nikolskij inequality by taking $\alpha_1 = \alpha_2 = 0$, $r = s$ and by using the embedding $\ell_q \hookrightarrow \ell_s$.

In the previous lemma we have not treated the case $s \leqslant q$. The next lemma gives a positive answer.

LEMMA 3.14. *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $0 < s,p,q,r \leqslant \infty$. We suppose that $\alpha_1 + n/s > 0$, $0 < s \leqslant q \leqslant \infty$ and $\alpha_2 > \alpha_1 + n/s - n/q$. Then there exist a positive constant c independent of R such that for all $f \in \dot{K}_q^{\alpha_2,p} \cap \mathscr{S}'(\mathbb{R}^n)$ with* supp $\mathscr{F}f \subset \overline{B}(0,R)$, *we have*

$$\left\| f \mid \dot{K}_s^{\alpha_1,r} \right\| \leqslant c\, R^{n/q-n/s+\alpha_2-\alpha_1} \left\| f \mid \dot{K}_q^{\alpha_2,p} \right\|.$$

Proof. We employ the notations II_R and I_R from (3.11). The estimate of I_R follows easily from the previous lemma. We only need to estimate the part II_R. Hölder's inequality gives

$$\left\| f\chi_k \right\|_s \leqslant c\, 2^{kn(1/s-1/q)} \left\| f\chi_k \right\|_q. \tag{3.15}$$

Therefore,

$$\begin{aligned} II_R &\leqslant \sum_{k=1-[\log_2 R]}^{\infty} 2^{k(n/s-n/q+\alpha_2-\alpha_1)r} 2^{k\alpha_2 r} \|f\chi_k\|_q^r \\ &\leqslant \sup_{k\in\mathbb{Z}} \left(2^{k\alpha_2} \|f\chi_k\|_q\right)^r \sum_{k=1-[\log_2 R]}^{\infty} 2^{k(n/s-n/q-\alpha_2+\alpha_1)r} \\ &\leqslant c\, R^{(n/q-n/s+\alpha_2-\alpha_1)r} \left\|f \mid \dot{K}_q^{\alpha_2,p}\right\|^r \sum_{k=1-[\log_2 R]}^{\infty} \left(2^k R\right)^{(n/s-n/q-\alpha_2+\alpha_1)r} \\ &\leqslant c\, R^{(n/q-n/s+\alpha_2-\alpha_1)r} \left\|f \mid \dot{K}_q^{\alpha_2,p}\right\|^r, \end{aligned}$$

since $2^k R > 1$. The proof is completed. □

Using the estimate (3.15), we easily obtain that the previous lemma is true for $\alpha_2 = \alpha_1 + n/s - n/q, s \leqslant q, r = p$ and any $f \in \dot{K}_q^{\alpha_2,p}$.

The following statement can be found in [3], that plays an essential role later on.

LEMMA 3.16. *Let real numbers $s_1 < s_0$ be given, and $\sigma \in]0,1[$. For $0 < q \leqslant \infty$ there is $c > 0$ such that*

$$\left\|\left\{2^{(\sigma s_0+(1-\sigma)s_1)j} a_j\right\}_{j\in\mathbb{N}_0} \mid \ell_q\right\| \leqslant c \left\|\left\{2^{s_0 j} a_j\right\}_{j\in\mathbb{N}_0} \mid \ell_\infty\right\|^\sigma \left\|\left\{2^{s_1 j} a_j\right\}_{j\in\mathbb{N}_0} \mid \ell_\infty\right\|^{1-\sigma}$$

holds for all complex sequences $\left\{2^{s_0 j} a_j\right\}_{j\in\mathbb{N}_0}$ in ℓ_∞.

4. Function spaces

In this section we present the Fourier analytical definition of Herz-type Besov spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and Herz-type Triebel-Lizorkin spaces $\dot{K}_q^{\alpha,p} F_\beta^s$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity.

DEFINITION 4.1. Let Ψ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leqslant 1$ and $\Psi(x) = 0$ for $|x| \geqslant \frac{3}{2}$. We put $\varphi_0(x) = \Psi(x)$, $\varphi_1(x) = \Psi(x/2) - \Psi(x)$ and

$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \quad \textit{for} \quad j = 2, 3, \ldots.$$

Then we have supp $\varphi_j \subset \left\{x \in \mathbb{R}^n : 2^{j-1} \leqslant |x| \leqslant 3\cdot 2^{j-1}\right\}$, $\varphi_j(x) = 1$ for $3\cdot 2^{j-2} \leqslant |x| \leqslant 2^j$ and $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. The system of functions $\left\{\varphi_j\right\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j by the following:

$$\Delta_j f = \mathscr{F}^{-1}\varphi_j * f, \quad j \in \mathbb{N} \quad \textit{and} \quad \Delta_0 f = \mathscr{F}^{-1}\Psi * f, \quad f \in \mathscr{S}'(\mathbb{R}^n).$$

Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty} \Delta_j f$ of all $f \in \mathscr{S}'(\mathbb{R}^n)$ (convergence in $\mathscr{S}'(\mathbb{R}^n)$).

We are now in a position to state the definitions of Herz-type Besov and Triebel-Lizorkin spaces.

DEFINITION 4.2. (i) Let $\alpha, s \in \mathbb{R}$ and $0 < p, q, \beta \leqslant \infty$. The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid \dot{K}_q^{\alpha,p} B_\beta^s \right\| = \left(\sum_{j=0}^{\infty} 2^{js\beta} \left\| \Delta_j f \mid \dot{K}_q^{\alpha,p} \right\|^\beta \right)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.
(ii) Let $\alpha, s \in \mathbb{R}, 0 < p, q < \infty$ and $0 < \beta \leqslant \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p} F_\beta^s$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid \dot{K}_q^{\alpha,p} F_\beta^s \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} \left| \Delta_j f \right|^\beta \right)^{1/\beta} \mid \dot{K}_q^{\alpha,p} \right\| < \infty,$$

with the obvious modification if $\beta = \infty$.

REMARK 4.3. Let $s \in \mathbb{R}, 0 < p, q, \beta \leqslant \infty$ (with $0 < p, q < \infty$ for $\dot{K}_q^{\alpha,p} F_\beta^s$ spaces) and $\alpha > -n/q$. The spaces $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions (in the sense of equivalent quasi-norms). In particular both $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi-Banach spaces and if $p, q, \beta \geqslant 1$, then both $\dot{K}_q^{\alpha,p} B_\beta^s$ and $\dot{K}_q^{\alpha,p} F_\beta^s$ are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [22], [23] and [25].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

DEFINITION 4.4. (i) Let $s \in \mathbb{R}$ and $0 < p, \beta \leqslant \infty$. The Besov space $B_{p,\beta}^s$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid B_{p,\beta}^s \right\| = \left(\sum_{j=0}^{\infty} 2^{js\beta} \left\| \Delta_j f \right\|_p^\beta \right)^{1/\beta} < \infty.$$

(ii) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \leqslant \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\left\| f \mid F_{p,\beta}^s \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{js\beta} \left| \Delta_j f \right|^\beta \right)^{1/\beta} \right\|_p < \infty.$$

The theory of the spaces $B^s_{p,q}$ and $F^s_{p,\beta}$ has been developed in detail in [17], [18] and [19] but has a longer history already including many contributors; we do not want to discuss this here. In particular, with $p=q=\infty, s>0$, one recovers Hölder-Zygmund spaces $\mathscr{C}^s = B^s_{\infty,\infty}$, cf. [17, Thm. 2.5.12]. Clearly, for $s\in\mathbb{R}, 0<p\leqslant\infty$ ($0<p<\infty$ for the $\dot{K}^{0,p}_p F^s_\beta$ spaces) and $0<\beta\leqslant\infty$,

$$\dot{K}^{0,p}_p B^s_\beta = B^s_{p,\beta} \quad \text{and} \quad \dot{K}^{0,p}_p F^s_\beta = F^s_{p,\beta}.$$

5. Embeddings

The following theorem gives basic embeddings of the spaces $\dot{K}^{\alpha,p}_q B^s_\beta$ and $\dot{K}^{\alpha,p}_q F^s_\beta$. For $\dot{K}^{\alpha,p}_q F^s_\beta$ spaces these results are proved in [22, p, 649]. Their arguments are true for $\dot{K}^{\alpha,p}_q B^s_\beta$ spaces.

THEOREM 5.1. *Let $s\in\mathbb{R}, 0<p,q\leqslant\infty$ and $\alpha>-n/q$.*
(i) *If $0<\beta_1\leqslant\beta_2\leqslant\infty$, then*

$$\dot{K}^{\alpha,p}_q B^s_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q B^s_{\beta_2} \quad \textit{and} \quad \dot{K}^{\alpha,p}_q F^s_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q F^s_{\beta_2}.$$

(ii) *If $0<\beta_1,\beta_2\leqslant\infty$ and $\varepsilon>0$, then*

$$\dot{K}^{\alpha,p}_q B^{s+\varepsilon}_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q B^s_{\beta_2} \quad \textit{and} \quad \dot{K}^{\alpha,p}_q F^{s+\varepsilon}_{\beta_1} \hookrightarrow \dot{K}^{\alpha,p}_q F^s_{\beta_2}.$$

(iii) *If $0<p_1\leqslant p_2\leqslant\infty$, then*

$$\dot{K}^{\alpha,p_1}_q B^s_\beta \hookrightarrow \dot{K}^{\alpha,p_2}_q B^s_\beta \quad \textit{and} \quad \dot{K}^{\alpha,p_1}_q F^s_\beta \hookrightarrow \dot{K}^{\alpha,p_2}_q F^s_\beta.$$

(iv) *$0<q_1\leqslant q_2\leqslant\infty$, then*

$$\dot{K}^{\alpha,p}_{q_2} B^s_\beta \hookrightarrow \dot{K}^{r,p}_{q_1} B^s_\beta \quad \textit{and} \quad \dot{K}^{\alpha,p}_{q_2} F^s_\beta \hookrightarrow \dot{K}^{r,p}_{q_1} F^s_\beta,$$

where $r=\alpha-n(1/q_1-1/q_2)$.

The following theorem gives basic embeddings between the spaces $\dot{K}^{\alpha,p}_q B^s_\beta$ and $\dot{K}^{\alpha,p}_q F^s_\beta$.

THEOREM 5.2. *Let $s\in\mathbb{R}, 0<p,q<\infty, 0<\beta\leqslant\infty$ and $\alpha>-n/q$. Then*

$$\dot{K}^{\alpha,p}_q B^s_{\min(\beta,p,q)} \hookrightarrow \dot{K}^{\alpha,p}_q F^s_\beta \hookrightarrow \dot{K}^{\alpha,p}_q B^s_{\max(\beta,p,q)}. \tag{5.3}$$

Proof. Our proof is based upon ideas found in [17, Prop. 2.3.2/2]. We begin with the following equalities

$$\max(\beta,p,q) = \max(\beta,\max(p,q)) \text{ and } \min(\beta,p,q) = \min(\beta,\min(p,q)).$$

We first consider $\beta \geqslant \max(p,q)$; thus the right-hand embedding of (5.3) reduces to $\dot{K}_q^{\alpha,p}F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p}B_\beta^s$. Let $f \in \dot{K}_q^{\alpha,p}F_\beta^s$, then the generalised triangle inequality yields that

$$
\begin{aligned}
&\left(\sum_{j=0}^{\infty} 2^{js\beta} \left\|\Delta_j f \mid \dot{K}_q^{\alpha,p}\right\|^\beta\right)^{1/\beta} \\
&= \left(\sum_{j=0}^{\infty} 2^{js\beta} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\|\Delta_j f \chi_k\right\|_q^p\right)^{\frac{\beta}{p}}\right)^{1/\beta} \\
&\leqslant \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=0}^{\infty} 2^{js\beta} \left\|\Delta_j f \chi_k\right\|_q^\beta\right)^{p/\beta}\right)^{1/p} \\
&\leqslant \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\|\left(\sum_{j=0}^{\infty} 2^{js\beta} \left|\Delta_j f\right|^\beta \chi_k\right)^{1/\beta}\right\|_q^p\right)^{1/p} \\
&\leqslant c \left\|f \mid \dot{K}_q^{\alpha,p}F_\beta^s\right\|.
\end{aligned}
$$

Conversely, when $\beta < \max(p,q)$, we have $\dot{K}_q^{\alpha,p}F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p}F_{\max(p,q)}^s \hookrightarrow \dot{K}_q^{\alpha,p}B_{\max(p,q)}^s$. This completes that

$$\dot{K}_q^{\alpha,p}F_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p}B_{\max(\beta,p,q)}^s.$$

Concerning the left-hand embedding of (5.3). We first consider $\beta \leqslant \min(p,q)$; thus the left-hand embedding of (5.3) reduces to $\dot{K}_q^{\alpha,p}B_\beta^s \hookrightarrow \dot{K}_q^{\alpha,p}F_\beta^s$. Let $f \in \dot{K}_q^{\alpha,p}B_\beta^s$, then the generalised triangle inequality yields that

$$
\begin{aligned}
&\left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\|\left(\sum_{j=0}^{\infty} 2^{js\beta} \left|\Delta_j f\right|^\beta\right)^{1/\beta} \chi_k\right\|_q^p\right)^{1/p} \\
&= \left(\sum_{k=-\infty}^{\infty} \left\|\sum_{j=0}^{\infty} 2^{js\beta+k\alpha\beta} \left|\Delta_j f\right|^\beta \chi_k\right\|_{q/\beta}^{p/\beta}\right)^{1/p} \\
&\leqslant \left(\sum_{k=-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \left\|2^{js\beta+k\alpha\beta} \left|\Delta_j f\right|^\beta \chi_k\right\|_{q/\beta}\right)^{p/\beta}\right)^{1/p} \\
&\leqslant \left(\sum_{j=0}^{\infty} \left(\sum_{k=-\infty}^{\infty} \left\|2^{js\beta+k\alpha\beta} \left|\Delta_j f\right|^\beta \chi_k\right\|_{q/\beta}^{p/\beta}\right)^{\beta/p}\right)^{1/\beta}
\end{aligned}
$$

$$= \left(\sum_{j=0}^{\infty} 2^{js\beta} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left\| \Delta_j f \chi_k \right\|_q^p \right)^{\beta/p} \right)^{1/\beta}$$
$$\leqslant c \left\| f \mid \dot{K}_q^{\alpha,p} B_\beta^s \right\|.$$

Now let $\beta > \min(p,q)$. From the case $\beta \leqslant \min(p,q)$ we immediately obtain the embeddings $\dot{K}_q^{\alpha,p} B^s_{\min(p,q)} \hookrightarrow \dot{K}_q^{\alpha,p} F^s_{\min(p,q)} \hookrightarrow \dot{K}_q^{\alpha,p} F_\beta^s$. This completes the proof. □

REMARK 5.4. Theorem 5.1 when $\alpha = 0$, $p = q$ generalizes the corresponding results on Besov and Triebel-Lizorkin spaces established in [17], Section 2.3.

THEOREM 5.5. (i) *Let $s \in \mathbb{R}, 0 < p,q,\beta \leqslant \infty$ and $\alpha > -n/q$. The Herz-type Besov space $\dot{K}_q^{\alpha,p} B_\beta^s$ is a quasi-Banach space. Furthermore,*

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_q^{\alpha,p} B_\beta^s \hookrightarrow \mathscr{S}'(\mathbb{R}^n). \tag{5.6}$$

If $0 < p,q,\beta < \infty, s \in \mathbb{R}$ and $\alpha > -n/q$, then $\mathscr{S}(\mathbb{R}^n)$ is dense in $\dot{K}_q^{\alpha,p} B_\beta^s$.
(ii) *Let $s \in \mathbb{R}, 0 < p,q < \infty, 0 < \beta \leqslant \infty$ and $\alpha > -n/q$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p} F_\beta^s$ is a quasi-Banach space. Furthermore,*

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_q^{\alpha,p} F_\beta^s \hookrightarrow \mathscr{S}'(\mathbb{R}^n). \tag{5.7}$$

If $s \in \mathbb{R}, 0 < p,q,\beta < \infty$ and $\alpha > -n/q$, then $\mathscr{S}(\mathbb{R}^n)$ is dense in $\dot{K}_q^{\alpha,p} F_\beta^s$.

Proof. The proof of the left-hand embedding of (5.7) and the density of $\mathscr{S}(\mathbb{R}^n)$ in $\dot{K}_q^{\alpha,p} F_\beta^s$ are given in [22, Theorem 4.2] (a similar argument is valid to prove the density of $\mathscr{S}(\mathbb{R}^n)$ in $\dot{K}_q^{\alpha,p} B_\beta^s$) and then Theorem 5.1(ii) and the right-hand embedding of (5.3) yield the left-hand embedding of (5.6). Now we prove that

$$\dot{K}_q^{\alpha,p} B_\infty^s \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be the smooth dyadic resolution of unity. We put $\omega_j = \sum_{i=j-1}^{i=j+1} \varphi_i$ if $j = 1,2,\ldots$ (with $\varphi_{-1} = 0$). If $f \in \dot{K}_q^{\alpha,p} B_\infty^s$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, then $f(\psi)$ denotes the value of the functional f of $\mathscr{S}'(\mathbb{R}^n)$ for the test function ψ. We obtain

$$|f(\psi)| \leqslant \sum_{j=0}^{\infty} \left| \Delta_j f(\mathscr{F}^{-1}\omega_j * \psi) \right| = \sum_{j=0}^{\infty} \left\| \Delta_j f \cdot (\mathscr{F}^{-1}\omega_j * \psi) \right\|_1$$
$$= \sum_{j=0}^{\infty} \left\| \Delta_j f \cdot (\mathscr{F}^{-1}\omega_j * \psi) \mid \dot{K}_1^{0,1} \right\|.$$

Recalling the definition of $\dot{K}_1^{0,1}$ spaces, this sum can be rewritten as

$$\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left\| \Delta_j f \cdot \mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k \right\|_1$$

$$\leqslant \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sup_{x \in B(0,2^k)} |\Delta_j f(x)| \, \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 .$$

We divide the last sum into two parts

$$\sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} \cdots + \sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} \cdots .$$

Lemma 3.3, gives for any $0 < d < 1/\max(1/q, 1/q + \alpha/n)$ and any $N > n/d - n/q - \alpha$

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} \sup_{x \in B(0,2^{-j})} |\Delta_j f(x)| \, \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant \sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} 2^{(n/q+\alpha)j} \|\Delta_j f \mid \dot{K}_q^{\alpha,p}\| \, \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant c \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| \sum_{j=0}^{\infty} \sum_{k=-\infty}^{-j-1} 2^{(\alpha+n/q-s)j} \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant c \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| \left\| \psi \mid B_{1,1}^{\alpha+n/q-s} \right\| .
\end{aligned}$$

Using again Lemma 3.3, we have for any $0 < d < 1/\max(1/q, 1/q + \alpha/n)$

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} \sup_{x \in B(0,2^k)} |\Delta_j f(x)| \, \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant \sum_{j=0}^{\infty} \sum_{k=-j}^{\infty} 2^{jn/d} 2^{(n/d-n/q-\alpha)k} \|\Delta_j f \mid \dot{K}_q^{\alpha,p}\| \, \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant c \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| \sum_{j=0}^{\infty} 2^{(n/d-s)j} \sum_{k=-j}^{\infty} 2^{(n/d-n/q-\alpha)k} \|\mathscr{F}^{-1}\omega_j * \psi \cdot \chi_k\|_1 \\
&\leqslant c \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| \left\| \psi \mid \dot{K}_1^{n/d-n/q-\alpha,1} B_1^{n/d-s} \right\| .
\end{aligned}$$

Consequently

$$\begin{aligned}
&|f(\psi)| \\
&\leqslant c \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| \max\left(\left\| \psi \mid B_{1,1}^{\alpha+n/q-s} \right\|, \left\| \psi \mid \dot{K}_1^{n/d-n/q-\alpha,1} B_1^{n/d-s} \right\| \right).
\end{aligned}$$

By our assumption on d we have $\mathscr{S}(\mathbb{R}^n) \hookrightarrow \dot{K}_1^{n/d-n/q-\alpha,1} B_1^{n/d-s}$. From this and the embedding $\mathscr{S}(\mathbb{R}^n) \hookrightarrow B_{1,1}^{\alpha+n/q-s}$ we obtain

$$|f(\psi)| \leqslant c \, p_N(\psi) \|f \mid \dot{K}_q^{\alpha,p} B_\infty^s\| .$$

This proves that $\dot{K}_q^{\alpha,p} B_\infty^s$ is continuously embedded in $\mathscr{S}'(\mathbb{R}^n)$. Hence the right-hand embeddings of (5.6) and (5.7) are proved in view of the embeddings (5.3). □

We next consider embeddings of Sobolev-type. It is well-known that

$$B^{s_2}_{q,\beta} \hookrightarrow B^{s_1}_{s,\beta}, \quad F^{s_2}_{q,\infty} \hookrightarrow F^{s_1}_{s,\beta} \tag{5.8}$$

if $s_1 - n/s = s_2 - n/q$, where $0 < q \leqslant s \leqslant \infty$ ($0 < q \leqslant s < \infty$ for the Triebel-Lizorkin spaces), $s_1 \leqslant s_2$ and $0 < \beta \leqslant \infty$ (see e.g. [17, Theorem 2.7.1]). In the following theorem we generalize these embeddings to Herz-type Besov and Triebel-Lizorkin spaces.

THEOREM 5.9. *Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p, q, r, \beta \leqslant \infty, \alpha_1 > -n/s$ and $\alpha_2 > -n/q$. We suppose that*

$$s_1 - n/s - \alpha_1 \leqslant s_2 - n/q - \alpha_2. \tag{5.10}$$

(i) *Let $0 < q \leqslant s \leqslant \infty$ and $\alpha_2 \geqslant \alpha_1$ or $0 < s \leqslant q \leqslant \infty$ and*

$$\alpha_2 + n/q \geqslant \alpha_1 + n/s. \tag{5.11}$$

Then

$$\dot{K}^{\alpha_2,\theta}_q B^{s_2}_\beta \hookrightarrow \dot{K}^{\alpha_1,r}_s B^{s_1}_\beta, \tag{5.12}$$

where

$$\theta = \begin{cases} r \text{ if } \alpha_2 + n/q = \alpha_1 + n/s, s \leqslant q \text{ or } \alpha_2 = \alpha_1, q \leqslant s \\ p \text{ if } \alpha_2 + n/q > \alpha_1 + n/s, s \leqslant q \text{ or } \alpha_2 > \alpha_1, q \leqslant s. \end{cases}$$

(ii) *Let $0 < s, p, q, r < \infty$ and $0 < \beta \leqslant \infty$. We suppose that*

$$\alpha_2 + n/q = \alpha_1 + n/s, \quad 0 < s \leqslant q < \infty$$

or

$$\alpha_2 + n/q > \alpha_1 + n/s, \quad \max(0, \alpha_1 s/q) \leqslant \alpha_2 \leqslant (\alpha_1 + n/s) r/p - n/q.$$

Then

$$\dot{K}^{\alpha_2,p}_q F^{s_2}_\infty \hookrightarrow \dot{K}^{\alpha_1,r}_s F^{s_1}_\beta. \tag{5.13}$$

Proof. (i). Let $f \in \dot{K}^{\alpha_2,\theta}_q B^{s_2}_\beta$. Lemmas 3.10 and 3.14 give

$$\left\|\Delta_j f \mid \dot{K}^{\alpha_1,r}_s\right\| \leqslant c\, 2^{j(\alpha_2+n/q-n/s-\alpha_1)} \left\|\Delta_j f \mid \dot{K}^{\alpha_2,\theta}_q\right\|,$$

where $c > 0$ is independent of $j \in \mathbb{N}_0$. However the desired embedding is an immediate consequence of this estimate.

(ii). Let $f \in \dot{K}^{\alpha_2,p}_q F^{s_2}_\infty$. The case $\alpha_2 + n/q = \alpha_1 + n/s$ and $s \leqslant q$ can be found in [22]. The idea of the second case is from [8]. Let $s_0 = s_2 - n/q - \alpha_2$. Since $\alpha_2 + n/q > \alpha_1 + n/s$ there is some $\sigma \in]0,1[$ so that $\sigma(n/q + \alpha_2) = \alpha_1 + n/s$. Hence

$$\begin{aligned} \sigma s_2 + (1-\sigma) s_0 &= \sigma s_2 + (1-\sigma)(s_2 - n/q - \alpha_2) \\ &\geqslant s_1 - n/s - \alpha_1 + \sigma(n/q + \alpha_2) \\ &= s_1. \end{aligned}$$

Lemma 3.16 gives for any $k \in \mathbb{Z}$ and $x \in C_k$

$$\left\| \left\{ 2^{s_1 j} \Delta_j f(x) \right\}_{j \in \mathbb{N}_0} \mid \ell_\beta \right\| \leqslant \left\| \left\{ 2^{s_2 j} \Delta_j f(x) \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\|^\sigma \left\| \left\{ 2^{s_0 j} \Delta_j f(x) \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\|^{1-\sigma}.$$

By Lemma 3.10 we obtain for any $x \in \mathbb{R}^n$, $\alpha_2 \geqslant 0$ and $0 < p, q \leqslant \infty$

$$\left| \Delta_j f(x) \right| \leqslant c\, 2^{j(\alpha_2 + n/q)} \left\| \Delta_j f \mid \dot{K}_q^{\alpha_2, p} \right\|.$$

Therefore,

$$\left\| \left\{ 2^{s_0 j} \Delta_j f(x) \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\| \leqslant c \left\| f \mid \dot{K}_q^{\alpha_2, p} F_\infty^{s_2} \right\|.$$

Since $1/\sigma s = 1/q + \frac{\alpha_2 - \alpha_1/\sigma}{n}$ and $\alpha_2 \geqslant \alpha_1/\sigma$, the Hölder inequality gives

$$\begin{aligned} 2^{k\alpha_1} \left\| \left\| \left\{ 2^{s_2 j} \Delta_j f \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\|^\sigma \chi_k \right\|_s &= 2^{k\alpha_1} \left\| \left\| \left\{ 2^{s_2 j} \Delta_j f \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\| \chi_k \right\|_{\sigma s}^\sigma \\ &\leqslant 2^{k\sigma\alpha_2} \left\| \left\| \left\{ 2^{s_2 j} \Delta_j f \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\| \chi_k \right\|_q^\sigma. \end{aligned}$$

The last expression in ℓ_r is bounded by

$$\begin{aligned} \left\| \left\| \left\{ 2^{s_2 j} \Delta_j f \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\| \mid \dot{K}_q^{\alpha_2, \sigma r} \right\|^\sigma &\leqslant c \left\| \left\| \left\{ 2^{s_2 j} \Delta_j f \right\}_{j \in \mathbb{N}_0} \mid \ell_\infty \right\| \mid \dot{K}_q^{\alpha_2, p} \right\|^\sigma \\ &\leqslant c \left\| f \mid \dot{K}_q^{\alpha_2, p} F_\infty^{s_2} \right\|^\sigma, \end{aligned}$$

by the embedding $\dot{K}_q^{\alpha_2, p} \hookrightarrow \dot{K}_q^{\alpha_2, \sigma r}$ (because of $p(\alpha_2 + n/q) \leqslant r(\alpha_1 + n/s)$). □

If $\alpha_1 = \alpha_2 = 0$, $p = q$ and $r = s$, Theorem 5.9 reduces to the known results on $B_{p,q}^s$ and $F_{p,q}^s$; see (5.8) (by using the embedding $\ell_q \hookrightarrow \ell_s$ in the case of $B_{p,q}^s$ spaces). Also under the hypothesis of this theorem, we have $s_1 \leqslant s_2$.

Let us prove that (5.10) and (5.11) are necessary. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be a function such that supp $\mathscr{F}\varphi \subset \{\xi \in \mathbb{R}^n : 3/4 < |\xi| < 1\}$. For $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$ we put $f_N(x) = \varphi(2^N x)$. First we have $\varphi \in \dot{K}_q^{\alpha_2, \theta} \cap \dot{K}_s^{\alpha_1, r}$. Due to the support properties of the function φ we have for any $j \in \mathbb{N}_0$

$$\Delta_j f_N = \begin{cases} f_N, & j = N \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \left\| f_N \mid \dot{K}_s^{\alpha_1, r} B_\beta^{s_1} \right\| &= 2^{s_1 N} \left\| f_N \mid \dot{K}_s^{\alpha_1, r} \right\| = 2^{s_1 N} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \left\| f_N \chi_k \right\|_s^r \right)^{1/r} \\ &= 2^{(s_1 - n/s)N} \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 r} \left\| \varphi \chi_{k+N} \right\|_s^r \right)^{1/r} \\ &= 2^{(s_1 - \alpha_1 - n/s)N} \left\| \varphi \mid \dot{K}_s^{\alpha_1, r} \right\|. \end{aligned}$$

The same arguments give

$$\left\| f_N \mid \dot{K}_s^{\alpha_1,r} F_\beta^{s_1} \right\| = 2^{(s_1-\alpha_1-n/s)N} \left\| \varphi \mid \dot{K}_s^{\alpha_1,r} \right\|,$$
$$\left\| f_N \mid \dot{K}_q^{\alpha_2,p} F_\infty^{s_2} \right\| = 2^{(s_2-\alpha_2-n/q)N} \left\| \varphi \mid \dot{K}_q^{\alpha_2,p} \right\|$$

and

$$\left\| f_N \mid \dot{K}_q^{\alpha_2,\theta} B_\beta^{s_2} \right\| = 2^{(s_2-\alpha_2-n/q)N} \left\| \varphi \mid \dot{K}_q^{\alpha_2,\theta} \right\|.$$

If the embeddings (5.12) and (5.13) hold then for any $N \in \mathbb{N}$

$$2^{(s_1-s_2-\alpha_1+\alpha_2-n/s+n/q)N} \leqslant c.$$

Thus, we conclude that (5.10) must necessarily hold by letting $N \to +\infty$.

Let now $\psi \in \mathscr{S}(\mathbb{R}^n)$ be a function such that supp $\mathscr{F}\psi \subset \{\xi \in \mathbb{R}^n : |\xi| < 1\}$. For $x \in \mathbb{R}^n$ and $N \in \mathbb{Z}\backslash\mathbb{N}$ we put $f_N(x) = \psi(2^N x)$. We have $\psi \in \dot{K}_q^{\alpha_2,\theta} \cap \dot{K}_s^{\alpha_1,r}$. It easy to see that

$$\Delta_j f_N = \begin{cases} f_N, & j=0 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\left\| f_N \mid \dot{K}_s^{\alpha_1,r} B_\beta^{s_1} \right\| = \left\| f_N \mid \dot{K}_s^{\alpha_1,r} \right\| = 2^{-(\alpha_1+n/s)N} \left\| \psi \mid \dot{K}_s^{\alpha_1,r} \right\|.$$

The same arguments give

$$\left\| f_N \mid \dot{K}_s^{\alpha_1,r} F_\beta^{s_1} \right\| = 2^{-(\alpha_1+n/s)N} \left\| \psi \mid \dot{K}_s^{\alpha_1,r} \right\|$$
$$\left\| f_N \mid \dot{K}_q^{\alpha_2,p} F_\infty^{s_2} \right\| = 2^{-(\alpha_2+n/q)N} \left\| \psi \mid \dot{K}_q^{\alpha_2,p} \right\|$$

and

$$\left\| f_N \mid \dot{K}_q^{\alpha_2,\theta} B_\beta^{s_2} \right\| = 2^{-(\alpha_2+n/q)N} \left\| \psi \mid \dot{K}_q^{\alpha_2,\theta} \right\|.$$

If the embeddings (5.12) and (5.13) hold then for any $N \in \mathbb{Z}\backslash\mathbb{N}$

$$2^{-(\alpha_1-\alpha_2+n/s-n/q)N} \leqslant c.$$

Thus, we conclude that (5.11) must necessarily hold by letting $N \to -\infty$.

From Theorem 5.9 and the fact that $\dot{K}_s^{0,s} B_\beta^{s_1} = B_{s,\beta}^{s_1}$ and $\dot{K}_s^{0,s} F_\beta^{s_1} = F_{s,\beta}^{s_1}$ we immediately arrive at the following result.

THEOREM 5.14. *Let* $\alpha, s_1, s_2 \in \mathbb{R}, 0 < s, p, q \leqslant \infty, s_1 - n/s \leqslant s_2 - n/q - \alpha$ *and* $0 < \beta \leqslant \infty$.
(i) *If* $\alpha \geqslant 0, 0 < q \leqslant s \leqslant \infty$ *or* $\alpha + n/q \geqslant n/s$ *and* $0 < s \leqslant q \leqslant \infty$, *then*

$$\dot{K}_q^{\alpha,\theta} B_\beta^{s_2} \hookrightarrow B_{s,\beta}^{s_1},$$

where

$$\theta = \begin{cases} s \text{ if } \alpha + n/q = n/s, s \leqslant q \text{ or } \alpha = 0, q \leqslant s \\ p \text{ if } \alpha + n/q > n/s, s \leqslant q \text{ or } \alpha > 0, q \leqslant s. \end{cases} \tag{5.15}$$

(ii) *If* $0 < s, p, q < \infty$ *and* $\alpha \geqslant 0$ *such that* $n/s - n/q < \alpha \leqslant n/p - n/q$*, then*

$$\dot{K}_q^{\alpha,p} F_\infty^{s_2} \hookrightarrow F_{s,\beta}^{s_1}.$$

Using this result, we have the following useful consequence.

COROLLARY 5.16. *Let* $s_1, s_2 \in \mathbb{R}, 0 < s, p, q \leqslant \infty, s_1 - n/s \leqslant s_2 - n/q$ *and* $0 < \beta \leqslant \infty$*. Then*

$$B_{q,\beta}^{s_2} \hookrightarrow \dot{K}_q^{0,s} B_\beta^{s_2} \hookrightarrow B_{s,\beta}^{s_1}, \quad 0 < q \leqslant s \leqslant \infty.$$

To prove this it sufficient to take in Theorem 5.14, $\theta = s$ and $\alpha = 0$. However the desired embeddings are an immediate consequence of the fact that

$$B_{q,\beta}^{s_2} = \dot{K}_q^{0,q} B_\beta^{s_2} \hookrightarrow \dot{K}_q^{0,s} B_\beta^{s_2}.$$

Let us define

$$\sigma_q = n\left(\frac{1}{\min(1,q)} - 1\right) \quad \text{and} \quad \overline{q} = \max(1,q).$$

By Theorem 5.14, the embedding $F_{q,\beta}^{s_1} \hookrightarrow B_{q,\max(q,\beta)}^{s_1}$ and the Sobolev-type embeddings (5.8), we get

$$\begin{aligned} &\dot{K}_q^{\alpha,p} B_\beta^{s_2} \hookrightarrow B_{q,\beta}^{s_1} \hookrightarrow B_{\overline{q},1}^{0}, \quad \alpha > 0, \sigma_q < s_1 \leqslant s_2 - \alpha \\ &\dot{K}_q^{\alpha,p} F_\beta^{s_2} \hookrightarrow F_{q,\beta}^{s_1} \hookrightarrow B_{\overline{q},1}^{0}, \quad 0 < \alpha \leqslant n/p - n/q, \sigma_q < s_1 \leqslant s_2 - \alpha \end{aligned}$$

for any $0 < p, q, \beta \leqslant \infty$ (with $0 < p, q < \infty$ in the F-case). In addition from the fact that $F_{q,2}^0 = L^q(\mathbb{R}^n)$ for any $1 < q < \infty$, we have

$$\dot{K}_q^{\alpha,p} F_\beta^{s_2} \hookrightarrow L^q(\mathbb{R}^n), \quad 0 < \alpha \leqslant n/p - n/q, s_2 \geqslant \alpha.$$

We further conclude that

$$\|f\|_{\overline{q}} \leqslant \sum_{j=0}^{\infty} \|\Delta_j f\|_{\overline{q}} = \|f \mid B_{\overline{q},1}^0\| \leqslant c \left\|f \mid \dot{K}_q^{\alpha,p} B_\beta^{s_2}\right\|$$

and

$$\|f\|_{\overline{q}} \leqslant c \left\|f \mid \dot{K}_q^{\alpha,p} F_\beta^{s_2}\right\|.$$

This shows that under the above assumptions the elements from $\dot{K}_q^{\alpha,p} B_\beta^{s_2}$ and $\dot{K}_q^{\alpha,p} F_\beta^{s_2}$ are regular distributions.

PROPOSITION 5.17. *Let* $\alpha > 0, 0 < s, p, q \leqslant \infty$ *and* $0 < \beta \leqslant \infty$.

(i) *If* $s > \sigma_q + \alpha$*, then*

$$\dot{K}_q^{\alpha,p} B_\beta^{s} \hookrightarrow L^{\overline{q}}(\mathbb{R}^n).$$

(ii) *Let* $0 < p, q < \infty$ *and* $0 < \alpha \leqslant n/p - n/q$*. If* $s > n/q - n + \alpha$ *and* $0 < q \leqslant 1$ *or* $s \geqslant \alpha$ *and* $1 < q < \infty$*, then*

$$\dot{K}_q^{\alpha,p} F_\beta^{s} \hookrightarrow L^{\overline{q}}(\mathbb{R}^n).$$

Let C_u be the space of all bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup norm. Concerning embeddings into C_u, we have the following result.

COROLLARY 5.18. *Let* $\alpha \geqslant 0$ *and* $0 < p,q \leqslant \infty$. *Then*

$$\dot{K}_q^{\alpha,\theta} B_1^{\alpha+n/q} \hookrightarrow C_u,$$

where

$$\theta = \begin{cases} \infty \text{ if } \alpha = 0 \\ p \text{ if } \alpha > 0. \end{cases}$$

Proof. It follows from Theorem 5.14 that

$$\dot{K}_q^{\alpha,\theta} B_1^{\alpha+n/q} \hookrightarrow B_{\infty,1}^0.$$

Hence the result follows by the embedding $B_{\infty,1}^0 \hookrightarrow C_u$, see [17, Proposition 2.5.7]. □

The following statement holds by Theorem 5.9 and the fact that $\dot{K}_q^{0,q} B_\beta^{s_2} = B_{q,\beta}^{s_2}$ and $\dot{K}_q^{0,q} F_\infty^{s_2} = F_{q,\infty}^{s_2}$.

THEOREM 5.19. *Let* $\alpha, s_1, s_2 \in \mathbb{R}, 0 < s,q,r \leqslant \infty, s_1 - n/s - \alpha \leqslant s_2 - n/q$ *and* $0 < \beta, \theta \leqslant \infty$.
(i) *If* $-n/s < \alpha \leqslant 0, 0 < q \leqslant s \leqslant \infty$ *or* $-n/s < \alpha \leqslant n/q - n/s, 0 < s \leqslant q \leqslant \infty$, *then*

$$B_{q,\beta}^{s_2} \hookrightarrow \dot{K}_s^{\alpha,\theta} B_\beta^{s_1},$$

where

$$\theta = \begin{cases} q \text{ if } \quad \alpha = n/q - n/s, s \leqslant q \text{ or } \alpha = 0, q \leqslant s \\ r \text{ if } -n/s < \alpha < n/q - n/s, s \leqslant q \text{ or } -n/s < \alpha < 0, q \leqslant s. \end{cases}$$

(ii) *If* $\alpha = n/q - n/s, 0 < s \leqslant q < \infty, 0 < r < \infty$ *or* $0 < s \leqslant r < \infty, 0 < q < \infty, n/r - n/s \leqslant \alpha \leqslant 0$ *and* $\alpha < n/q - n/s$, *then*

$$F_{q,\infty}^{s_2} \hookrightarrow \dot{K}_s^{\alpha,r} F_\beta^{s_1}.$$

Using this result, we obtain:

COROLLARY 5.20. *Let* $s_1, s_2 \in \mathbb{R}, 0 < s,p,q \leqslant \infty, s_1 - n/s \leqslant s_2 - n/q$ *and* $0 < \beta \leqslant \infty$. *Then*

$$B_{q,\beta}^{s_2} \hookrightarrow \dot{K}_s^{0,q} B_\beta^{s_1} \hookrightarrow B_{s,\beta}^{s_1}, \quad 0 < q \leqslant s \leqslant \infty.$$

To prove this it sufficient to take in Theorem 5.19, $\theta = q$ and $\alpha = 0$. Then the desired embedding is an immediate consequence of the fact that $\dot{K}_s^{0,q} B_\beta^{s_1} \hookrightarrow \dot{K}_s^{0,s} B_\beta^{s_1} = B_{s,\beta}^{s_1}$.

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