Mathematical
Inequalities
& Applications
Volume 16, Number 2 (2013), 461–476 doi:10.7153/mia-16-33

NON–ISOTROPIC SINGULAR INTEGRALS AND MAXIMAL OPERATORS ALONG SURFACES OF REVOLUTION

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(*Communicated by B. Opic*)

Abstract. The authors establish the L^p-mapping properties for a class of non-isotropic singular integrals along surfaces of revolution as well as the related maximal operators, where the integral kernels are given by functions Ω in $L(\log^+ L)^\alpha(\Sigma)$.

1. Introduction

Let $\mathbb{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space with a non-isotropic dilation. Precisely, let P be an $n \times n$ real matrix whose eigenvalues have positive real parts and let $\gamma = \operatorname{trace} P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^P = \exp((\log t)P)$. There is a non-negative function r on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$. The function r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; furthermore it satisfies:

(i) $r(A_t x) = t r(x)$, for all $t > 0$ and $x \in \mathbb{R}^n$;

(ii) $r(x+y) \leqslant C(r(x) + r(y))$ for some $C > 0$;

(iii) if $\Sigma = \{x \in \mathbb{R}^n : \ r(x) = 1\}$, then $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta \rangle = 1\}$ for a positive symmetric matrix B, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. And then, the Lebesgue measure can be written as $dx = t^{\gamma-1} d\sigma dt$, that is,

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \int_\Sigma f(A_t\theta)t^{\gamma-1} d\sigma(\theta)dt$$

for appropriate functions f, where $d\sigma$ is a C^∞ measure on Σ. See [4, 9, 12] for more details.

Let Ω be locally integrable in $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$. We assume that

$$\int_\Sigma \Omega(\theta) d\sigma(\theta) = 0. \tag{1.1}$$

Mathematics subject classification (2010): Primary 42B20; Secondary 42B25, 42B15.

Keywords and phrases: Non-isotropic dilations, singular integrals, maximal functions, surfaces of revolution, rough kernels.

First author partially supported by the NNSF of China (Grant No.s 10931001, 10871173). Second author partially supported by the NNSF of China (Grant No. 11071200) and the NSF of Fujian Province of China (No. 2010J01013).

Paper MIA-16-33

Let $L(\log^+ L)^\alpha(\Sigma)$ (for $\alpha > 0$) denote the space of all those functions Ω on Σ which satisfy

$$\int_\Sigma |\Omega(\theta)|\log^\alpha(2+|\Omega(\theta)|)d\sigma(\theta) < \infty.$$

For $m \in \mathbb{N}$, let $\Gamma: [0,\infty) \longrightarrow \mathbb{R}^m$ be a continuous mapping satisfying $\Gamma(0) = 0$, $h: [0,\infty) \longrightarrow \mathbb{C}$ be a measurable function. We define a singular integral operator along the surface $(y, \Gamma(r(y))$ by

$$T_{\Omega,\Gamma,h}(f)(x,z) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y, z-\Gamma(r(y))K_{\Omega,h}(y)dy, \tag{1.2}$$

where $K_{\Omega,h}(y) = \Omega(y')r(y)^\gamma h(r(y))$, $y' = A_{r(y)^{-1}}y$. We assume that the principal value integral in (1.2) exists for every $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ and $f \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^m)$ (the Schwartz class).

For $\Gamma \equiv 0$, the operator $T_{\Omega,\Gamma,h}$ essentially reduces to the following lower dimensional non-isotropic singular integral

$$S(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K_{\Omega,h}(y)dy.$$

The early study on the non-isotropic singular integral operator S can be found in Stein and Wainger's work [13] and Riviere's work [9] for $h \equiv 1$. Subsequently, Duoandikoetxea and Rubio de Francia [7] showed that S is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $\Omega \in L^q(\Sigma)$ for some $q > 1$ and $h \in \Delta_2(\mathbb{R}_+)$, where, for $\nu > 1$, $\Delta_\nu(\mathbb{R}_+)$ denotes the set of all measurable functions h on $\mathbb{R}_+$ satisfying the condition

$$\sup_{R>0}\left(R^{-1}\int_0^R |h(t)|^\nu dt\right)^{1/\nu} < \infty,$$

and $\Delta_\infty(\mathbb{R}_+) = L^\infty(\mathbb{R}_+)$. It is easy to show $\Delta_\infty(\mathbb{R}_+) \subset \Delta_{\nu_2}(\mathbb{R}_+) \subset \Delta_{\nu_1}(\mathbb{R}_+)$, which are proper inclusions, for $\nu_1 < \nu_2 < \infty$. Recently, S. Sato [10] improved the above result to the case $\Omega \in L\log^+ L(\Sigma)$ and $h \in \mathscr{L}_a$, which is more general than $\Delta_\nu(\mathbb{R}_+)$ for $\nu > 1$ (see [10, Theorem 4] for the details). Furthermore, for the general mapping Γ, S. Sato [10] established the following:

THEOREM A. *Let $T_{\Omega,\Gamma,h}$ be given by* (1.2). *Suppose that $\Omega \in L\log^+ L(\Sigma)$ and $h \in \Delta_\nu(\mathbb{R}_+)$ for some $\nu > 1$. Then $T_{\Omega,\Gamma,h}$ is bounded on $L^p(\mathbb{R}^{n+m})$ provided that the maximal operator M_Γ given by*

$$M_\Gamma g(z) = \sup_{R>0} R^{-1}\int_0^R |g(z-\Gamma(t))|dt \tag{1.3}$$

is bounded on $L^q(\mathbb{R}^m)$ for all $q > 1$ and $|1/p - 1/2| < \min\{1/\nu', 1/2\}$, where $1/\nu + 1/\nu' = 1$.

Clearly, the range of p given in Theorem A is the full range $(1,\infty)$ when $\nu \geqslant 2$. However, the range of p becomes a tiny open interval around 2 as ν approaches 1. It is natural to ask whether the L^p-boundedness of $T_{\Omega,\Gamma,h}$ holds for p outside the range $|1/p-1/2|<\min\{1/2,1/\nu'\}$ for $1<\nu<2$.

In this paper we will focus on the solution of the above question. By imposing a more restrictive condition on h, we obtain a affirmative answer. Precisely, for $\nu>1$, we define $\mathscr{H}_\nu(\mathbb{R}_+)$ to be the set of all measurable function h on $\mathbb{R}_+$ satisfying the condition

$$\|h\|_{\mathscr{H}_\nu(\mathbb{R}_+)} := \|h\|_{L^\nu(\mathbb{R}_+,t^{-1}dt)} = \left(\int_{\mathbb{R}^+} |h(t)|^\nu t^{-1}dt\right)^{1/\nu} < \infty,$$

and define $\mathscr{H}_\infty(\mathbb{R}_+) = L^\infty(\mathbb{R}_+,t^{-1}dt)$. It is easy to see that $\mathscr{H}_\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+)$, and $\mathscr{H}_\nu(\mathbb{R}_+) \subset \Delta_\nu(\mathbb{R}_+)$, which is a proper inclusion, for $1<\nu<\infty$. We will establish the following

THEOREM 1.1. *Let $T_{\Omega,\Gamma,h}$ be given by (1.2) and M_Γ given by (1.3). Suppose that M_Γ is bounded on $L^q(\mathbb{R}^m)$ for all $q>1$, $h\in\mathscr{H}_\nu(\mathbb{R}_+)$ for some $1<\nu\leqslant\infty$ and $\Omega\in L(\log^+L)^{1/\nu'}(\Sigma)$ satisfying (1.1). Then $T_{\Omega,\Gamma,h}$ is bounded on $L^p(\mathbb{R}^{n+m})$ for $1<p<\infty$.*

REMARK 1.1. Notice that $L(\log^+L)^\alpha(\Sigma)\subset L(\log^+L)^\beta(\Sigma)$ for $0<\beta<\alpha$, which is proper, the condition on Ω in Theorem 1.1 is much weaker than those in Theorem A, and the range of p in Theorem 1.1 is extended to the full range $(1,\infty)$.

REMARK 1.2. We remark that such functions Γ satisfying the condition in Theorem 1.1 exist. See [11, 12] for examples. When $r(x)=|x|$ (the Euclidean norm), $m=1$ and Γ is a C^2, convex and increasing function satisfying $\Gamma(0)=0$, Theorem 1.1 was proved by Al-Qassem (see [2, Theorem 1.3]). And the corresponding result for $h\in\Delta_\nu(\mathbb{R}_+)$ ($\nu>1$) and $\Omega\in L\log^+L(\Sigma)$ was proved by Al-Salman and Pan (see [3, Theorem 4.1]). Therefore, our result is also the generalization of Al-Qassem [2], Al-Salman and Pan [3], even in the case $r(x)=|x|$.

To prove Theorem 1.1, we will use the following L^p-mapping properties of the related maximal operator $\mathscr{M}^{(\nu)}_{\Omega,\Gamma}$ given by

$$\mathscr{M}^{(\nu)}_{\Omega,\Gamma}f(x,z) = \sup_{\|h\|_{\mathscr{H}_\nu(\mathbb{R}_+)}\leqslant 1} \left|T_{\Omega,\Gamma,h}(x,z)\right|. \tag{1.4}$$

THEOREM 1.2. *Let $\mathscr{M}^{(\nu)}_{\Omega,\Gamma}$, M_Γ be as above. Suppose that M_Γ is bounded on $L^q(\mathbb{R}^m)$ for all $q>1$, $\Omega\in L(\log^+L)^{1/\nu'}(\Sigma)$ and satisfies (1.1). Then $\mathscr{M}^{(\nu)}_{\Omega,\Gamma}$ is bounded on $L^p(\mathbb{R}^{n+m})$ for $\nu'\leqslant p<\infty$ and $1<\nu\leqslant 2$, and it is bounded on $L^\infty(\mathbb{R}^{n+m})$ for $\nu=1$.*

REMARK 1.3. Theorem 1.2 has itself interesting. In the case of Euclidean norm, the study of the maximal operator $\mathscr{M}^{(\nu)}_{\Omega,\Gamma}$ began by Chen and Lin in [5], and subsequently by many other authors [2, 6, 8]. It is still an open problem whether the L^p-boundedness of $\mathscr{M}^{(\nu)}_{\Omega,\Gamma}$ holds for $2<\nu<\infty$, even for the case $m=1$ and $\Gamma\equiv 0$.

This paper is organized as follows. At first we will give some preliminary lemmas in Section 2. Then the proof of Theorem 1.2 will be given in Section 3. Finally, we will prove Theorem 1.1 in Section 4. We remark that some ideas of our arguments are taken from [2, 10], but we must establish some non-trivial estimates by new techniques.

Throughout this paper, we always use the letter C to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2. Preliminary lemmas

Following the notation in [10], let P^* denote the adjoint of the matrix P. Then $A_t^* = \exp((\log t)P^*)$. We write $A_t^* = B_t$. We can define a non-negative function s from $\{B_t\}$ exactly in the same way as we define r from $\{A_t\}$.

We will use the following estimates (see [12]):

$$c_1|x|^{\alpha_1} < r(x) < c_2|x|^{\alpha_2}, \quad \text{if } r(x) \geqslant 1, \tag{2.1}$$

$$c_3|x|^{\beta_1} < r(x) < c_2|x|^{\beta_2}, \quad \text{if } 0 < r(x) \leqslant 1; \tag{2.2}$$

and

$$d_1|\xi|^{a_1} < s(\xi) < d_2|\xi|^{a_2}, \quad \text{if } s(\xi) \geqslant 1, \tag{2.3}$$

$$d_3|\xi|^{b_1} < s(\xi) < d_4|\xi|^{b_2}, \quad \text{if } 0 < s(\xi) \leqslant 1, \tag{2.4}$$

where c_j, d_j $(j = 1, 2, 3, 4)$, $\alpha_k, \beta_k, a_k, b_k$ $(k = 1, 2)$ are positive constants.

LEMMA 2.1. (cf. [10]) *Let L be the degree of the minimal polynomial of P. Then, for $\eta, \zeta \in \mathbb{R}^n \backslash \{0\}$, we have*

$$\left| \int_1^2 \exp(i\langle B_t\eta, \zeta\rangle) t^{-1} dt \right| \leqslant C|\langle \eta, P\zeta\rangle|^{-1/L}$$

for some positive constant C independent of η and ζ.

Let $\Omega \in L(\log L)^{\alpha}(\Sigma)$ for $\alpha > 0$ and satisfy (1.1). Following the notation in [1], let $E_k := \{x' \in \Sigma : 2^k \leqslant |\Omega(x')| < 2^{k+1}\}$ for $k \in \mathbb{N}$ and let $E_0 := \{x' \in \Sigma : |\Omega(x')| < 2\}$. Set $\Lambda := \{k \in \mathbb{N} : \sigma(E_k) > 2^{-4k}\}$, and for $k \geqslant 1$,

$$\Omega_k(x') = \Omega(x')\chi_{E_k}(x') - \sigma(\Sigma)^{-1} \int_{E_k} \Omega(x') d\sigma(x'),$$

and $\Omega_0(x') = \Omega(x') - \sum_{k\in\Lambda} \Omega_k(x')$. It is easy to check that

$$\int_{\Sigma} \Omega_k(x') d\sigma(x') = 0, \quad k \geqslant 0, \tag{2.5}$$

$$\|\Omega_k\|_{L^1(\Sigma)} \leqslant 2\|\Omega\chi_{E_k}\|_{L^1(\Sigma)} := 2G_k, \quad k \in \Lambda, \tag{2.6}$$

$$\|\Omega_0\|_{L^1(\Sigma)} \leqslant C\|\Omega_0\|_{L^2(\Sigma)} \leqslant C, \tag{2.7}$$

$$\Omega(x') = \sum_{k\in\Lambda\cup\{0\}} \Omega_k(x'), \tag{2.8}$$

$$\sum_{k\in\Lambda\cup\{0\}}(k+1)^{\alpha}G_k\leqslant C\|\Omega\|_{L(\log L)^{\alpha}(\Sigma)}\quad\text{for any}\ \alpha>0, \tag{2.9}$$

where $G_k:=\|\Omega\chi_{E_k}\|_{L^1(\Sigma)}$ for $k\in\Lambda$ and $G_0=1$.

For each Ω_k, $k\in\Lambda\cup\{0\}$, we define a sequence of Borel measures $\{\tau_{k,j}:\ j\in\mathbb{Z}\}$ on $\mathbb{R}^{n+m}$ by

$$\widehat{\tau}_{k,j}(\xi,\eta)=\int_{D_{k,j}}e^{-2\pi i[y\cdot\xi+\Gamma(r(y))\cdot\eta]}\Omega_k(y')h(r(y))r(y)^{-\gamma}dy,$$

where $D_{k,j}=\{x\in\mathbb{R}^n: 2^{(k+1)j}<r(x)\leqslant 2^{(k+1)(j+1)}\}$, $\widehat{\tau}_{k,j}$ denotes the Fourier transform of $\tau_{k,j}$ defined by $\widehat{\tau}_{k,j}(\xi,\eta)=\int e^{-2\pi i[x\cdot\xi+z\cdot\eta]}d\tau_{k,j}(x,z)$. Here and below, the notation $u\cdot v$ denotes the inner product of u and v in $\mathbb{R}^n$ or $\mathbb{R}^m$. Then

$$T_{\Omega,\Gamma,h}(f)(x,z)=\sum_{k\in\Lambda\cup\{0\}}\sum_{j\in\mathbb{Z}}\tau_{k,j}*f(x,z):=\sum_{k\in\Lambda\cup\{0\}}T_{\Omega_k,\Gamma,h}(f)(x,z). \tag{2.10}$$

Also, we define the maximal function $\tau^*_{\Gamma,k}$ by

$$\tau^*_{\Gamma,k}(f)(x,z)=\sup_{j\in\mathbb{Z}}\left||\tau_{k,j}|*f(x,z)\right|,$$

where $|\tau_{k,j}|$ is defined by

$$\widehat{|\tau_{k,j}|}(\xi,\eta)=\int_{D_{k,j}}e^{-2\pi i[\xi\cdot y+\eta\cdot\Gamma(r(y))]}|\Omega_k(y')h(r(y))|r(y)^{-\gamma}dy.$$

In what follows, we will establish some lemmas, which will play key roles in the proofs of our main theorems.

LEMMA 2.2. *Let $k\in\Lambda\cup\{0\}$, Ω_k be as above. For $j\in\mathbb{Z}$, define*

$$\lambda_{k,j}(\xi,\eta)=\left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\left|\int_{\Sigma}\Omega_k(y')e^{-2\pi i[\xi\cdot A_r y'+\eta\cdot\Gamma(r)]}d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/2}.$$

Then for every $(\xi,\eta)\in\mathbb{R}^{n+m}$, there exists a positive number $\delta>0$ such that

$$\left|\lambda_{k,j}(\xi,\eta)\right|\leqslant C(k+1)^{1/2}G_k\min\left\{\left[2^{(k+1)j}s(\xi)\right]^{\delta/(k+1)},\left[2^{(k+1)j}s(\xi)\right]^{-\delta/(k+1)}\right\}. \tag{2.11}$$

Proof. By the definition, it is easy to see that

$$\left|\lambda_{k,j}(\xi,\eta)\right|\leqslant C(k+1)^{1/2}G_k. \tag{2.12}$$

On the other hand, it follows from the cancellation condition (2.5) of Ω_k and a change of variables that

$$\begin{aligned}|\lambda_{k,j}(\xi,\eta)| &\leqslant \Bigg(\int_1^{2^{k+1}}\Bigg(\int_\Sigma \Big|e^{-2\pi i[A_{2^{(k+1)j}r}y'\cdot\xi+\Gamma(2^{(k+1)j}r)\cdot\eta]}-e^{-2\pi i\Gamma(2^{(k+1)j}r)\cdot\eta}\Big| \\ &\qquad\times |\Omega_k(y')|d\sigma(y')\Bigg)^2\frac{dr}{r}\Bigg)^{1/2} \\ &\leqslant C\left(\int_1^{2^{k+1}}\left|\int_\Sigma |A_r y'||B_{2^{(k+1)j}}\xi||\Omega_k(y')|d\sigma(y')\right)^2\frac{dr}{r}\right)^{1/2}.\end{aligned}$$

Note that $|A_r y'|=|y|$ and the estimates (2.1)–(2.4), there are two positive constants α_1 and α_2 such that

$$|A_r y'|\leqslant Cr^{\alpha_1} \qquad\text{and}\qquad \left|B_{2^{(k+1)j}}\xi\right|\leqslant C\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}. \tag{2.13}$$

Therefore,

$$\begin{aligned}\left|\lambda_{k,j}(\xi,\eta)\right| &\leqslant C\|\Omega_k\|_{L^1(\Sigma)}\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}\left(\int_1^{2^{k+1}} r^{2\alpha_1}\frac{dr}{r}\right)^{1/2} \\ &\leqslant CG_k(k+1)^{1/2}2^{(k+1)\alpha_1}\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}.\end{aligned} \tag{2.14}$$

Interpolating between (2.12) and (2.14), we get

$$\left|\lambda_{k,j}(\xi,\eta)\right|\leqslant C(k+1)^{1/2}G_k\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2/(k+1)}. \tag{2.15}$$

It remains to prove the second estimate. Notice that

$$\begin{aligned}&\left|\int_\Sigma \Omega_k(y')e^{-2\pi i[A_{2^{(k+1)j}r}y'\cdot\xi+\Gamma(2^{(k+1)j}r)\cdot\eta]}d\sigma(y')\right|^2 \\ &\qquad = \int_{\Sigma\times\Sigma}\Omega_k(y')\overline{\Omega_k(u')}e^{-2\pi i(y'-u')\cdot B_{2^{(k+1)j}r}\xi}d\sigma(y')d\sigma(u'),\end{aligned} \tag{2.16}$$

which implies

$$\begin{aligned}\left|\lambda_{k,j}(\xi,\eta)\right|^2 &= \int_{\Sigma\times\Sigma}\Omega_k(y')\overline{\Omega_k(u')}\left(\int_1^{2^{k+1}} e^{-2\pi i(y'-u')\cdot B_{2^{(k+1)j}r}\xi}\frac{dr}{r}\right)d\sigma(y')d\sigma(u') \\ &= \sum_{l=0}^{k}\int_{\Sigma\times\Sigma}\Omega_k(y')\overline{\Omega_k(u')}\left(\int_1^2 e^{-2\pi i(y'-u')\cdot B_{2^{(k+1)j}2^l r}\xi}\frac{dr}{r}\right)d\sigma(y')d\sigma(u').\end{aligned}$$

By Lemma 2.1, we have

$$\left|\int_1^2 e^{-2\pi i(y'-u')\cdot B_{2^{(k+1)j}2^l r}\xi}\frac{dr}{r}\right|\leqslant C\left|P(y'-u')\cdot B_{2^{(k+1)j}2^l}\xi\right|^{-\varepsilon},$$

where $0 < \varepsilon \leqslant 1/L$, and L is the degree of the minimal polynomial of P. Using Hölder's inequality, if $0 < \varepsilon < \min\{1/4, 1/L\}$, we get

$$\int\int_{\Sigma\times\Sigma} |\Omega_k(y')\overline{\Omega_k(u')}| \left|P(y'-u')\cdot B_{2^{(k+1)j}2^l}\xi\right|^{-\varepsilon} d\sigma(y')d\sigma(u')$$
$$\leqslant \left(\int\int_{\Sigma\times\Sigma} \left|(y'-u')\cdot P^* B_{2^{(k+1)j}2^l}\xi\right|^{-2\varepsilon} d\sigma(y')d\sigma(u')\right)^{1/2} \|\Omega_k\|_2^2$$
$$\leqslant C\|\Omega_k\|_2^2 \left|B_{2^{(k+1)j}2^l}\xi\right|^{-\varepsilon},$$

where the last inequality follows from [7, p.533] (also see [10, the proof of Lemma 1]). Therefore

$$\left|\lambda_{k,j}(\xi,\eta)\right|^2 \leqslant C\|\Omega_k\|_2^2 \sum_{l=0}^{k} \left|B_{2^{(k+1)j}2^l}\xi\right|^{-\varepsilon} \quad (0<\varepsilon<\min\{1/4,1/L\}).$$

By (2.3) and (2.4), there exists a positive number $\varepsilon_0 > 0$ such that

$$\left|B_{2^{(k+1)j}2^l}\xi\right| \geqslant C\left[2^{(k+1)j}2^l s(\xi)\right]^{\varepsilon_0}, \quad l = 0, 1, \cdots k,$$

which leads to

$$\sum_{l=0}^{k} \left|B_{2^{(k+1)j}2^l}\xi\right|^{-\varepsilon} \leqslant \sum_{l=0}^{k} C\left[2^{(k+1)j}2^l s(\xi)\right]^{-\varepsilon\varepsilon_0} \leqslant C(k+1)\left[2^{(k+1)j}s(\xi)\right]^{-\varepsilon\varepsilon_0}.$$

Consequently,

$$\left|\lambda_{k,j}(\xi,\eta)\right| \leqslant C(k+1)^{1/2}\|\Omega_k\|_2 \left[2^{(k+1)j}s(\xi)\right]^{-\varepsilon\varepsilon_0/2}.$$

Recall $\|\Omega_0\|_2 \leqslant C = CG_0$, and for $k\in\Lambda$, $G_k \geqslant C2^k\sigma(E_k) \geqslant C2^{-3k}$, we have

$$\|\Omega_k\|_2 \leqslant C2^{k+1}\sigma(E_k)^{1/2} \leqslant C2^{2(k+1)}G_k.$$

Thus,

$$\left|\lambda_{k,j}(\xi,\eta)\right| \leqslant C(k+1)^{1/2}2^{2(k+1)}G_k \left[2^{(k+1)j}s(\xi)\right]^{-\varepsilon\varepsilon_0/2}.$$

This together with (2.12) and an application of interpolation theorem implies

$$\left|\lambda_{k,j}(\xi,\eta)\right| \leqslant C(k+1)^{1/2}G_k \left[2^{(k+1)j}s(\xi)\right]^{-\varepsilon\varepsilon_0/2(k+1)}$$

and completes the proof of Lemma 2.2. □

REMARK 2.1. Define

$$\widetilde{\lambda}_{k,j}(\xi,\eta) = \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \left|\int_\Sigma |\Omega_k(y')| e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right|^2 \frac{dr}{r}\right)^{1/2}.$$

Then by the same arguments as those used in the proof of Lemma 2.2, we can get

$$\left|\widetilde{\lambda}_{k,j}(\xi,\eta)\right| \leqslant CG_k(k+1)^{1/2}\min\left\{1, \left[2^{(k+1)j}s(\xi)\right]^{-\delta/(k+1)}\right\}$$

with δ as in (2.11).

LEMMA 2.3. *Let* $k \in \Lambda \cup \{0\}$, $h \in \mathscr{H}_\nu(\mathbb{R}_+)$ *for some* $\nu > 1$. *Then, for every* $(\xi,\eta) \in \mathbb{R}^{n+m}$, *there exists* $\delta > 0$ *such that*

$$|\widehat{\tau}_{k,j}(\xi,\eta)| \leqslant C(k+1)^{1/\nu'} G_k \min\left\{\left[2^{(k+1)j}s(\xi)\right]^{\delta/\nu'(k+1)}, \left[2^{(k+1)j}s(\xi)\right]^{-\delta/\nu'(k+1)}\right\}, \tag{2.17}$$

$$\left|\widehat{|\tau_{k,j}|}(\xi,\eta)\right| \leqslant C(k+1)^{1/\nu'} G_k \min\left\{1, \left[2^{(k+1)j}s(\xi)\right]^{-\delta/\nu'(k+1)}\right\}, \tag{2.18}$$

$$\left|\widehat{|\tau_{k,j}|}(\xi,\eta) - \widehat{|\tau_{k,j}|}(0,\eta)\right| \leqslant C(k+1)^{1/\nu'} G_k \left[2^{(k+1)j}s(\xi)\right]^{\delta/\nu'(k+1)}. \tag{2.19}$$

Proof. At first, we will prove (2.17). By the definition of $\widehat{\tau}_{k,j}$ and the Hölder inequality, we have

$$\begin{aligned}
\left|\widehat{\tau}_{k,j}(\xi,\eta)\right| &\leqslant \int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} |h(r)| \left|\int_\Sigma \Omega_k(y') e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right| \frac{dr}{r} \\
&\leqslant \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \left|\int_\Sigma \Omega_k(y') e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right|^{\nu'} \frac{dr}{r}\right)^{1/\nu'} \\
&\quad \times \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} |h(r)|^\nu \frac{dr}{r}\right)^{1/\nu} \\
&\leqslant C\left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \left|\int_\Sigma \Omega_k(y') e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right|^{\nu'} \frac{dr}{r}\right)^{1/\nu'}.
\end{aligned}$$

In what follows, we will consider cases: $2 \leqslant \nu' < \infty$ and $1 < \nu' < 2$, respectively. If $2 \leqslant \nu' < \infty$, then

$$\begin{aligned}
\left|\widehat{\tau}_{k,j}(\xi,\eta)\right| &\leqslant C\|\Omega_k\|_1^{1-2/\nu'} \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \left|\int_\Sigma \Omega_k(y') e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right|^2 \frac{dr}{r}\right)^{1/\nu'} \\
&\leqslant C G_k^{1-2/\nu'} \left|\lambda_{k,j}(\xi,\eta)\right|^{2/\nu'}.
\end{aligned} \tag{2.20}$$

If $1 < \nu' < 2$, then, by Hölder's inequality, we get

$$\begin{aligned}
\left|\widehat{\tau}_{k,j}(\xi,\eta)\right| &\leqslant \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \left|\int_\Sigma \Omega_k(y') e^{-2\pi i[\xi\cdot A_r y' + \eta\cdot\Gamma(r)]} d\sigma(y')\right|^2 \frac{dr}{r}\right)^{1/2} \\
&\quad \times \left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \frac{dr}{r}\right)^{1/\nu'-1/2} \\
&= (k+1)^{1/\nu'-1/2} \left|\lambda_{k,j}(\xi,\eta)\right|.
\end{aligned} \tag{2.21}$$

Thus, (2.17) follows from (2.20)–(2.21) and (2.11).

Similarly, by Remark 2.1, we can get (2.18).

It remains to prove (2.19). By Hölder's inequality, it is easy to see that

$$\begin{aligned}\left|\widehat{\tau}_{k,j}(\xi,\eta)-\widehat{\tau}_{k,j}(0,\eta)\right| &\leqslant \|h\|_{L^{\nu}(\mathbb{R}_+,r^{-1}dr)}\|\Omega_k\|_1\left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\frac{dr}{r}\right)^{1/\nu'} \\ &\leqslant C(k+1)^{1/\nu'}G_k.\end{aligned} \tag{2.22}$$

On the other hand, we have

$$\begin{aligned}&\left|\widehat{\tau}_{k,j}(\xi,\eta)-\widehat{\tau}_{k,j}(0,\eta)\right| \\ &\quad\leqslant\left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\left|\int_{\Sigma}|\Omega_k(y')|e^{-2\pi i\eta\cdot\Gamma(r)}\left[e^{-2\pi i\xi\cdot A_ry'}-1\right]d\sigma(y')\right|^{\nu'}\frac{dr}{r}\right)^{1/\nu'} \\ &\quad\leqslant\left(\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\left|\int_{\Sigma}|\Omega_k(y')|\left|e^{-2\pi i\xi\cdot A_ry'}-1\right|d\sigma(y')\right|^{\nu'}\frac{dr}{r}\right)^{1/\nu'} \\ &\quad\leqslant\left(\int_{1}^{2^{k+1}}\left|\int_{\Sigma}|\Omega_k(y')|\left|e^{-2\pi iA_ry'\cdot B_{2^{(k+1)j}}\xi}-1\right|d\sigma(y')\right|^{\nu'}\frac{dr}{r}\right)^{1/\nu'} \\ &\quad\leqslant C\left(\int_{1}^{2^{k+1}}\left|\int_{\Sigma}|\Omega_k(y')||A_ry'|\left|B_{2^{(k+1)j}}\xi\right|d\sigma(y')\right|^{\nu'}\frac{dr}{r}\right)^{1/\nu'}.\end{aligned}$$

Invoking (2.13) leads to

$$\begin{aligned}\left|\widehat{\tau}_{k,j}(\xi,\eta)-\widehat{\tau}_{k,j}(0,\eta)\right| &\leqslant C\left(\int_{1}^{2^{k+1}}\left|\int_{\Sigma}|\Omega_k(y')|r^{\alpha_1}\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}d\sigma(y')\right|^{\nu'}\frac{dr}{r}\right)^{1/\nu'} \\ &\leqslant C\|\Omega_k\|_1\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}\left(\int_{1}^{2^{k+1}}r^{\nu'\alpha_2}\frac{dr}{r}\right)^{1/\nu'} \\ &\leqslant CG_k(k+1)^{1/\nu'}2^{(k+1)\alpha_1}\left[2^{(k+1)j}s(\xi)\right]^{\alpha_2}.\end{aligned}$$

This together with (2.22) implies (2.19) and completes the proof of Lemma 2.3. □

LEMMA 2.4. *Let* $k\in\Lambda\cup\{0\}$, $h\in\mathscr{H}_{\nu}(\mathbb{R}_+)$ *for some* $\nu>1$. *If* M_Γ *given by* (1.3) *is bounded on* $L^q(\mathbb{R}^m)$ *for all* $q>1$, *then, for* $\nu'<p\leqslant\infty$,

$$\left\|\tau^*_{\Gamma,k}(f)\right\|_{L^p(\mathbb{R}^{n+m})}\leqslant C(k+1)^{1/\nu'}G_k\|f\|_{L^p(\mathbb{R}^{n+m})}.$$

Proof. Let

$$\int_{\mathbb{R}^{n+m}}fd\mu_{k,j}=\int_{D_{k,j}}\Omega_k(y')f(y,\Gamma(r(y))r(y)^{-\gamma}dy,$$

and $\mu^*_{\Gamma,k}(f)(x,z)=\sup_{j\in\mathbb{Z}}\left||\mu_{k,j}|*f(x,z)\right|$. Then by Hölder's inequality,

$$\begin{aligned}\tau^*_{\Gamma,k}(f)(x,z) &\leqslant\left(\int_{I_{k,j}}|h(r)|^{\nu}\frac{dr}{r}\right)^{1/\nu}G_k^{1/\nu}\left(\mu^*_{\Gamma,k}(|f|^{\nu'})(x,z)\right)^{1/\nu'} \\ &\leqslant CG_k^{1/\nu}\left(\mu^*_{\Gamma,k}(|f|^{\nu'})(x,z)\right)^{1/\nu'}.\end{aligned}$$

Also, by Lemma 2.3's (2.18) and (2.19) (invoking the results of the case $\nu' = 1$) and the same arguments as those used in the proof of Proposition 3's (1) in [10], it is easy to show that

$$\left\|\mu^*_{\Gamma,k}(f)\right\|_{L^p(\mathbb{R}^{n+m})} \leqslant C(k+1)G_k\|f\|_{L^p(\mathbb{R}^{n+m})}, \quad 1 < p \leqslant \infty. \tag{2.23}$$

Therefore, for $\nu' < p \leqslant \infty$, we get

$$\|\tau^*_{\Gamma,k}(f)\|_{L^p(\mathbb{R}^{n+m})} \leqslant CG_k^{1/\nu}\left\|\mu^*_{\Gamma,k}(|f|^{\nu'})\right\|^{1/\nu'}_{L^{p/\nu'}(\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/\nu'}G_k\|f\|_{L^p(\mathbb{R}^{n+m})},$$

and completes the proof of Lemma 2.4. □

LEMMA 2.5. *Let $k \in \Lambda \cup \{0\}$ and $h \in \mathscr{H}_\nu(\mathbb{R}_+)$ for some $\nu \geqslant 2$. Suppose that M_Γ is bounded on $L^q(\mathbb{R}^m)$ for all $q > 1$. Then, for $\nu' < p < \infty$, there exists a positive constant C_p which is independent of k such that*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left|\tau_{k,j} * g_j\right|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{n+m})} \leqslant C_p(k+1)^{1/\nu'}G_k\left\|\left(\sum_{j\in\mathbb{Z}}|g_j|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^{n+m})}$$

holds for arbitrary measurable functions $\{g_j\}$ on $\mathbb{R}^{n+m}$.

Proof. Following the proof of Lemma 2.6 in [2], let $\nu' < p < \infty$, by Hölder's inequality and the condition on h, we get

$$\left|\tau_{k,j} * g_k(x,z)\right|^{\nu'} \leqslant CG_k^{\nu'-1}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\int_\Sigma |\Omega_k(y')||g_j(x-A_ry', z-\Gamma(r))|^{\nu'}d\sigma(y')\frac{dr}{r}.$$

Let $d = p/\nu'$. For $\{g_j\} \in L^d(\mathbb{R}^{n+m}, l^2)$, by duality, there exists a nonnegative function $w \in L^{d'}(\mathbb{R}^{n+m})$ such that $\|w\|_{L^{d'}} \leqslant 1$ and

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\tau_{k,j} * g_k|^{\nu'}\right)^{1/\nu'}\right\|_p^{\nu'} = \int_{\mathbb{R}^{n+m}}\sum_{j\in\mathbb{Z}}|\tau_{k,j} * g_j(x,z)|^{\nu'}w(x,z)dxdz.$$

This together with a change of variables yields

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\tau_{k,j} * g_j|^{\nu'}\right)^{1/\nu'}\right\|_p^{\nu'} \leqslant CG_k^{\nu'-1}\int_{\mathbb{R}^{n+m}}\sum_{j\in\mathbb{Z}}|g_k(x,z)|^{\nu'}\mu^*_{\Gamma,k}(\widetilde{w})(-x,-z)dxdz,$$

where $\widetilde{w}(x,z) = w(-x,-z)$, $\mu^*_{\Gamma,k}$ is as in the proof of Lemma 2.4. By the $L^{d'}$-boundedness of $\mu^*_{\Gamma,k}$ and Hölder's inequality, we obtain

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\tau_{k,j} * g_j|^{\nu'}\right)^{1/\nu'}\right\|_p \leqslant C(k+1)^{1/\nu'}G_k\left\|\left(\sum_{j\in\mathbb{Z}}|g_j|^{\nu'}\right)^{1/\nu'}\right\|_p. \tag{2.24}$$

Moreover, by Lemma 2.4, we have

$$\left\|\sup_{j\in\mathbb{Z}}|\tau_{k,j}*g_j|\right\|_p \leqslant \left\|\tau^*_{\Gamma,k}\left(\sup_{j\in\mathbb{Z}}|g_j|\right)\right\|_p \leqslant C(k+1)^{1/\nu'}G_k\left\|\left(\sup_{j\in\mathbb{Z}}|g_j|\right)\right\|_p. \tag{2.25}$$

Note that $\nu'\in[1,2]$, interpolating between (2.24) and (2.25), we complete the proof of Lemma 2.5. $\square$

3. Proof of Theorem 1.2

Our arguments are similar to those in the proof of Theorem 1.6 of [2]. By (2.8), we have $\mathscr{M}^{(\nu)}_{\Gamma,\Omega}(f)(x,z)\leqslant\sum_{k\in\Lambda\cup\{0\}}\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}(f)(x,z)$, where $\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}$ is defined as in (1.4) only replaced Ω by Ω_k. Hence, by (2.9), to prove Theorem 1.2, it suffices to show that

$$\left\|\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}(f)\right\|_p \leqslant C(k+1)^{1/\nu'}G_k\|f\|_p \tag{3.1}$$

holds for $\nu'\leqslant p<\infty$ if $1<\nu\leqslant 2$ and for $p=\infty$ if $\nu=1$. We will consider the following three cases.

Case 1 ($\nu=2$). By duality, we have

$$\begin{aligned}\mathscr{M}^{(2)}_{\Gamma,\Omega_k}(f)(x,z) &= \sup_{\|h\|_{\mathscr{H}_2(\mathbb{R}_+)}\leqslant 1}\left|\int_0^\infty h(r)\int_\Sigma \Omega_k(y')f(x-A_ry',z-\Gamma(r))d\sigma(y')\frac{dr}{r}\right|\\ &\leqslant \left(\int_0^\infty\left|\int_\Sigma \Omega_k(y')f(x-A_ry',z-\Gamma(r))d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/2}\\ &= \left(\sum_{j\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\left|\int_\Sigma \Omega_k(y')f(x-A_ry',z-\Gamma(r))d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/2}.\end{aligned}$$

Let $\{\psi_{k,j}\}^\infty_\infty$ be a sequence in $C^\infty((0,\infty))$ such that

$$\operatorname{supp}(\psi_{k,j})\subset[2^{(k+1)(-j-1)},2^{(k+1)(-j+1)}],\qquad \sum_{j\in\mathbb{Z}}\psi_{k,j}(r)=1,$$

$$\left|(d/dr)^l\psi_{k,j}(r)\right|\leqslant C_l/r^l,\quad (l=1,2,\cdots),$$

where C_l is independent of k. Define the multiplier operators $S_{k,j}$ in $\mathbb{R}^{n+m}$ by

$$\widehat{S_{k,j}}(f)(\xi,\eta)=\psi_{k,j}(s(\xi))\widehat{f}(\xi,\eta)\qquad\text{for}\quad(\xi,\eta)\in\mathbb{R}^{n+m}. \tag{3.2}$$

Then for any $f\in\mathscr{S}(\mathbb{R}^{n+m})$ and $j\in\mathbb{Z}$, we have $f(x,z)=\sum_j S_{k,j+l}(f)(x,z)$. Thus, by

(3.2) and Minkowski's inequality, we get

$$\begin{aligned}\mathscr{M}^{(2)}_{\Gamma,\Omega_k}(f)(x,z) &\leqslant \left(\sum_{\text{[redacted]}\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\left|\sum_{l\in\mathbb{Z}}H_{k,j+l,r}(f)(x,z)\right|^2\frac{dr}{r}\right)^{1/2}\\ &\leqslant \sum_{l\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}|H_{k,j+l,r}(f)(x,z)|^2\frac{dr}{r}\right)^{1/2}\\ &:= \sum_{l\in\mathbb{Z}}T_{k,l}(f)(x,z),\end{aligned} \tag{3.3}$$

where

$$H_{k,l,r}(f)(x,z) = \int_{\Sigma}S_{k,l}(f)(x-A_ry',z-\Gamma(r))\Omega_k(y')d\sigma(y'),$$

$$T_{k,l}(f)(x,z) = \left(\sum_{j\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}|H_{k,j+l,r}(f)(x,z)|^2\frac{dr}{r}\right)^{1/2}.$$

Thus, to prove (3.1) for $\nu=2$, it suffices to show

$$\left\|T_{k,l}(f)\right\|_{L^p(\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/2}2^{-\theta|l|}G_k\|f\|_{L^p(\mathbb{R}^{n+m})} \tag{3.4}$$

for some positive number θ and for all $2\leqslant p<\infty$.

By Plancherel's theorem and Lemma 2.2, we have

$$\begin{aligned}\|T_{k,l}(f)\|_2^2 &= \int_{\mathbb{R}^{n+m}}\sum_{j\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}|H_{k,j+l,r}(f)(x,z)|^2\frac{dr}{r}dxdz\\ &= \sum_{j\in\mathbb{Z}}\int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}}\int_{\mathbb{R}^{n+m}}\left|\widehat{H_{k,j+l,r}}(f)(\xi,\eta)\right|^2 d\xi d\eta\frac{dr}{r}\\ &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{n+m}}\left|\widehat{S_{k,j+l}}(f)(\xi,\eta)\right|^2\left|\lambda_{k,j}(\xi,\eta)\right|^2 d\xi d\eta\\ &\leqslant \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^m}\int_{\widetilde{D}_{k,j+l}}|\widehat{\lambda}_{k,j}(\xi,\eta)|^2|\widehat{f}(\xi,\eta)|^2 d\xi d\eta\\ &\leqslant C(k+1)2^{-2\delta|l|}G_k^2\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^m}\int_{\widetilde{D}_{k,j+l}}|\widehat{f}(\xi,\eta)|^2 d\xi d\eta,\end{aligned}$$

where

$$\widetilde{D}_{k,j} = \{\xi\in\mathbb{R}^n:\ 2^{(k+1)(-j-1)}\leqslant s(\xi)\leqslant 2^{(k+1)(-j+1)}\}. \tag{3.5}$$

Noting that $\widetilde{D}_{k,j}$'s are finitely overlapping, we get

$$\|T_{k,l}(f)\|_2 \leqslant C(k+1)^{1/2}G_k2^{-\delta|l|}\|f\|_2. \tag{3.6}$$

On the other hand, by duality, for $p>2$, there exists a function $g\in L^{(p/2)'}(\mathbb{R}^{n+m})$

with $\|g\|_{(p/2)'} \leqslant 1$ such that

$$\begin{aligned}
\|T_{k,l}(f)\|_p^2 &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{n+m}} \int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} |H_{k,j+l,r}(f)(x,z)|^2 \frac{dr}{r} |g(x,z)| dxdz \\
&\leqslant \|\Omega_k\|_1 \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{n+m}} \int_{2^{(k+1)j}}^{2^{(k+1)(j+1)}} \int_\Sigma |\Omega_k(y')||g(x+A_r y', z+\Gamma(r))| \\
&\quad \times \left|S_{k,j+l}(f)(x,z)\right|^2 d\sigma(y') \tfrac{dr}{r} dxdz \\
&\leqslant CG_k \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{n+m}} \left|S_{k,j+l}(f)(x,z)\right|^2 \mu^*_{\Gamma,k}(\widetilde{g})(-x,-z) dxdz \\
&\leqslant CG_k \left\| \sum_{j\in\mathbb{Z}} |S_{k,j+l}(f)|^2 \right\|_{(p/2)} \|\mu^*_{\Gamma,k}(\widetilde{g})\|_{(p/2)'},
\end{aligned}$$

where $\widetilde{g}(x,z) = g(-x,-z)$. Therefore, by (2.23) and the Littlewood-Paley theory, we have

$$\|T_{k,l}(f)\|_p \leqslant C(k+1)^{1/2} G_k \|f\|_p \qquad \text{for} \quad 2 \leqslant p < \infty. \tag{3.7}$$

Consequently, (3.4) follows from the interpolation between (3.5) and (3.7). This completes the proof of (3.1) for $\nu = 2$.

Case 2 ($\nu = 1$). For $f \in L^\infty(\mathbb{R}^{n+m})$ and $h \in \mathscr{H}_1(\mathbb{R}_+)$, we have

$$\begin{aligned}
\left|T_{\Gamma,\Omega_k,h}(f)(x,z)\right| &= \left| \int_0^\infty h(r) \int_\Sigma \Omega_k(y') f(x - A_r y', z - \Gamma(r)) d\sigma(y') \frac{dr}{r} \right| \\
&\leqslant CG_k \|h\|_{\mathscr{H}_1(\mathbb{R}_+)} \|f\|_{L^\infty(\mathbb{R}^{n+m})}
\end{aligned}$$

holds for every (x,z). Consequently, for every $(x,z) \in \mathbb{R}^{n+m}$, we get

$$\left|\mathscr{M}^{(1)}_{\Gamma,\Omega_k}(f)(x,z)\right| = \sup_{\|h\|_{\mathscr{H}_1} \leqslant 1} \left|T_{\Gamma,\Omega_k,h}(f)(x,z)\right| \leqslant CG_k \|f\|_{L^\infty(\mathbb{R}^{n+m})},$$

which implies

$$\left\|\mathscr{M}^{(1)}_{\Gamma,\Omega_k}(f)\right\|_{L^\infty(\mathbb{R}^{n+m})} \leqslant CG_k \|f\|_{L^\infty(\mathbb{R}^{n+m})}. \tag{3.8}$$

(3.1) is proved for $\nu = 1$.

Case 3 ($1 < \nu < 2$). By duality, it is easy to see that

$$\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}(f)(x,z) = \left\| \int_\Sigma \Omega(y') f(x - A_r y', z - \Gamma(r)) d\sigma(y') \right\|_{L^{\nu'}(\mathbb{R}_+, r^{-1}dr)}. \tag{3.9}$$

Write

$$S_{\Gamma,\Omega_k}(f)(x,z,r) = \int_\Sigma \Omega_k(y') f(x - A_r y', z - \Gamma(r)) d\sigma(y').$$

Then,

$$\left\|\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}(f)\right\|_{L^p(\mathbb{R}^{n+m})} = \left\|S_{\Gamma,\Omega_k}(f)\right\|_{L^p(L^{\nu'}(\mathbb{R}_+, r^{-1}dr), \mathbb{R}^{n+m})}.$$

Hence, it follows from (3.1) (for $\nu=2$) and (3.8) that

$$\big\|S_{\Gamma,\Omega_k}(f)\big\|_{L^p(L^2(\mathbb{R}_+,r^{-1}dr),\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/2}G_k\|f\|_{L^p(\mathbb{R}^{n+m})}, \qquad 2\leqslant p<\infty, \quad (3.10)$$

and

$$\big\|S_{\Gamma,\Omega_k}(f)\big\|_{L^\infty(L^\infty(\mathbb{R}_+,r^{-1}dr),\mathbb{R}^{n+m})} \leqslant CG_k\|f\|_{L^\infty(\mathbb{R}^{n+m})}. \quad (3.11)$$

The real interpolation theorem for the Lebesgue mixed norm spaces tells us that, for $1<\nu<2$,

$$\big\|S_{\Gamma,\Omega_k}(f)\big\|_{L^p(L^{\nu'}(\mathbb{R}_+,r^{-1}dr),\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/\nu'}G_k\|f\|_{L^p(\mathbb{R}^{n+m})}, \qquad \nu'\leqslant p<\infty,$$

that is,

$$\Big\|\mathscr{M}^{(\nu)}_{\Gamma,\Omega_k}(f)\Big\|_{L^p(\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/\nu'}G_k\|f\|_{L^p(\mathbb{R}^{n+m})},$$

$\nu'\leqslant p<\infty$. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

Notice that $\mathscr{H}_\infty(\mathbb{R}_+)=\Delta_\infty(\mathbb{R}_+)$, Theorem 1.1 directly follows from Theorem A for $\nu=\infty$. Therefore, we need only to prove Theorem 1.1 in the following two cases.

Case 1 ($1<\nu\leqslant 2$): Without loss of generality, we may assume that $\|h\|_{L^\nu(\mathbb{R}_+,r^{-1}dr)}=1$. Then,

$$\big|T_{\Gamma,\Omega,h}(f)(x,z)\big| \leqslant \mathscr{M}^{(\nu)}_{\Gamma,\Omega}(f)(x,z).$$

Therefore, by Theorem 1.2, we get

$$\big\|T_{\Gamma,\Omega,h}(f)\big\|_{L^p(\mathbb{R}^{n+m})} \leqslant \Big\|\mathscr{M}^{(\nu)}_{\Gamma,\Omega}(f)\Big\|_{L^p(\mathbb{R}^{n+m})} \leqslant C\|f\|_{L^p(\mathbb{R}^{n+m})} \qquad \text{for} \quad \nu'\leqslant p<\infty. \quad (4.1)$$

From this inequality and a standard duality argument, we also obtain

$$\big\|T_{\Gamma,\Omega,h}(f)\big\|_{L^p(\mathbb{R}^{n+m})} \leqslant C\|f\|_{L^p(\mathbb{R}^{n+m})} \qquad \text{for} \quad 1<p\leqslant\nu. \quad (4.2)$$

Thus, for $\nu=2$, we have

$$\big\|T_{\Gamma,\Omega,h}(f)\big\|_{L^p(\mathbb{R}^{n+m})} \leqslant C\|f\|_{L^p(\mathbb{R}^{n+m})} \qquad \text{for} \quad 1<p<\infty.$$

For $1<\nu<2$, interpolating between (4.1) and (4.2) gives the L^p-boundedness of $T_{\Gamma,\Omega,h}$ for the remaining range of p: $\nu<p<\nu'$.

Case 2 ($2<\nu<\infty$): By (2.9) and (2.10), it suffices to show that

$$\big\|T_{\Gamma,\Omega_k,h}(f)\big\|_{L^p(\mathbb{R}^{n+m})} \leqslant C(k+1)^{1/\nu'}G_k\|f\|_{L^p(\mathbb{R}^{n+m})} \qquad \text{for} \quad 1<p<\infty. \quad (4.3)$$

Let $\{\phi_{k,j}\}_{j\in\mathbb{Z}}$ be a sequence in $C^\infty((0,\infty))$ such that

$$\operatorname{supp}(\phi_{k,j})\subset[2^{(k+1)(-j-1)},2^{(k+1)(-j+1)}], \qquad \sum_{j\in\mathbb{Z}}\phi_{k,j}(t)^2=1,$$

$$\left|(d/dt)^{\beta}\phi_{k,j}(t)\right| \leqslant C/t^{\beta}, \quad (\beta = 1,2,\cdots).$$

Define $\widetilde{S}_{k,j}$ by $\widehat{\widetilde{S}_{k,j}(f)}(\xi,\eta) = \phi_{k,j}(s(\xi))\widehat{f}(\xi,\eta)$. Then we can write

$$T_{\Gamma,\Omega_k,h}(f)(x,z) = \sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\widetilde{S}_{k,j+l}\left(\tau_{k,j} * \widetilde{S}_{j+l}(f)\right)(x,z) := \sum_{l\in\mathbb{Z}} V_{k,l}(f)(x,z).$$

In what follows, we estimate $\|V_{k,l}(f)\|_{L^p(\mathbb{R}^{n+m})}$. Firstly, by Lemma 2.4 and Littlewood-Paley inequality, for $\nu' < p < \infty$,

$$\begin{aligned}\left\|V_{k,l}(f)\right\|_p &\leqslant C\left\|\left(\sum_{j\in\mathbb{Z}}\left|\tau_{k,j} * \widetilde{S}_{k,j+l}(f)\right|^2\right)^{1/2}\right\|_p \\ &\leqslant C(k+1)^{1/\nu'}G_k\left\|\left(\sum_{j\in\mathbb{Z}}|\widetilde{S}_{k,j+l}(f)|^2\right)^{1/2}\right\|_p \\ &\leqslant C(k+1)^{1/\nu'}G_k\|f\|_p.\end{aligned} \tag{4.4}$$

Secondly, we estimate $\|V_{k,l}(f)\|_2$:

$$\begin{aligned}\|V_{k,l}(f)\|_2^2 &\leqslant \sum_{j\in\mathbb{Z}}\left\|\widetilde{S}_{k,j+l}\left(\tau_{k,j} * \widetilde{S}_{k,j+l}(f)\right)\right\|_2^2 \\ &\leqslant \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^m}\int_{\widetilde{D}_{k,j+l}}\left|\widehat{\tau}_{k,j}(\xi,\eta)\right|^2|\widehat{f}(\xi,\eta)|^2 d\xi d\eta,\end{aligned}$$

where $\widetilde{D}_{k,j}$ are as in (3.4). This together with (2.17) implies

$$\|V_{k,l}(f)\|_2 \leqslant C(k+1)^{1/\nu'}G_k 2^{-\delta|l|}\|f\|_2. \tag{4.5}$$

Interpolating between (4.4) and (4.5), we get

$$\|V_{k,l}(f)\|_p \leqslant C(k+1)^{1/\nu'}G_k 2^{-\delta'|l|}\|f\|_p, \quad \nu' < p < \infty.$$

Hence, for $\nu' < p < \infty$,

$$\left\|T_{\Gamma,\Omega_k,h}(f)\right\|_p \leqslant \sum_{l\in\mathbb{Z}}\|V_{k,l}(f)\|_p \leqslant C(k+1)^{1/\nu'}G_k\|f\|_p\sum_{l\in\mathbb{Z}}2^{-\delta'|l|} \leqslant C(k+1)^{1/\nu'}G_k\|f\|_p.$$

Noting that $1 < \nu' < 2$, by duality, we have

$$\left\|T_{\Gamma,\Omega_k,h}(f)\right\|_p \leqslant C(k+1)^{1/\nu'}G_k\|f\|_p, \qquad 1 < p \leqslant 2,$$

which completes the proof of (4.3). Theorem 1.1 is proved.

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(Received October 21, 2011)

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