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NORM INEQUALITIES FOR SOME ONE–SIDED OPERATORS

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Abstract. We show that the one-sided maximal operators associated with Borel measures are of strong type (p,p), $1<p<\infty$, with constant p^*, and the related one-sided geometric maximal operators are of strong type (p,p), $0<p<\infty$, with constant $e^{1/p}$. We also investigate norm inequalities for integral operators with three measures on the cone of nonnegative nonincreasing functions. Our results show that if we restrict the measures in the inequalities to some particular classes, then a simple characterization for these inequalities to hold can be obtained.

1. Introduction

Let μ be a non-negative Borel measure on $\mathbb{R}$ which is finite on bounded sets. Let ϕ be a nonnegative Borel function defined on $\mathscr{D}=\{(x,t):-\infty<t\leqslant x<\infty\}$ such that $\phi(x,\cdot)$ is locally $\mu-$integrable for each $x\in\mathbb{R}$. The one-sided maximal operator $M^-_{\phi,\mu}$ is defined for locally μ-integrable f by

$$M^-_{\phi,\mu}f(x):=\sup_{s<x}\frac{1}{\int_{(s,x]}\phi(x,t)d\mu(t)}\int_{(s,x]}\phi(x,t)|f(t)|d\mu(t).$$

In the case $\phi\equiv 1$, we write M^-_μ instead of $M^-_{\phi,\mu}$. When μ is the usual Lebesgue measure, it is well-known that M^-_μ is of weak type $(1,1)$ and strong type (p,p), where $1<p\leqslant\infty$. See for example [13], [15]. Weighted inequalities for M^-_μ was established by Sawyer [30] in the case that μ is the Lebesgue measure, and by Martín-Reyes et al. [23] in the case that μ is absolutely continuous with respect to Lebesgue measure with positive derivative. When μ is a positive Borel measure which is finite on bounded sets, Bernal [5] showed that M^-_μ is of weak type $(1,1)$ with constant 1, and Andersen [1] characterized weights for which M^-_μ is of weak type (p,p), $1\leqslant p<\infty$, and of strong type (p,p), $1<p<\infty$. In this paper we use the method in the proof of [1, Theorem 1] to obtain

$$\|M^-_{\phi,\mu}f\|_{p,\mu}\leqslant p^*\|f\|_{p,\mu} \tag{1.1}$$

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for $1<p<\infty$ under a nondecreasing condition on ϕ. Here $\|f\|_{p,\mu}=(\int_{\mathbb{R}}|f|^p d\mu)^{1/p}$. We also consider the one-sided geometric maximal operator and its related limiting operator

$$G^-_{\phi,\mu}f(x):=\sup_{s<x}\exp\left(\frac{1}{\int_{(s,x]}\phi(x,t)d\mu(t)}\int_{(s,x]}\phi(x,t)\log|f(t)|d\mu(t)\right),$$
$$G^{-*}_{\phi,\mu}f(x):=\lim_{\varepsilon\searrow 0}\sup_{s<x}\left(\frac{1}{\int_{(s,x]}\phi(x,t)d\mu(t)}\int_{(s,x]}\phi(x,t)|f(t)|^{\varepsilon}d\mu(t)\right)^{1/\varepsilon}.$$

Weighted inequalities for geometric maximal operator have been investigated in these papers [10], [11], [32], [36]. In [29] weighted inequalities for $G^-_{\phi,\mu}$ with $\phi\equiv 1$ was established when μ is the Lebesgue measure. Here we apply (1.1) to obtain

$$\|Tf\|_{p,\mu}\leqslant e^{1/p}\|f\|_{p,\mu},\quad T=G^-_{\phi,\mu},\,G^{-*}_{\phi,\mu}. \tag{1.2}$$

for $0<p<\infty$.

We also consider the integral operator $T_{\phi,\mu}$ which is defined as

$$T_{\phi,\mu}f(x)=\frac{1}{\Phi(x,x)}\int_{(0,x]}\phi(x,t)f(t)d\mu(t),\quad x>0,$$

where $\Phi(x,x)=\int_{(0,x]}\phi(x,t)d\mu(t)$, and the inequality of the form

$$\|T_{\phi,\mu}f\|_{q,\eta}\leqslant C\|f\|_{p,\nu} \tag{1.3}$$

for all $f\geqslant 0$ or $0\leqslant f\downarrow$, where $0<p,q<\infty$, η and ν are Borel measures on $(0,\infty)$. Here we restrict the domain of the kernel ϕ to the set $\mathscr{D}^+=\{(x,t):0<t\leqslant x<\infty\}$. We use $0\leqslant f\downarrow$ as a symbol for a nonnegative and nonincreasing function f on $(0,\infty)$, and $\|f\|_{p,\nu}=(\int_{(0,\infty)}|f|^p d\nu)^{1/p}$. If $d\mu=dx$, then we write T_ϕ instead of $T_{\phi,\mu}$, and if $d\nu=vdx$, where v is a weight, then we write $\|f\|_{p,v}$ instead of $\|f\|_{p,\nu}$. Inequalities of the form (1.3) have been widely studied in many literatures. Consider the case $d\mu=dx$, $d\eta=udx$, and $d\nu=vdx$ of (1.3), where u and v are weights. In [31] Sawyer showed that the Hardy-Littlewood maximal operator M is bounded from classical Lorentz spaces $\Lambda_p(v)$ to $\Lambda_q(u)$, $1<p,q<\infty$, if and only if (1.3) with $\phi\equiv 1$ holds for all $0\leqslant f\downarrow$, and a characterization was also given. In [34] Stepanov gave an alternative proof of Sawyer's result and established similar results for the cases $0<q<1<p<\infty$, $0<p\leqslant q<\infty$, and $0<p<1$. See also [4], [16], [18], [27], and references given there. Inequality (1.3) for general Borel measures were discussed in [17], [28], and [33]. Several known results showed that, in many cases, the following condition

$$\|f\|_{q,\eta}\leqslant A\|f\|_{p,\mu}\quad \text{for all } 0\leqslant f\downarrow \tag{1.4}$$

is necessary for (1.3) to hold for all $0\leqslant f\downarrow$. In particular, Persson et al. [28] considered (1.3) for $1\leqslant p<\infty$, $0<q<\infty$, and $\nu=\mu$. The authors [28, theorem 2.3] showed that if $\phi(x,\cdot)$ is nonincreasing on $(0,x]$ for each $x>0$, then (1.3) holds for all

$f \geqslant 0$ if and only if this inequality holds for all $0 \leqslant f \downarrow$. Moreover, by [28, Theorem 2.14] we see that if ϕ belongs to the Oinarov's class and $0 \leqslant T_{\phi,\mu} f \downarrow$ for all $0 \leqslant f \downarrow$, and if $\mathbb{A}_0 < \infty$, where $\mathbb{A}_0$ is given in [28, Proposition 2.10], then in the case $1 < p < \infty$, $0 < q < \infty$, and $\nu = \mu$, inequality (1.3) holds for all $f \geqslant 0$ if and only if (1.4) holds. A similar result for $\phi \equiv 1$ can be found in [33, Theorem 4.1]. In general it is not easy to establish inequalities of the form (1.3) for $0 \leqslant f \downarrow$ with three measures. See [17] for the case $\phi \equiv 1$. Our results show that if ϕ can be written in the form ψh, where $h(x,\cdot)$ is nondecreasing and left continuous on $(0,x]$ for each $x > 0$ and $0 \leqslant T_{\psi,\mu} f \downarrow$ for all $0 \leqslant f \downarrow$, and if $\|T_{\psi,\mu} f\|_{p,\nu} \leqslant C_1 \|f\|_{p,\nu}$ for all $0 \leqslant f \downarrow$ or $\|T_{\psi,\mu} f\|_{q,\eta} \leqslant C_1 \|f\|_{q,\eta}$ for all $0 \leqslant f \downarrow$, then (1.3) holds for all $0 \leqslant f \downarrow$ if and only if (1.4) holds. Many well known results in [2], [3], [7], [19], and [26] can be applied with our theorems to establish (1.3) for $0 \leqslant f \downarrow$. Therefore if we restrict the measures in (1.3) to some particular classes, then the problem of proving (1.3) for $0 \leqslant f \downarrow$ is more simple.

It is of independent interest to investigate inequalities of the form

$$\|T_1 f\|_{q,u} \leqslant C \|T_2 f\|_{p,v} \quad \text{for all } 0 \leqslant f \downarrow \tag{1.5}$$

and the condition

$$\mathbb{B} = \sup_{r>0} \frac{\|T_1 \chi_{(0,r]}\|_{q,u}}{\|T_2 \chi_{(0,r]}\|_{p,v}} < \infty. \tag{1.6}$$

Here T_1 and T_2 are two operators and u, v are weights. The constant $\mathbb{B}$ given in (1.6) plays an important role in many results. In the cases $p = q$, $u = v$, $T_1 = T_\phi$, and $T_2 = I$, where I is the identity operator, inequality (1.5) is reduced to

$$\|T_\phi f\|_{p,v} \leqslant C \|f\|_{p,v} \quad \text{for all } 0 \leqslant f \downarrow . \tag{1.7}$$

In [3] Ariño and Muckenhoupt showed that the Hardy-Littlewood maximal operator M is bounded on classical Lorentz space $\Lambda_p(v)$, $1 \leqslant p < \infty$, if and only if (1.7) holds for $1 \leqslant p < \infty$ and $T_\phi = H$, where H is the Hardy averaging operator defined by $Hf(x) = x^{-1} \int_0^x f(t)dt$, $x > 0$, and the necessary and sufficient condition is that $v \in B_p$, that is, there exists a constant B such that

$$\int_r^\infty t^{-p} v(t)dt \leqslant B r^{-p} \int_0^r v(t)dt, \quad r > 0. \tag{1.8}$$

It is easy to see that $v \in B_p$ is equivalent to (1.6) with $p = q$, $u = v$, $T_1 = H$, and $T_2 = I$. Similar results for more general integral operators T_ϕ can be found in [2], [7], [19], and [26]. We see that, in these results, inequality (1.7) holds if and only if it holds on characteristic functions of the form $\chi_{(0,r]}$, $r > 0$. Moreover, the following results can be found in [7], [8], [14], [18], [19], [21], [22], [24], [25], [28], [33], [34], [35], and references given there.

THEOREM P. *Inequalities* (1.5) *and* (1.6) *are equivalent in the following cases.*

(i) $0 < p \leqslant 1 \leqslant q < \infty$, $T_1 = T_\phi$, *and* $T_2 = T_\psi$.

(ii) $0 < p \leqslant q < \infty$, $1 \leqslant q < \infty$, $T_1 = I$, *and* $T_2 = T_\psi$.

(iii) $0 < p \leqslant 1$, $p \leqslant q < \infty$, $T_1 = T_\phi$, *and* $T_2 = I$.

(iv) $0 < p \leqslant q < \infty$, $T_1 = I$, *and* $T_2 = I$.

(v) $0 < p \leqslant q < \infty$, $T_1 = H$, *and* $T_2 = H$.

Moreover, in these cases we have

$$\sup_{0 \leqslant f\downarrow} \frac{\|T_1 f\|_{q,u}}{\|T_2 f\|_{p,v}} = \sup_{r>0} \frac{\|T_1 \chi_{(0,r)}\|_{q,u}}{\|T_2 \chi_{(0,r)}\|_{p,v}}.$$

The result of Theorem P(iii) is not true in the case $1 < p \leqslant q < \infty$ and Theorem P(iv) does not hold if $q < p$. Examples can be found in [6, Remark 1.2.13]. In this paper we extend Theorem P(v) from H to more general integral operators. As an application, we also establish (1.5) for $T_1 = T_\phi$ and $T_2 = T_\psi$.

Throughout this paper we suppose that all functions are Borel measurable. A weight on $(0,\infty)$ is a nonnegative locally integrable function defined on $(0,\infty)$. For $1 \leqslant z \leqslant \infty$, we define z^* by $1/z + 1/z^* = 1$. We also take $\exp(-\infty) = 0$, $\log 0 = -\infty$, $0^0 = \infty^0 = 1$, and $\infty/\infty = 0/0 = 0 \cdot \infty = 0$.

2. Inequalities for maximal operators

We first consider M_μ^-. The following Lemma 2.1 can be obtained by the method given in the proof of [1, Theorem 1].

LEMMA 2.1. *Let* $\lambda \geqslant 0$. *If* f *is locally* μ*-integrable on* $\mathbb{R}$, *then*

$$\int_{\{x|M_\mu^- f(x) > \lambda\}} d\mu \leqslant \frac{1}{\lambda} \int_{\{x|M_\mu^- f(x) > \lambda\}} |f| d\mu. \tag{2.1}$$

Proof of Lemma 2.1. We first show that if $x_0 \in \{x|M_\mu^- f(x) > \lambda\}$, then there exists $x_1 > x_0$ such that $[x_0, x_1) \subset \{x|M_\mu^- f(x) > \lambda\}$. For $x \in \mathbb{R}$, let $B(x)$ be the set

$$B(x) = \left\{ s < x \,\Big|\, \int_{(s,x]} d\mu > 0 \text{ and } \int_{(s,x]} |f| d\mu > \lambda \int_{(s,x]} d\mu \right\}.$$

Consider the case $\lambda > 0$. Since $M_\mu^- f(x_0) > \lambda$, there exists $c \in B(x_0)$. Choose $\varepsilon > 0$ such that $\int_{(c,x_0]} |f| d\mu > \lambda \left(\int_{(c,x_0]} d\mu + \varepsilon \right)$. Since μ is finite on bounded sets, we have $\int_{(c,z)} d\mu \to \int_{(c,x_0]} d\mu$ as $z \to x_0^+$. Hence there exists $x_1 > x_0$ such that $\int_{(c,x_0]} d\mu \leqslant \int_{(c,x_1)} d\mu < \int_{(c,x_0]} d\mu + \varepsilon$. Then for any $x \in (x_0, x_1)$,

$$\int_{(c,x]} |f| d\mu \geqslant \int_{(c,x_0]} |f| d\mu > \lambda \left(\int_{(c,x_0]} d\mu + \varepsilon \right) > \lambda \int_{(c,x_1)} d\mu \geqslant \lambda \int_{(c,x]} d\mu.$$

This implies $M_\mu^- f(x) > \lambda$. The case $\lambda = 0$ is trivial. Thus we have $[x_0, x_1) \subset \{x|M_\mu^- f(x) > \lambda\}$. The set $\{x|M_\mu^- f(x) > \lambda\}$ then can be written as a countable union of disjoint intervals $\{I_i\}$, where each I_i is of the form $[a,b)$ for $-\infty < a < b \leqslant \infty$ or (a,b) for

$-\infty \leqslant a < b \leqslant \infty$, and moreover, for each I_i there is no larger interval J of the form $[a,b)$ or (a,b) such that $I_i \subset J \subseteq \{x|M_\mu^- f(x) > \lambda\}$. We claim that if $I \in \{I_i\}$, then

$$\int_I |f|d\mu \geqslant \lambda \int_I d\mu. \tag{2.2}$$

First suppose that $I = [a,b)$ for $-\infty < a < b \leqslant \infty$. There exists an increasing sequence $\{a_j\}$ such that $a_j \to a^-$ as $j \to \infty$ and $a_j \notin \{x|M_\mu^- f(x) > \lambda\}$ for each j. Since $M_\mu^- f(a_j) \leqslant \lambda$, $\int_{(c,a_j]} |f|d\mu \leqslant \lambda \int_{(c,a_j]} d\mu$ for all $c < a_j < a$. Let $a_j \to a^-$, we have

$$\int_{(c,a)} |f|d\mu \leqslant \lambda \int_{(c,a)} d\mu \quad \text{ for all } c < a.$$

On the other hand, for $x \in [a,b)$, $M_\mu^- f(x) > \lambda$, then $B(x) \neq \emptyset$. Let $c^* = \inf B(x)$. We claim that $c^* < a$. This is trivial for the case $x = a$. For $a < x < b$, if $c^* \geqslant a$ then $c^* \in [a,b)$, $M_\mu^- f(c^*) > \lambda$ and so there exists $c_1 \in B(c^*)$. This implies

$$\int_{(c_1,x]} |f|d\mu = \int_{(c_1,c^*]} |f|d\mu + \int_{(c^*,x]} |f|d\mu > \lambda \int_{(c_1,c^*]} d\mu + \lambda \int_{(c^*,x]} d\mu = \lambda \int_{(c_1,x]} d\mu,$$

a contradiction. Therefore we may choose $c < a$ such that $c \in B(x)$. Hence

$$\int_{[a,x]} |f|d\mu = \int_{(c,x]} |f|d\mu - \int_{(c,a)} |f|d\mu > \lambda \int_{(c,x]} d\mu - \lambda \int_{(c,a)} d\mu = \lambda \int_{[a,x]} d\mu.$$

Let $x \to b^-$, we have $\int_{[a,b)} |f|d\mu \geqslant \lambda \int_{[a,b)} d\mu$.

Now we consider the case $I = (a,b)$ for $-\infty \leqslant a < b \leqslant \infty$. First assume that $a > -\infty$. Since $M_\mu^- f(a) \leqslant \lambda$, we have

$$\int_{(c,a]} |f|d\mu \leqslant \lambda \int_{(c,a]} d\mu \quad \text{ for all } c < a.$$

For $x \in (a,b)$, $M_\mu^- f(x) > \lambda$ and $B(x) \neq \emptyset$. It is easy to see that $c^* = \inf B(x) \leqslant a$. If $c^* = a$, then $\int_{(a,x]} |f|d\mu \geqslant \lambda \int_{(a,x]} d\mu$. If $c^* < a$, then we may choose $c \in (c^*,a)$ so that $\int_{(c,x]} |f|d\mu > \lambda \int_{(c,x]} d\mu$. Hence

$$\int_{(a,x]} |f|d\mu = \int_{(c,x]} |f|d\mu - \int_{(c,a]} |f|d\mu > \lambda \int_{(a,x]} d\mu.$$

In both cases, let $x \to b^-$, we have $\int_{(a,b)} |f|d\mu \geqslant \lambda \int_{(a,b)} d\mu$. In the case $a = -\infty$, for $x \in (-\infty,b)$, $M_\mu^- f(x) > \lambda$ and $B(x) \neq \emptyset$. It is easy to verify that $\inf B(x) = -\infty$ and hence $\int_{(-\infty,x]} |f|d\mu \geqslant \lambda \int_{(-\infty,x]} d\mu$. Let $x \to b^-$, we have $\int_{(-\infty,b)} |f|d\mu \geqslant \lambda \int_{(-\infty,b)} d\mu$.

We have proved that (2.2) holds for each interval $I \in \{I_i\}$. Therefore

$$\int_{\{x|M_\mu^- f(x)>\lambda\}} |f|d\mu = \sum_{i=1}^{\infty} \int_{I_i} |f|d\mu \geqslant \lambda \sum_{i=1}^{\infty} \int_{I_i} d\mu = \lambda \int_{\{x|M_\mu^- f(x)>\lambda\}} d\mu$$

and we have (2.1). $\square$

The proof of Theorem 2.2 is standard and is similar to that given in [15, Theorem 21.76].

THEOREM 2.2. *Let* $1 < p < \infty$. *Then*

$$\|M_\mu^- f\|_{p,\mu} \leqslant p^* \|f\|_{p,\mu}. \tag{2.3}$$

Proof of Theorem 2.2. We may assume that f is nonnegative and $\int_{\mathbb{R}} f^p d\mu < \infty$. We first show that $\int_{\mathbb{R}} (M_\mu^- f)^p d\mu < \infty$. Write $f = f_a + f^a$ for $a \geqslant 0$, where

$$f_a(x) = \begin{cases} f(x), & \text{if } 0 \leqslant f(x) \leqslant a, \\ a, & \text{if } f(x) > a, \end{cases} \qquad \text{and} \qquad f^a(x) = f(x) - f_a(x).$$

For any $\lambda > 0$, choose $a = \lambda/2$, then

$$M_\mu^- f(x) \leqslant M_\mu^-(f^{\lambda/2})(x) + M_\mu^-(f_{\lambda/2})(x) \leqslant M_\mu^-(f^{\lambda/2})(x) + \frac{\lambda}{2}.$$

Thus by (2.1) we have

$$\int_{\{x | M_\mu^- f(x) > \lambda\}} d\mu \leqslant \int_{\{x | M_\mu^-(f^{\lambda/2})(x) > \lambda/2\}} d\mu$$
$$\leqslant \frac{2}{\lambda} \int_{\{x | M_\mu^-(f^{\lambda/2})(x) > \lambda/2\}} f^{\lambda/2}(x)\, d\mu \leqslant \frac{2}{\lambda} \int_{\{x | f(x) > \lambda/2\}} f(x) - \frac{\lambda}{2}\, d\mu.$$

This implies

$$\int_{\mathbb{R}} M_\mu^- f(x)^p d\mu = p \int_0^\infty \lambda^{p-1} \left(\int_{\{x | M_\mu^- f(x) > \lambda\}} d\mu \right) d\lambda$$
$$\leqslant p \int_0^\infty 2\lambda^{p-2} \left(\int_{\{x | f(x) > \lambda/2\}} f(x) - \frac{\lambda}{2}\, d\mu \right) d\lambda$$
$$\leqslant 2p \int_{\mathbb{R}} \left(\int_0^{2f(x)} \lambda^{p-2} d\lambda \right) f(x)\, d\mu = 2^p p^* \int_{\mathbb{R}} f(x)^p d\mu < \infty.$$

Therefore by Lemma 2.1,

$$\|M_\mu^- f\|_{p,\mu}^p \leqslant p \int_0^\infty \lambda^{p-2} \left(\int_{\{x | M_\mu^- f(x) > \lambda\}} f(x)\, d\mu \right) d\lambda$$
$$= p \int_{\mathbb{R}} \left(\int_0^{M_\mu^- f(x)} \lambda^{p-2} d\lambda \right) f(x)\, d\mu = p^* \int_{\mathbb{R}} M_\mu^- f(x)^{p-1} f(x)\, d\mu$$
$$\leqslant p^* \|M_\mu^- f\|_{p,\mu}^{p/p^*} \|f\|_{p,\mu}$$

and this leads us to (2.3). $\square$

Let ϕ and ψ be nonnegative Borel functions defined on $\mathscr{D}$ such that $\phi(x,\cdot)$ and $\psi(x,\cdot)$ are locally $\mu-$integrable for each $x \in \mathbb{R}$. The following lemma gives a pointwise estimate of $M_{\phi,\mu}^- f(x)$ in terms of $M_{\psi,\mu}^- f(x)$.

LEMMA 2.3. *If $\phi = \psi h$, and $h(x,\cdot)$ is nondecreasing and left continuous on $(-\infty,x]$ for each $x \in \mathbb{R}$, then*

$$M^-_{\phi,\mu} f(x) \leqslant M^-_{\psi,\mu} f(x). \tag{2.4}$$

Proof of Lemma 2.3. Let $\Lambda_{h(x,\cdot)}$ be the Lebesgue-Stieltjes measure on $(-\infty,x]$ generated by $h(x,\cdot)$ defined by $\Lambda_{h(x,\cdot)}([a,b)) = h(x,b) - h(x,a)$ for $a < b \leqslant x$. Let $a < x$. Then $h(x,t) = h(x,a) + \int_{[a,t)} d\Lambda_{h(x,\cdot)}$ for all $a < t \leqslant x$. By Fubini's Theorem we see that

$$\begin{aligned}
&\int_{(a,x]} \phi(x,t)|f(t)|d\mu(t)\\
=&h(x,a)\int_{(a,x]} \psi(x,t)|f(t)|d\mu(t) + \int_{[a,x)} \left(\int_{(s,x]} \psi(x,t)|f(t)|d\mu(t)\right) d\Lambda_{h(x,\cdot)}\\
\leqslant&\left\{h(x,a)\int_{(a,x]} \psi(x,t)d\mu(t) + \int_{[a,x)} \left(\int_{(s,x]} \psi(x,t)d\mu(t)\right) d\Lambda_{h(x,\cdot)}\right\} M^-_{\psi,\mu} f(x)\\
=&\left(\int_{(a,x]} \phi(x,t)d\mu(t)\right) M^-_{\psi,\mu} f(x).
\end{aligned}$$

Therefore we have

$$\frac{1}{\int_{(a,x]} \phi(x,t)d\mu(t)} \int_{(a,x]} \phi(x,t)|f(t)|d\mu(t) \leqslant M^-_{\psi,\mu} f(x)$$

and this implies (2.4). □

The following Corollary 2.4 can be obtained by Lemma 2.3 with the case $\psi \equiv 1$ and Theorem 2.2.

COROLLARY 2.4. *Let $1 < p < \infty$. Suppose that $\phi(x,\cdot)$ is nondecreasing and left continuous on $(-\infty,x]$ for each $x \in \mathbb{R}$. Then*

$$\|M^-_{\phi,\mu} f\|_{p,\mu} \leqslant p^* \|f\|_{p,\mu}. \tag{2.5}$$

By a limiting process, we have Corollary 2.5.

COROLLARY 2.5. *Let $0 < p < \infty$ and let ϕ be given as in Corollary 2.4. Then*

$$\|Tf\|_{p,\mu} \leqslant e^{1/p} \|f\|_{p,\mu}, \quad T = G^-_{\phi,\mu}, G^{-*}_{\phi,\mu}. \tag{2.6}$$

Proof of Corollary 2.5. Since for any $\varepsilon > 0$,

$$G^-_{\phi,\mu} f(x) = \{G^-_{\phi,\mu}(|f|^\varepsilon)(x)\}^{1/\varepsilon} \leqslant \{M^-_{\phi,\mu}(|f|^\varepsilon)(x)\}^{1/\varepsilon},$$

we have

$$\|G^-_{\phi,\mu} f\|_{p,\mu} \leqslant \|M^-_{\phi,\mu}(|f|^\varepsilon)\|^{1/\varepsilon}_{p/\varepsilon,\mu} \leqslant \{(p/\varepsilon)^*\}^{1/\varepsilon} \|f\|_{p,\mu}$$

for any $0<\varepsilon<p$. By letting $\varepsilon \searrow 0$ we obtain

$$\|G^{-}_{\phi,\mu}f\|_{p,\mu} \leqslant e^{1/p}\|f\|_{p,\mu}.$$

On the other hand,

$$G^{-*}_{\phi,\mu}f(x) = \lim_{\varepsilon\searrow 0}\{M^{-}_{\phi,\mu}(|f|^{\varepsilon})(x)\}^{1/\varepsilon},$$

and therefore

$$\|G^{-*}_{\phi,\mu}f\|_{p,\mu} \leqslant \liminf_{\varepsilon\searrow 0}\|M^{-}_{\phi,\mu}(|f|^{\varepsilon})\|^{1/\varepsilon}_{p/\varepsilon,\mu} \leqslant \liminf_{\varepsilon\searrow 0}\{(p/\varepsilon)^{*}\}^{1/\varepsilon}\|f\|_{p,\mu} = e^{1/p}\|f\|_{p,\mu}. \qquad \square$$

3. Inequalities for integral operators

In this section we consider norm inequalities for integral operators $T_{\phi,\mu}$ and $T_{\psi,\mu}$. We restrict the domain of kernels ϕ and ψ to the set $\mathscr{D}^{+}$. Since $T_{\phi,\mu}f(x) \leqslant M^{-}_{\phi,\mu}f(x)$ for $f \geqslant 0$ and Corollary 2.4 still holds if we set $\|f\|_{p,\mu} = (\int_{(0,\infty)}|f|^{p}d\mu)^{1/p}$, we see that if $\phi(x,\cdot)$ is nondecreasing and left continuous on $(0,x]$ for each $x>0$, then for $1<p<\infty$,

$$\|T_{\phi,\mu}f\|_{p,\mu} \leqslant p^{*}\|f\|_{p,\mu}. \tag{3.1}$$

In [28], Persson et al. considered the following inequality with two measures

$$\|T_{\phi,\mu}f\|_{q,\eta} \leqslant C\|f\|_{p,\mu}. \tag{3.2}$$

They showed that if $\phi(x,\cdot)$ is nonincreasing on $(0,x]$, then for $1 \leqslant p < \infty$ and $0<q<\infty$, inequality (3.2) holds for all $f \geqslant 0$ if and only if it holds for all $0 \leqslant f\downarrow$. Moreover, by [28, Theorem 2.14] we see that if ϕ satisfies some suitable conditions, then (3.2) holds for all $f \geqslant 0$ if and only if

$$\|f\|_{q,\eta} \leqslant A\|f\|_{p,\mu} \quad \text{for all } 0 \leqslant f\downarrow. \tag{3.3}$$

The idea of the proof of [28, Theorem 2.14] can be applied to prove inequalities of the form (1.3) with three measures. By the proof of Lemma 2.3 we have a pointwise estimate of $T_{\phi,\mu}f(x)$ for $0 \leqslant f\downarrow$ in terms of $T_{\psi,\mu}f(x)$.

LEMMA 3.1. *Let $\phi=\psi h$, and $h(x,\cdot)$ is nondecreasing and left continuous on $(0,x]$ for each $x>0$. Then*

$$T_{\phi,\mu}f(x) \leqslant T_{\psi,\mu}f(x) \quad \textit{for all } 0 \leqslant f\downarrow.$$

The following Theorem 3.2 shows that if we can establish one of the following inequalities

$$\|T_{\psi,\mu}f\|_{p,\nu} \leqslant C_1\|f\|_{p,\nu} \quad \text{for all } 0 \leqslant f\downarrow; \tag{3.4}$$

$$\|T_{\psi,\mu}f\|_{q,\eta} \leqslant C_1\|f\|_{q,\eta} \quad \text{for all } 0 \leqslant f\downarrow, \tag{3.5}$$

then we have a simple condition for (1.3) to hold for all $0 \leqslant f\downarrow$.

THEOREM 3.2. *Let* $0 < p,q < \infty$ *and let* ϕ *and* ψ *be given as in Lemma 3.1. Suppose that* $0 \leqslant T_{\psi,\mu} f \downarrow$ *for all* $0 \leqslant f \downarrow$. *If* (3.4) *or* (3.5) *holds, then*

$$\|T_{\phi,\mu} f\|_{q,\eta} \leqslant C\|f\|_{p,\nu} \quad \text{for all } 0 \leqslant f \downarrow \tag{3.6}$$

if and only if

$$\|f\|_{q,\eta} \leqslant A\|f\|_{p,\nu} \quad \text{for all } 0 \leqslant f \downarrow . \tag{3.7}$$

Moreover, we have $A \leqslant C \leqslant AC_1$.

Proof of Theorem 3.2. Since $T_{\phi,\mu} f(x) \geqslant f(x)$ for all $0 \leqslant f \downarrow$, by Lemma 3.1 we see that

$$\|f\|_{q,\eta} \leqslant \|T_{\phi,\mu} f\|_{q,\eta} \leqslant \|T_{\psi,\mu} f\|_{q,\eta}, \quad 0 \leqslant f \downarrow .$$

If (3.6) holds then

$$\|f\|_{q,\eta} \leqslant C\|f\|_{p,\nu} \quad \text{for all } 0 \leqslant f \downarrow .$$

Conversely, suppose that (3.7) holds. If (3.4) is satisfied, then

$$\|T_{\phi,\mu} f\|_{q,\eta} \leqslant A\|T_{\psi,\mu} f\|_{p,\nu} \leqslant AC_1\|f\|_{p,\nu}.$$

If (3.5) is satisfied, then

$$\|T_{\phi,\mu} f\|_{q,\eta} \leqslant C_1\|f\|_{q,\eta} \leqslant AC_1\|f\|_{p,\nu}. \quad \square$$

In [28] Persson et al. showed that $0 \leqslant T_{\psi,\mu} f \downarrow$ for all $0 \leqslant f \downarrow$ if and only if $\Psi(x,r)/\Psi(x,x)$ is nonincreasing in x when $x \geqslant r$, where $\Psi(x,r) = \int_{(0,r]} \psi(x,t)\,d\mu(t)$ for $0 < r \leqslant x$. By [33] it is known that for $0 < p \leqslant q < \infty$,

$$\sup_{0 \leqslant f \downarrow} \frac{\|f\|_{q,\eta}}{\|f\|_{p,\nu}} = \sup_{r>0} \frac{\|\chi_{(0,r]}\|_{q,\eta}}{\|\chi_{(0,r]}\|_{p,\nu}}, \tag{3.8}$$

and for $0 < q < p < \infty$ and $1/r = 1/q - 1/p$,

$$\sup_{0 \leqslant f \downarrow} \frac{\|f\|_{q,\eta}}{\|f\|_{p,\nu}} \approx \left\{ \int_{(0,\infty)} \left(\int_{[x,\infty)} \frac{1}{\int_{(0,t]} d\nu}\, d\eta \right)^{r/q} d\nu \right\}^{1/r}. \tag{3.9}$$

In general, it is not easy to prove (3.6). Theorem 3.2 shows that if μ, ν satisfy (3.4) or μ, η satisfy (3.5), then we have simple characterizations (3.8) – (3.9) for (3.6) to hold. By [33, Theorem 1.1], we see that in the case $\psi \equiv 1$ and $1 < p < \infty$, $\|T_{\psi,\mu} f\|_{p,\mu} \leqslant p^*\|f\|_{p,\mu}$ holds for all $f \geqslant 0$. This implies the following corollary.

COROLLARY 3.3. *Suppose that* $\phi(x,\cdot)$ *is nondecreasing and left continuous on* $(0,x]$ *for each* $x > 0$. *Then for* $1 < p < \infty$, $0 < q < \infty$, *and* $\nu = \mu$, *inequalities* (3.6) *and* (3.7) *are equivalent with* $A \leqslant C \leqslant p^*A$. *Similarly,* (3.6) *and* (3.7) *are also equivalent with* $A \leqslant C \leqslant q^*A$ *in the case* $0 < p < \infty$, $1 < q < \infty$, *and* $\eta = \mu$.

By [3, Theorem (1.7)], we see that in the case $\psi \equiv 1$ and $d\mu = dx$, inequality (3.4) holds if $1 \leqslant p < \infty$ and $d\nu = v\,dx$, where $v \in B_p$, and (3.5) holds if $1 \leqslant q < \infty$ and $d\eta = u\,dx$, where $u \in B_q$. Hence we have Corollary 3.4.

COROLLARY 3.4. *Suppose that* $d\mu = dx$ *and* $\phi(x,\cdot)$ *is nondecreasing and left continuous on* $(0,x]$ *for each* $x > 0$. *Then* (3.6) *and* (3.7) *are equivalent in the following cases.*

(i) $1 \leqslant p < \infty$, $0 < q < \infty$, $d\nu = vdx$, *where* $v \in B_p$.

(ii) $0 < p < \infty$, $1 \leqslant q < \infty$, $d\eta = udx$, *where* $u \in B_q$.

Several equivalent characterizations on B_p-weights can be found in [9] and [12]. In particular, any nonincreasing weight belongs to class B_p for $1 < p < \infty$. Several known results (see [2], [3], [7], [19], and [26]) can also be applied to established (3.6).

Let g be a positive function defined on $(0,\infty)$ such that $0 < G(x) = \int_0^x g(t)dt < \infty$ for all $x > 0$. Define

$$H_g f(x) = \frac{1}{G(x)} \int_0^x g(t)f(t)dt.$$

In the following corollary we consider the case that $\phi(x,\cdot)/g$ is nondecreasing and left continuous on $(0,x]$ for each $x > 0$. We choose $\psi(x,t) = g(t)$ and then $0 \leqslant T_\psi f = H_g f \downarrow$ for all $0 \leqslant f \downarrow$. By Theorem 3.2 and [2, Theorem 3] we have Corollary 3.5.

COROLLARY 3.5. *Let* $0 < p,q < \infty$. *Let* g *be given as above and* u, v *be weights. Suppose that* $\phi(x,\cdot)/g$ *is nondecreasing and left continuous on* $(0,x]$ *for each* $x > 0$. *If*

$$\sup_{r>0} \frac{\|H_g \chi_{(0,r]}\|_{q,u}}{\|\chi_{(0,r]}\|_{q,u}} < \infty \quad \text{or} \quad \sup_{r>0} \frac{\|H_g \chi_{(0,r]}\|_{p,v}}{\|\chi_{(0,r]}\|_{p,v}} < \infty,$$

then (3.6) *and* (3.7) *are equivalent for* $d\mu = dx$, $d\eta = udx$, *and* $d\nu = vdx$.

Now consider the case that ψ is homogeneous of degree -1 and $\int_0^1 \psi(1,y)dy < \infty$. Then $0 \leqslant T_\psi f \downarrow$ for all $0 \leqslant f \downarrow$. By [20, Theorem 1] and Theorem 3.2, we have the following corollary.

COROLLARY 3.6. *Let* $0 < p,q < \infty$ *and* u, v *be weights. Let* ϕ *and* ψ *be given as in Lemma 3.1, and let* ψ *be homogeneous of degree* -1 *such that* $\int_0^1 \psi(1,y)dy < \infty$.

(i) *If* $q \geqslant 1$, $d\mu = dx$, $d\eta = udx$, *where* u *satisfies* $u(ab) \leqslant u(a)u(b)$ *for all* $a,b > 0$, *and* $\int_0^1 \psi(1,y)y^{-1/q}u(y^{-1})^{1/q}dy < \infty$, *then* (3.6) *and* (3.7) *are equivalent and we have* $A \leqslant C \leqslant A\int_0^1 \psi(1,y)y^{-1/q}u(y^{-1})^{1/q}dy$.

(ii) *If* $p \geqslant 1$, $d\mu = dx$, $d\nu = vdx$, *where* v *satisfies* $v(ab) \leqslant v(a)v(b)$ *for all* $a,b > 0$, *and* $\int_0^1 \psi(1,y)y^{-1/p}v(y^{-1})^{1/p}dy < \infty$, *then* (3.6) *and* (3.7) *are equivalent and we have* $A \leqslant C \leqslant A\int_0^1 \psi(1,y)y^{-1/p}v(y^{-1})^{1/p}dy$.

The following corollary considers the case that $\phi(x,\cdot)$ is nonincreasing for each $x > 0$, and it can be obtained by Theorem 3.2 and [28, Theorem 2.3].

COROLLARY 3.7. *Let* $1 \leqslant p < \infty$ *and* $0 < q < \infty$. *Let* ϕ *and* ψ *be given as in Lemma 3.1. Suppose that* $\phi(x,\cdot)$ *is nonincreasing for each* $x > 0$ *and* $0 \leqslant T_{\psi,\mu} f \downarrow$ *for* *all* $0 \leqslant f \downarrow$. *If* (3.4) *for* $\nu = \mu$ *or* (3.5) *is satisfied, then* (3.2) *holds for all* $f \geqslant 0$ *if and only if* (3.3) *holds. Moreover, we have* $A \leqslant C \leqslant AC_1$.

In the following we extend Theorem P(v) from H to more general integral operator T_ϕ. Consider the condition:

$$B_1\Phi(x,t)\Phi(t,r) \leqslant \Phi(x,r) \leqslant B_2\Phi(x,t)\Phi(t,r), \quad 0 < r \leqslant t \leqslant x. \tag{3.10}$$

Here B_1 and B_2 are positive constants and $\Phi(x,r) = \int_0^r \phi(x,t)dt$, $0 < r \leqslant x$. If ϕ satisfies the right-hand inequality of (3.10) and $0 \leqslant T_\phi f \downarrow$ for all $0 \leqslant f \downarrow$, then Lai [19, Theorem 3.1] showed that for $1 \leqslant p < \infty$, inequality (1.5) and (1.6) are equivalent in the cases $p = q$, $u = v$, $T_1 = T_\phi$, and $T_2 = I$. Moreover, we have $\mathbb{B} \leqslant C \leqslant \mathbb{B}^q B_2^{q(q-1)}$. The method of the proof used in [19, Theorem 3.1] can be applied to prove Theorem 3.8.

THEOREM 3.8. *Let* $1 \leqslant p \leqslant q < \infty$ *and* u, v *be weights. If* (3.10) *is satisfied and* $0 \leqslant T_\phi f \downarrow$ *for all* $0 \leqslant f \downarrow$, *then*

$$\|T_\phi f\|_{q,u} \leqslant C\|T_\phi f\|_{p,v} \quad \textit{for all } 0 \leqslant f \downarrow \tag{3.11}$$

if and only if

$$\mathbb{B} = \sup_{r>0} \frac{\|T_\phi \chi_{(0,r]}\|_{q,u}}{\|T_\phi \chi_{(0,r]}\|_{p,v}} < \infty. \tag{3.12}$$

Moreover, we have $\mathbb{B} \leqslant C \leqslant (B_2/B_1)^{1/q^*}\mathbb{B}$.

Proof of Theorem 3.8. If (3.11) holds, then by choosing $f(x) = \chi_{(0,r]}(x)$ for $r > 0$, we have (3.12) with $\mathbb{B} \leqslant C$. Conversely, suppose that (3.12) holds. Then

$$\begin{aligned}&\int_0^r \Phi(x,x)^q u(x)dx + \int_r^\infty \Phi(x,r)^q u(x)dx \\ \leqslant &\mathbb{B}^q \left\{ \int_0^r \Phi(x,x)^p v(x)dx + \int_r^\infty \Phi(x,r)^p v(x)dx \right\}^{q/p}.\end{aligned}$$

Let $r = g(y)$, where g is a nonincreasing homeomorphism. Then

$$\begin{aligned}L \equiv &\int_0^\infty \left\{ \int_0^{g(y)} \Phi(x,x)^q u(x)dx + \int_{g(y)}^\infty \Phi(x,g(y))^q u(x)dx \right\} dy \\ \leqslant &\mathbb{B}^q \int_0^\infty \left\{ \int_0^{g(y)} \Phi(x,x)^p v(x)dx + \int_{g(y)}^\infty \Phi(x,g(y))^p v(x)dx \right\}^{q/p} dy \equiv \mathbb{B}^q R.\end{aligned} \tag{3.13}$$

Since a general monotone function can be approximated by homeomorphisms, we may suppose that $0 \leqslant f \downarrow$ is a homeomorphism when proving (3.11). Note also that (3.10) implies

$$B_1\Phi(x,t)\int_0^t \phi(t,y)f(y)dy \leqslant \int_0^t \phi(x,y)f(y)dy \leqslant B_2\Phi(x,t)\int_0^t \phi(t,y)f(y)dy.$$

Choose g so that $g^{-1}(t)=T_\phi f(t)^{q-1}f(t)$. The proof of [19, Theorem 3.1] shows that

$$L \geqslant B_2^{1-q}\|T_\phi f\|_{q,u}^q.$$

On the other hand, since $p\leqslant q$, by Minkowski's inequality we see that

$$\begin{aligned}R^{p/q} &\leqslant \int_0^\infty \left\{\int_0^\infty \left(\chi_{(0,g(y))}(x)\Phi(x,x)^p+\chi_{(g(y),\infty)}(x)\Phi(x,g(y))^p\right)^{q/p}v(x)^{q/p}dy\right\}^{p/q}dx\\ &=\int_0^\infty\left\{\int_0^{g^{-1}(x)}\Phi(x,x)^q dy+\int_{g^{-1}(x)}^\infty \Phi(x,g(y))^q dy\right\}^{p/q}v(x)dx.\end{aligned}$$

Follow the proof of [19, Theorem 3.1] we have

$$R^{p/q}\leqslant \int_0^\infty\left\{q\int_0^x g^{-1}(t)\Phi(x,t)^{q-1}\phi(x,t)dt\right\}^{p/q}v(x)dx.$$

By the choice of $g^{-1}(t)$ we have

$$\begin{aligned}R^{p/q}&\leqslant \int_0^\infty\left\{q\int_0^x T_\phi f(t)^{q-1}f(t)\Phi(x,t)^{q-1}\phi(x,t)dt\right\}^{p/q}v(x)dx\\ &\leqslant B_1^{-p/q^*}\int_0^\infty\left\{q\int_0^x\left(\int_0^t\phi(x,y)f(y)dy\right)^{q-1}\phi(x,t)f(t)dt\right\}^{p/q}v(x)dx\\ &=B_1^{-p/q^*}\int_0^\infty\left(\int_0^x\phi(x,t)f(t)dt\right)^p v(x)dx=B_1^{-p/q^*}\|T_\phi f\|_{p,v}^p.\end{aligned}$$

Hence $B_2^{1-q}\|T_\phi f\|_{q,u}^q\leqslant \mathbb{B}^q B_1^{-q/q^*}\|T_\phi f\|_{p,v}^q$. This implies

$$\|T_\phi f\|_{q,u}\leqslant B_1^{-1/q^*}B_2^{1/q^*}\mathbb{B}\|T_\phi f\|_{p,v}. \quad \square$$

By Theorem P(i) we see that for $0<p\leqslant 1\leqslant q<\infty$, inequality (3.11) and (3.12) are equivalent and $\mathbb{B}=C$. Hence by Theorem 3.8 and Theorem P(i) we have (3.11) if and only if (3.12) holds for $0<p\leqslant q<\infty$.

Consider the case $T_\phi=H_g$, where $\phi(x,t)=g(t)$. Then (3.10) is satisfied with $B_1=B_2=1$ and $0\leqslant H_g f\downarrow$ for all $0\leqslant f\downarrow$. We have Corollary 3.9.

COROLLARY 3.9. *Let $0<p\leqslant q<\infty$. Then*

$$\sup_{0\leqslant f\downarrow}\frac{\|H_g f\|_{q,u}}{\|H_g f\|_{p,v}}=\sup_{r>0}\frac{\|H_g\chi_{(0,r]}\|_{q,u}}{\|H_g\chi_{(0,r]}\|_{p,v}}.$$

Corollary 3.9 extends Theorem P(v) from H to H_g. On the other hand, if ϕ is homogeneous of degree -1, then $0\leqslant T_\phi f\downarrow$ for all $0\leqslant f\downarrow$ and (3.10) is reduced to

$$B_1\Phi(1,s_1)\Phi(1,s_2)\leqslant\Phi(1,s_1s_2)\leqslant B_2\Phi(1,s_1)\Phi(1,s_2),\quad 0<s_1,s_2\leqslant 1. \tag{3.14}$$

The following corollary can also be obtained by Theorem 3.8 and Theorem P(i).

COROLLARY 3.10. *Let $0 < p \leqslant q < \infty$ and let ϕ be homogeneous of degree -1 such that $0 < \Phi(1,1) < \infty$. If* (3.14) *is satisfied for some constants B_1 and B_2, then* (3.11) *and* (3.12) *are equivalent.*

Putting Lemma 3.1 and Theorem 3.8 together yield a comparison of $\|T_\phi f\|_{q,u}$ and $\|T_\psi f\|_{p,v}$.

COROLLARY 3.11. *Let $1 \leqslant p \leqslant q < \infty$ and u, v be weights. Let ϕ and ψ be given as in Lemma 3.1. Suppose that ψ satisfies* (3.10), *and $0 \leqslant T_\psi f \downarrow$ for all $0 \leqslant f \downarrow$. If*

$$\mathbb{B}_1 = \sup_{r>0} \frac{\|T_\psi \chi_{(0,r]}\|_{q,u}}{\|T_\psi \chi_{(0,r]}\|_{p,v}} < \infty,$$

then we have

$$\|T_\phi f\|_{q,u} \leqslant (B_2/B_1)^{1/q^*} \mathbb{B}_1 \|T_\psi f\|_{p,v} \quad \text{for all } 0 \leqslant f \downarrow. \tag{3.15}$$

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