Mathematical
Inequalities
& Applications
Volume 16, Number 2 (2013), 577–586 doi:10.7153/mia-16-43

WILKER AND HUYGENS TYPE INEQUALITIES FOR THE LEMNISCATE FUNCTIONS, II

CHAO-PING CHEN

(*Communicated by I. Franjić*)

Abstract. In this paper, we establish new Wilker and Huygens type inequalities for the Lemniscate functions.

1. Introduction and Definitions

It is known in the literature that for $0 < |x| < \frac{\pi}{2}$,

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

and

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3. \tag{1.2}$$

Inequality (1.1) was presented without proof by Wilker [15]. Wilker inequality (1.1) has attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [4, 6, 8, 10, 11, 12, 14, 16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein). Inequality (1.2) is due Huygens [5].

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points (x, y) in the plane satisfying the equation $(x^2+y^2)^2 = x^2+y^2$. In polar coordinates (r, θ), the equation becomes $r^2 = \cos(2\theta)$ and its arc length is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1-t^4}}, \qquad |x| \leqslant 1, \tag{1.3}$$

where $\operatorname{arcsl} x$ is called the arc lemniscate sine function studied by C.F. Gauss in 1797–1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{\mathrm{d}t}{\sqrt{1+t^4}}, \qquad x \in \mathbb{R}. \tag{1.4}$$

Mathematics subject classification (2010): 33E05, 26D07.
Keywords and phrases: Inequalities, Lemniscate functions.

Functions (1.3) and (1.4) can be found (see [1, p. 259], [2, (2.5)–(2.6)], [7, 9] and [13, Ch. 1]).

Another pair of lemniscate functions, the arc lemniscate tangent arctl and the hyperbolic arc lemniscate tangent arctlh, have been introduced in [7, (3.1)–(3.2)]. Therein it has been proven that

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \qquad x \in \mathbb{R} \tag{1.5}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \qquad |x| < 1 \tag{1.6}$$

(see [7, Prop. 3.1]).

In [3], the author considered Wilker and Huygens type inequalities for the Lemniscate functions and proved the following result: for $0 < |x| < 1$,

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} > 2, \tag{1.7}$$

$$2\left(\frac{\operatorname{arcsl} x}{x}\right) + \frac{\operatorname{arctl} x}{x} > 3, \tag{1.8}$$

$$\frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2 > 2 \tag{1.9}$$

and

$$\frac{\operatorname{arcslh} x}{x} + 2\left(\frac{\operatorname{arctlh} x}{x}\right) > 3. \tag{1.10}$$

Moreover, the author pointed out that inequality (1.8) is sharper than inequality (1.7), and inequality (1.10) is sharper than inequality (1.9). Also in [3], the author proved that for $0 < |x| < 1$,

$$2 + \frac{1}{20}x^3 \operatorname{arctl} x < \left(\frac{\operatorname{arcsl} x}{x}\right)^2 + \frac{\operatorname{arctl} x}{x} \tag{1.11}$$

with the best possible constant $\frac{1}{20}$.

This paper is a continuation of our earlier work [3]. We establish new Wilker and Huygens type inequalities for the lemniscate functions. The results presented here are sharper than those derived in earlier work [3].

2. Lemmas

The following lemma is needed in the sequel.

LEMMA 2.1. (i) *For* $|x| < 1$,

$$\operatorname{arcsl} x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n+1} = x + \frac{1}{10}x^5 + \frac{1}{24}x^9 + \dots. \tag{2.12}$$

(ii) *Let* $p \geqslant 0$ *be an integer. Then for* $0 < x < 1$,

$$\sum_{k=0}^{2p-1} (-1)^k u_k(x) < \operatorname{arctl} x < \sum_{k=0}^{2p} (-1)^k u_k(x), \tag{2.13}$$

where

$$u_k(x) = \frac{\Gamma(k+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4k+1)\cdot k!} x^{4k+1}.$$

It follows from (2.12) and (2.13) that

$$\operatorname{arctl} x < x < \operatorname{arcsl} x \tag{2.14}$$

and

$$x - \frac{3}{20}x^5 < \operatorname{arctl} x < x - \frac{3}{20}x^5 + \frac{7}{96}x^9. \tag{2.15}$$

LEMMA 2.2. (i) *Let* $p \geqslant 0$ *be an integer. Then for* $0 < x < 1$,

$$\sum_{k=0}^{2p-1} (-1)^k v_k(x) < \operatorname{arcslh} x < \sum_{k=0}^{2p} (-1)^k v_k(x), \tag{2.16}$$

where

$$v_k(x) = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}(4k+1)\cdot n!} x^{4k+1}.$$

(ii) *For* $|x| < 1$,

$$\operatorname{arctlh} x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!} x^{4n+1} = x + \frac{3}{20}x^5 + \frac{7}{96}x^9 + \cdots. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\operatorname{arcslh} x < x < \operatorname{arctlh} x \tag{2.18}$$

and

$$x - \frac{1}{10}x^5 < \operatorname{arcslh} x < x - \frac{1}{10}x^5 + \frac{1}{24}x^9. \tag{2.19}$$

Lemmas 2.1 and 2.2 have been proved in [3].

3. Sharp Wilker type inequalities

Inequality (2.14) motivated us to introduce Theorem 3.1 below. Clearly, the lower bound in (3.20) is sharer than one in (1.11).

THEOREM 3.1. *For* $0<|x|<1$,

$$2+\frac{1}{20}x^3\operatorname{arcsl}x<\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x} \tag{3.20}$$

with the best possible constant $\frac{1}{20}$.

Proof. By using (2.12) and (2.15), we obtain

$$\begin{aligned}&\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-\left(2+\frac{1}{20}x^3\operatorname{arcsl}x\right)\\ &=\left(\frac{\operatorname{arcsl}x}{x}-\frac{1}{20}x^4\right)\frac{\operatorname{arcsl}x}{x}+\frac{\operatorname{arctl}x}{x}-2\\ &>\left(1+\frac{1}{10}x^4-\frac{1}{20}x^4\right)\left(1+\frac{1}{10}x^4\right)+\left(1-\frac{3}{20}x^4\right)-2\\ &=\frac{1}{200}x^8>0.\end{aligned}$$

Write (3.20) as

$$\frac{1}{20}<\frac{\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-2}{x^3\operatorname{arcsl}x}.$$

Elementary calculations show that

$$\lim_{x\to0^+}\frac{\left(\frac{\operatorname{arcsl}x}{x}\right)^2+\frac{\operatorname{arctl}x}{x}-2}{x^3\operatorname{arcsl}x}=\frac{1}{20}.$$

Hence, inequality (3.20) holds with best possible constant $\frac{1}{20}$. □

Theorem 3.2 solves a conjecture in [3].

THEOREM 3.2. *For* $0<|x|<1$,

$$2+\frac{1}{5}x^3\operatorname{arctlh}x<\frac{\operatorname{arcslh}x}{x}+\left(\frac{\operatorname{arctlh}x}{x}\right)^2 \tag{3.21}$$

with the best possible constant $\frac{1}{5}$.

Proof. By using (2.17) and (2.19), we obtain that $0 < x < 1$,

$$\begin{aligned}
&\frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2 - \left(2 + \frac{1}{5}x^3 \operatorname{arctlh} x\right) \\
&= \frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x} - \frac{1}{5}x^4\right)\frac{\operatorname{arctlh} x}{x} - 2 \\
&> 1 - \frac{1}{10}x^4 + \left(1 + \frac{3}{20}x^4 + \frac{7}{96}x^8 - \frac{1}{5}x^4\right)\left(1 + \frac{3}{20}x^4 + \frac{7}{96}x^8\right) - 2 \\
&= \frac{83}{600}x^8 + \frac{7}{960}x^{12} + \frac{49}{9216}x^{16} > 0.
\end{aligned}$$

Write (3.21) as

$$\frac{1}{5} < \frac{\frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2 - 2}{x^3 \operatorname{arctlh} x}.$$

Elementary calculations show that

$$\lim_{x \to 0^+} \frac{\frac{\operatorname{arcslh} x}{x} + \left(\frac{\operatorname{arctlh} x}{x}\right)^2 - 2}{x^3 \operatorname{arctlh} x} = \frac{1}{5}.$$

Hence, inequality (3.21) holds with best possible constant $\frac{1}{5}$. □

4. Inequalities for the arc lemniscate functions

Theorem 4.1 presents sharp inequalities.

THEOREM 4.1. *For* $0 < |x| < 1$,

$$a\left(\frac{\operatorname{arcsl} x}{x}\right)^2 + (1-a)\left(\frac{\operatorname{arctl} x}{x}\right) > 1, \tag{4.22}$$

$$b\left(\frac{\operatorname{arcsl} x}{x}\right) + (1-b)\left(\frac{\operatorname{arctl} x}{x}\right)^2 > 1 \tag{4.23}$$

and

$$c\left(\frac{\operatorname{arcsl} x}{x}\right) + (1-c)\left(\frac{\operatorname{arctl} x}{x}\right) > 1 \tag{4.24}$$

with the possible constants

$$a = \frac{3}{7}, \quad b = \frac{3}{4} \quad \text{and} \quad c = \frac{3}{5}. \tag{4.25}$$

In particular, we have for $0<|x|<1$,

$$\frac{3}{7}\left(\frac{\operatorname{arcsl} x}{x}\right)^2+\frac{4}{7}\left(\frac{\operatorname{arctl} x}{x}\right)>1, \tag{4.26}$$

$$\frac{3}{4}\left(\frac{\operatorname{arcsl} x}{x}\right)+\frac{1}{4}\left(\frac{\operatorname{arctl} x}{x}\right)^2>1 \tag{4.27}$$

and

$$\frac{3}{5}\left(\frac{\operatorname{arcsl} x}{x}\right)+\frac{2}{5}\left(\frac{\operatorname{arctl} x}{x}\right)>1. \tag{4.28}$$

Proof of Theorem 4.1. By using (2.12), we have for $0<|x|<1$,

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^3=\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n}\right)^3=1+\frac{3}{10}x^4+\frac{31}{200}x^8+\cdots. \tag{4.29}$$

By using (2.15) and (4.29), we obtain for $0<|x|<1$,

$$\begin{aligned}\left(\frac{\operatorname{arcsl} x}{x}\right)^3\left(\frac{\operatorname{arctl} x}{x}\right)^2&>\left(1+\frac{3}{10}x^4+\frac{31}{200}x^8\right)\left(1-\frac{3}{20}x^4\right)^2\\&=1+x^8\left(\frac{7}{80}-\frac{159}{4000}x^4+\frac{279}{80000}x^8\right)>1.\end{aligned} \tag{4.30}$$

By using the arithmetic–geometric mean inequality together with inequality (4.30), we obtain the inequalities (4.26)–(4.28). That is to say, when $a=\frac{3}{7}, b=\frac{3}{4}, c=\frac{3}{5}$, the inequalities (4.22)–(4.24) hold.

The inequalities (4.22)–(4.24) can be written as

$$a>\frac{1-(\operatorname{arctl} x/x)}{(\operatorname{arcsl} x/x)^2-(\operatorname{arctl} x/x)},\quad b>\frac{1-(\operatorname{arctl} x/x)^2}{(\operatorname{arcsl} x/x)-(\operatorname{arctl} x/x)^2}$$

and

$$c>\frac{x-\operatorname{arctl} x}{\operatorname{arcsl} x-\operatorname{arctl} x},$$

respectively. Elementary calculations show that

$$a\geqslant\lim_{x\to 0}\frac{1-(\operatorname{arctl} x/x)}{(\operatorname{arcsl} x/x)^2-(\operatorname{arctl} x/x)}=\frac{3}{7},\quad b\geqslant\lim_{x\to 0}\frac{1-(\operatorname{arctl} x/x)^2}{(\operatorname{arcsl} x/x)-(\operatorname{arctl} x/x)^2}=\frac{3}{4}$$

and

$$c\geqslant\lim_{x\to 0}\frac{x-\operatorname{arctl} x}{\operatorname{arcsl} x-\operatorname{arctl} x}=\frac{3}{5}.$$

This means that the inequalities (4.22)–(4.24) hold for $0<|x|<1$ with the best possible constants given in (4.25). The proof of Theorem 4.1 is complete. □

REMARK 4.1. Noting that $\frac{\operatorname{arcsl} x}{x} > 1$, we have, by (2.14),

$$\frac{(\operatorname{arcsl} x/x)^2 + \operatorname{arctl} x/x}{2} - \frac{3\,(\operatorname{arcsl} x/x)^2 + 4(\operatorname{arctl} x/x)}{7}$$
$$= \frac{1}{14}\left[\left(\frac{\operatorname{arcsl} x}{x}\right)^2 - \frac{\operatorname{arctl} x}{x}\right] > \frac{\operatorname{arcsl} x - \operatorname{arctl} x}{14x} > 0, \quad 0 < |x| < 1,$$

which shows that inequality (4.26) is sharper than inequality (1.7).

By (2.14), we have $0 < |x| < 1$,

$$\frac{2\,(\operatorname{arcsl} x/x) + \operatorname{arctl} x/x}{3} - \frac{3\,(\operatorname{arcsl} x/x) + 2(\operatorname{arctl} x/x)}{5} = \frac{\operatorname{arcsl} x - \operatorname{arctl} x}{15x} > 0,$$

which shows that inequality (4.28) is sharper than inequality (1.8).

5. Inequalities for the hyperbolic arc lemniscate functions

Theorem 5.1 is an interesting analogue of Theorem 4.1.

THEOREM 5.1. *For* $0 < |x| < 1$*,*

$$p\left(\frac{\operatorname{arcslh} x}{x}\right)^2 + (1-p)\left(\frac{\operatorname{arctlh} x}{x}\right) > 1, \tag{5.31}$$

$$q\left(\frac{\operatorname{arcslh} x}{x}\right) + (1-q)\left(\frac{\operatorname{arctlh} x}{x}\right)^2 > 1 \tag{5.32}$$

and

$$r\left(\frac{\operatorname{arcslh} x}{x}\right) + (1-r)\left(\frac{\operatorname{arctlh} x}{x}\right) > 1 \tag{5.33}$$

with the possible constants

$$p = \frac{3}{7}, \quad q = \frac{3}{4} \quad \text{and} \quad r = \frac{3}{5}. \tag{5.34}$$

In particular, we have for $0 < |x| < 1$,

$$\frac{3}{7}\left(\frac{\operatorname{arcslh} x}{x}\right)^2 + \frac{4}{7}\left(\frac{\operatorname{arctlh} x}{x}\right) > 1, \tag{5.35}$$

$$\frac{3}{4}\left(\frac{\operatorname{arcslh} x}{x}\right) + \frac{1}{4}\left(\frac{\operatorname{arctlh} x}{x}\right)^2 > 1 \tag{5.36}$$

and

$$\frac{3}{5}\left(\frac{\operatorname{arcslh} x}{x}\right) + \frac{2}{5}\left(\frac{\operatorname{arctlh} x}{x}\right) > 1. \tag{5.37}$$

Proof of Theorem 5.1. By using (2.17), we have for $0<|x|<1$,

$$\left(\frac{\operatorname{arctlh} x}{x}\right)^2=\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})\cdot(4n+1)\cdot n!}x^{4n}\right)^2=1+\frac{3}{10}x^4+\frac{101}{600}x^8+\dots. \tag{5.38}$$

By using (2.19) and (5.38), we obtain for $0<|x|<1$,

$$\begin{aligned}\left(\frac{\operatorname{arcslh} x}{x}\right)^3\left(\frac{\operatorname{arctlh} x}{x}\right)^2&>\left(1-\frac{1}{10}x^4\right)^3\left(1+\frac{3}{10}x^4+\frac{101}{600}x^8\right)\\&=1+x^8\left(\frac{13}{120}-\frac{17}{400}x^4\right)+x^{16}\left(\frac{19}{4000}-\frac{101}{600000}x^4\right)>1.\end{aligned} \tag{5.39}$$

By using the arithmetic–geometric mean inequality together with inequality (5.39), we obtain the inequalities (5.35)–(5.37). That is to say, when $p=\frac{3}{7},q=\frac{3}{4},r=\frac{3}{5}$, the inequalities (5.31)–(5.33) hold.

The inequalities (5.31)–(5.33) can be written as

$$p<\frac{(\operatorname{arctlh} x/x)-1}{(\operatorname{arctlh} x/x)-(\operatorname{arcslh} x/x)^2},\quad q<\frac{(\operatorname{arctlh} x/x)^2-1}{(\operatorname{arctlh} x/x)^2-(\operatorname{arcslh} x/x)}$$

and

$$r<\frac{\operatorname{arctlh} x-x}{\operatorname{arctlh} x-\operatorname{arcslh} x},$$

respectively. Elementary calculations show that

$$p\leqslant\lim_{x\to0}\frac{(\operatorname{arctlh} x/x)-1}{(\operatorname{arctlh} x/x)-(\operatorname{arcslh} x/x)^2}=\frac{3}{7},\quad q\leqslant\lim_{x\to0}\frac{(\operatorname{arctlh} x/x)^2-1}{(\operatorname{arctlh} x/x)^2-(\operatorname{arcslh} x/x)}=\frac{3}{4}$$

and

$$r\leqslant\lim_{x\to0}\frac{\operatorname{arctlh} x-x}{\operatorname{arctlh} x-\operatorname{arcslh} x}=\frac{3}{5}.$$

This means that the inequalities (5.31)–(5.33) hold for $0<|x|<1$ with the best possible constants given in (5.34). The proof of Theorem 5.1 is complete. □

REMARK 5.1. Noting that $\dfrac{\operatorname{arctlh} x}{x}>1$, we have, by (2.18),

$$\begin{aligned}&\frac{\operatorname{arcslh} x/x+(\operatorname{arctlh} x/x)^2}{2}-\frac{3(\operatorname{arcslh} x/x)+(\operatorname{arctlh} x/x)^2}{4}\\&=\frac{1}{4}\left[\left(\frac{\operatorname{arctlh} x}{x}\right)^2-\frac{\operatorname{arcslh} x}{x}\right]>\frac{\operatorname{arctlh} x-\operatorname{arcslh} x}{4x}>0,\quad 0<|x|<1,\end{aligned}$$

which shows that inequality (5.36) is sharper than inequality (1.9).

By (2.18), we have for $0 < |x| < 1$,

$$\begin{aligned}&\frac{\operatorname{arcslh} x/x + 2(\operatorname{arctlh} x/x)}{3} - \frac{3\,(\operatorname{arcslh} x/x) + 2(\operatorname{arctlh} x/x)}{5}\\ &= \frac{4(\operatorname{arctlh} x - \operatorname{arcslh} x)}{15x} > 0,\end{aligned}$$

which shows that inequality (5.37) is sharper than inequality (1.10).

REFERENCES

[1] J. M. BORWEIN AND P. B. BORWEIN, *Pi and the AGM: a study in the analytic number theory and computational complexity*, John Wiley and Sons, New York, 1987.

[2] B. C. CARLSON, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly **78** (1971), 496–505.

[3] C.-P. CHEN, *Wilker and Huygens type inequalities for the Lemniscate functions*, J. Math. Inequal., **6**, 4 (2012), 673–684.

[4] B.-N. GUO, B.-M. QIAO, F. QI AND W. LI, *On new proofs of Wilker inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), 19–22.

[5] C. HUYGENS, *Oeuvres Completes 1888-1940*, Société Hollondaise des Science, Haga.

[6] C. MORTICI, *The natural approach of Wilker-Cusa-Huygens inequalities*, Math. Inequal. Appl. **14** (2011), 535–541.

[7] E. NEUMAN, *On Gauss lemniscate functions and lemniscatic mean*, Math. Pannon, **18** (2007), 77–94.

[8] E. NEUMAN, *One- and two-sided inequalities for Jacobian elliptic functions and related results*, Integral Transforms Spec. Funct. **21** (2010), 399–407.

[9] E. NEUMAN, *Two-sided inequalities for the lemniscate functions*, J. Inequal. Spec. Funct. **1** (2010), 1–7.

[10] E. NEUMAN, *On Wilker and Huygens type inequalities*, Math. Inequal. Appl. **15** (2012) 271–279.

[11] E. NEUMAN AND J. SÁNDOR, *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*, Math. Inequal. Appl. **13** (2010), 715–723.

[12] I. PINELIS, *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*, Amer. Math. Monthly, **111** (2004), 905–909.

[13] C. L. SIEGEL, *Topics in Complex Function Theory*, Vol. 1, Wiley, New York, 1969.

[14] J. S. SUMNER, A. A. JAGERS, M. VOWE AND J. ANGLESIO, *Inequalities involving trigonometric functions*, Amer. Math. Monthly **98** (1991), 264–267.

[15] J. B. WILKER, *Problem E 3306*, Amer. Math. Monthly **96** (1989), 55.

[16] S.-H. WU, *On extension and refinement of Wilker's inequality*, Rocky Mountain J. Math. **39** (2009), 683–687.

[17] S.-H. WU AND H. M. SRIVASTAVA, *A weighted and exponential generalization of Wilker's inequality and its applications*, Integral Transform Spec. Funct. **18** (2007), 529–535.

[18] S.-H. WU AND H. M. SRIVASTAVA, *A further refinement of Wilker's inequality*, Integral Transform Spec. Funct. **19** (2008), 757–765.

[19] L. ZHANG AND L. ZHU, *A new elementary proof of Wilker's inequalities*, Math. Inequal. Appl. **11** (2008), 149–151.

[20] L. ZHU, *A new simple proof of Wilker's inequality*, Math. Inequal. Appl. **8** (2005), 749–750.

[21] L. ZHU, *Some new Wilker-type inequalities for circular and hyperbolic functions*, Abstr. Appl. Anal. (2009), Article ID 485842.

[22] L. ZHU, *A source of inequalities for circular functions* Comput. Math. Appl. **58** (2009), 1998–2004.

[23] L. ZHU, *Inequalities for Hyperbolic functions and their Applications*, J. Ineq. Appl. Vol. (2010), Article ID 130821.

(Received January 24, 2012)

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454003
Henan Province, China
e-mail: chenchaoping@sohu.com