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ON THE ARITHMETIC AND HARMONIC AVERAGES IN NONDISCRETE SETTING

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Abstract. In recent years, the almost sure central limit theorem attracted widespread attention in Probability Theory. It involves the harmonic (also called logarithmic) averages of a numerical sequence. In our previous paper [2], we investigated the convergence behavior of the sequence of harmonic averages of a given numerical sequence from the viewpoint of Summability Theory. In the present paper, our aim is to extend this investigation from the discrete to nondiscrete setting. Given a real or complex-valued function which is locally integrable on $\mathbb{R}_+$, we give necessary and sufficient conditions for the existence of the finite limit of its arithmetic or harmonic averages, respectively. These characterizations may be useful in the investigation of the almost sure behavior of Stochastic Processes.

1. Known results on the averages of numerical sequences

In the framework of Kolmogorov's axiomatic treatment of probability, one of the fundamental questions is the relationship between probability and relative frequency. The results of this investigation are called the *laws of large numbers*, which are formulated in terms of the arithmetic averages of random variables. See, e.g., in the book [3] by P. Révész.

In recent years, the so-called *almost sure central limit theorem* has attracted widespread attention in Probability Theory, in which the harmonic (also called logarithmic) averages of a sequence of indicator function values play a key role. See, e.g., in [1] and also the papers listed in its References.

Now, we briefly summarize the basic results on the convergence behavior of the sequences of the arithmetic and harmonic averages of sequences of numbers from the viewpoint of Summability Theory. Let $\{x_k : k = 1,2,\ldots\}$ be a sequence of real or complex numbers. We recall that the sequence $\{\sigma_n : n = 1,2,\ldots\}$ of its *arithmetic averages* is defined by

$$\sigma_n := \frac{1}{n}\sum_{k=1}^{n} x_k,$$

while the sequence $\{\tau_n\}$ of its *harmonic averages* is defined by

$$\tau_n := \frac{1}{\ell_n}\sum_{k=1}^{n} \frac{x_k}{k}, \quad \text{where} \quad \ell_n := \sum_{k=1}^{n} \frac{1}{k}, \quad n = 1,2,\ldots.$$

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Since

$$\lim_{n\to\infty} \frac{\ell_n}{\log n} = 1, \tag{1.1}$$

where the logarithm is to the natural base e, the sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ are equiconvergent with the same limit, where

$$\tau_n^* := \frac{1}{\log n} \sum_{k=1}^{n} \frac{x_k}{k}, \quad n = 2, 3, \ldots .$$

This fact explains that harmonic averages are often called *logarithmic* ones in the literature.

It is well known that if a sequence $\{x_k\}$ of numbers converges to a finite limit ξ, then the sequence $\{\sigma_n\}$ also converges to the same limit ξ. Furthermore, if $\{\sigma_n\}$ converges to a finite limit ξ, then $\{\tau_n\}$ also converges to the same ξ. On the other hand, in either case the converse statement is not true in general.

In the next two theorems, we characterize the convergence of $\{\sigma_n\}$ and $\{\tau_n\}$ to a finite limit as $n \to \infty$, in terms of the so-called *maximal moving averages* (see (1.2) and (1.3) below). The first theorem is a folklore.

THEOREM A. *The finite limit* $\lim_{n\to\infty} \sigma_n = \xi$ *exists if and only if*

$$\lim_{m\to\infty} \frac{1}{2^{m-1}} \max_{2^{m-1}<n\leqslant 2^m} \Big| \sum_{k=2^{m-1}+1}^{n} (x_k - \xi) \Big| = 0. \tag{1.2}$$

The second theorem was proved in [2, Theorem 2].

THEOREM B. *The finite limit* $\lim_{n\to\infty} \tau_n = \xi$ *exists if and only if*

$$\lim_{m\to\infty} \frac{1}{\ell_{\mu_m - \ell_{\mu_{m-1}}}} \max_{\mu_{m-1}<n\leqslant \mu_m} \Big| \sum_{k=\mu_{m-1}+1}^{n} \frac{x_k - \xi}{k} \Big| = 0, \tag{1.3}$$

where $\mu_m := 2^{2^m}$, $m = 0, 1, 2, \ldots$.

We note that by (1.1), the denominator $(\ell_{\mu_m} - \ell_{\mu_{m-1}})$ in (1.3) can be equivalently replaced by 2^{m-1}.

2. New results on the averages of integrable functions

In the present paper, our primary aim is to extend these investigations from the discrete to nondiscrete setting. To this effect, we consider real or complex-valued functions $x(t)$ that are locally integrable in Lebesgue's sense on the half-axis $\mathbb{R}_+ := [0, \infty)$ or on some infinite subinterval of $\mathbb{R}_+$.

As is well known, given a function $x(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$, its *arithmetic averages* are defined by

$$\sigma(T) := \frac{1}{T} \int_0^T x(t)dt, \quad T > 0.$$

Assume that $t^{-1}x(t) \in L^1_{\mathrm{loc}}([1,\infty))$, then the *harmonic averages* of the function $x(t)$ are defined by

$$\tau(T) := \frac{1}{\log T}\int_1^T \frac{x(t)}{t}dt, \quad T > 1,$$

Our new results stated in Theorems 1 and 2 below are the nondiscrete versions of Theorems A and B. They may be useful in the investigation of the almost sure behavior of Stochastic Processes.

THEOREM 1. *The finite limit* $\lim_{T\to\infty}\sigma(T) = \xi$ *exists if and only if*

$$\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} x(t)dt = \xi. \tag{2.1}$$

The ratio on the left-hand side of (2.1) may be called the *moving arithmetic average* of the function $x(t)$. The surprising fact is that in the characterization (2.1) we do not need maximal moving average (in contrast with (1.2)). Theorem 1 is a folklore, but we present its proof for the reader's convenience.

Proof. Necessity. In case $\lim_{T\to\infty}\sigma(T) = \xi$ we trivially have

$$\frac{1}{T}\int_T^{2T} x(t)dt = 2\sigma(2T) - \sigma(T) \to 2\xi - \xi = \xi \quad \text{as} \quad T \to \infty.$$

Sufficiency. By (2.1), for every $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon) > 0$ such that

$$d_0(T) := \left|\frac{1}{T}\int_T^{2T}(x(t)-\xi)dt\right| < \varepsilon \quad \text{if} \quad T > T_0. \tag{2.2}$$

Given $T > T_0$, let $m = m(T)$ be the integer for which

$$\frac{T}{2^{m+1}} \leqslant T_0 < \frac{T}{2^m}. \tag{2.3}$$

It follows that $T/2^m \leqslant 2T_0$. By (2.2) and (2.3), we estimate as follows:

$$\begin{aligned}
&\left|\frac{1}{T}\int_0^T (x(t)-\xi)dt\right| \\
&= \left|\frac{1}{T}\left\{\int_{T/2}^T + \int_{T/4}^{T/2} + \ldots + \int_{T/2^m}^{T/2^{m-1}} + \int_0^{T/2^m}\right\}(x(t)-\xi)dt\right| \\
&\leqslant \frac{1}{2}d_0\left(\frac{T}{2}\right) + \frac{1}{4}d_0\left(\frac{T}{4}\right) + \ldots + \frac{1}{2^m}d_0\left(\frac{T}{2^m}\right) + \frac{1}{T}\int_0^{2T_0}|x(t)-\xi|dt \\
&\leqslant \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^m}\right)\varepsilon + \varepsilon < 2\varepsilon,
\end{aligned} \tag{2.4}$$

provided that T is so large that the last term on the third line of (2.4) is less than ε. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T (x(t)-\xi)dt = 0.$$

This is equivalent to the existence of the limit $\lim_{T\to\infty}\sigma(T) = \xi$. □

THEOREM 2. *The finite limit* $\lim_{T\to\infty}\tau(T)=\xi$ *exists if and only if*

$$\lim_{T\to\infty}\frac{1}{T}\int_{e^T}^{e^{2T}}\frac{x(t)}{t}dt=\xi. \tag{2.5}$$

The ratio on the left-hand side of (2.5) may be called the *moving harmonic average* of $x(t)$. Again, in the characterization (2.5) we do not need maximal moving average (in constrast with (1.3)).

Proof. Necessity. In case $\lim_{T\to\infty}\tau(T)=\xi$, we trivially have

$$\frac{1}{T}\int_{e^T}^{e^{2T}}\frac{x(t)}{t}dt=2\tau(2T)-\tau(T)\to 2\xi-\xi=\xi \quad \text{as} \quad T\to\infty.$$

Sufficiency. By (2.5), for every $\varepsilon>0$ there exists $T_0=T_0(\varepsilon)>0$ such that

$$d_1(T):=\left|\frac{1}{T}\int_{e^T}^{e^{2T}}\frac{x(t)-\xi}{t}dt\right|<\varepsilon \quad \text{if} \quad T>T_0. \tag{2.6}$$

Given $T>T_0$, this time let $m=m(T)$ be the integer for which

$$\frac{e^T}{2^{m+1}}\leqslant T_0<\frac{e^T}{2^m}. \tag{2.7}$$

Clearly, it follows that $e^T/2^m\leqslant 2T_0$. By (2.6) and (2.7), we estimate as follows:

$$\begin{aligned}
&\left|\frac{1}{T}\int_1^{e^T}\frac{x(t)-\xi}{t}dt\right| \\
&=\left|\frac{1}{T}\left\{\int_{e^{T/2}}^{e^T}+\int_{e^{T/4}}^{e^{T/2}}+\ldots+\int_{e^{T/2^m}}^{e^{T/2^{m-1}}}+\int_1^{e^{T/2^m}}\right\}\frac{x(t)-\xi}{t}dt\right| \\
&\leqslant\frac{1}{2}d_1\left(\frac{T}{2}\right)+\frac{1}{4}d_1\left(\frac{T}{4}\right)+\ldots+\frac{1}{2^m}d_1\left(\frac{T}{2^m}\right)+\frac{1}{T}\int_1^{2T_0}\frac{|x(t)-\xi|}{t}dt \\
&\leqslant\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^m}\right)\varepsilon+\varepsilon<2\varepsilon,
\end{aligned} \tag{2.8}$$

provided that T is large enough. Since $\varepsilon>0$ is arbitrary, we conclude that

$$\lim_{T\to\infty}\frac{1}{T}\int_1^{e^T}\frac{x(t)-\xi}{t}dt=0, \tag{2.9}$$

which is equivalent to the existence of the limit $\lim_{T\to\infty}\tau(T)=\xi$. □

REMARK. It is easy to check that Theorems 1 and 2 remain valid if conditions (2.1) and (2.5) are replaced in them, respectively by the following ones:

$$\lim_{T\to\infty}\frac{1}{(a-1)T}\int_T^{aT}x(t)dt=\xi \tag{2.1$'$}$$

and

$$\lim_{T\to\infty}\frac{1}{(a-1)T}\int_{e^T}^{e^{aT}}\frac{x(t)}{t}dt=\xi, \tag{2.5$'$}$$

where $a>1$ is a given real number.

3. Higher order harmonic averages of integrable functions

Given a function $x(t)$ such that $(t\log t)^{-1}x(t) \in L^1_{\mathrm{loc}}([e,\infty))$, its *harmonic averages of second order* are defined by

$$\tau_2(T) := \frac{1}{\log\log T}\int_e^T \frac{x(t)}{t\log t}dt, \quad T > e.$$

The analogue of Theorem 2 reads as follows.

THEOREM 3. *The finite limit* $\lim_{T\to\infty}\tau_2(T) = \xi$ *exists if and only if*

$$\lim_{T\to\infty}\frac{1}{T}\int_{e^{e^T}}^{e^{e^{2T}}} \frac{x(t)}{t\log t}dt = \xi. \tag{3.1}$$

Proof. Necessity. In case $\lim_{T\to\infty}\tau_2(T) = \xi$ we trivially have

$$\frac{1}{T}\int_{e^{e^T}}^{e^{e^{2T}}} \frac{x(t)}{t\log t}dt = 2\tau_2(2T) - \tau_2(T) \to 2\xi - \xi = \xi \quad \text{as} \quad T\to\infty.$$

Sufficiency. It goes along the same lines as the proof of Theorem 2. Therefore, we only sketch it. By (3.1), for every $\varepsilon > 0$ there exists $T_0 = T_0(\varepsilon)$ such that

$$d_2(T) := \left|\frac{1}{T}\int_{e^{e^T}}^{e^{e^{2T}}} \frac{x(t)-\xi}{t\log t}dt\right| < \varepsilon \quad \text{if} \quad T > T_0. \tag{3.2}$$

Given $T > T_0$, this time let $m = m(T)$ be the integer for which

$$\frac{e^{e^T}}{2^{m+1}} \leqslant T_0 < \frac{e^{e^T}}{2^m}. \tag{3.3}$$

It follows that $e^{e^T}/2^m \leqslant 2T_0$. By (3.2) and (3.3), we conclude (in the same way as in the case of (2.8) and (2.9) that

$$\lim_{T\to\infty}\frac{1}{T}\int_e^{e^{e^T}} \frac{x(t)-\xi}{t\log t}dt = 0, \tag{3.4}$$

which is equivalent to the existence of the limit $\lim_{T\to\infty}\tau_2(T) = \xi$. □

REMARK. It is easy to check that Theorem 3 remains valid if condition (3.1) is replaced by the following one:

$$\lim_{T\to\infty}\frac{1}{(a-1)T}\int_{e^{e^T}}^{e^{e^{aT}}} \frac{x(t)}{t\log t}dt = \xi, \tag{3.1}$$

where $a > 1$ is a given real number.

One may also define for each integer $p \geqslant 3$ the *harmonic averages* $\tau_p(T)$ *of* p*th order* of a function $x(t)$ with an appropriate local integrability condition. In this perception, $\tau(T)$ defined in Section 2 may be considered as the harmonic average of first order of the function in question. Theorems analogous to Theorems 2 and 3 can be proved for each $p = 3,4,\ldots$.

4. Inclusions

It is easy to prove that if $x(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$, then the ordinary convergence

$$\lim_{T\to\infty} x(T) = \xi$$

implies the convergence of the arithmetic average to the same limit:

$$\lim_{T\to\infty} \sigma(T) = \xi. \tag{4.1}$$

In the next theorem, we will prove that (4.1) implies the convergence of the harmonic average to the same limit.

THEOREM 4. *Assume that* $x(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$ *and* $t^{-1}x(t) \in L^1_{\text{loc}}([1,\infty))$. *Then the implication (4.1)* $\Rightarrow$ *(4.2) holds true, where*

$$\lim_{T\to\infty} \tau(T) = \xi. \tag{4.2}$$

The converse implication is not true in general.

Proof. (i) First, we deal with the special case when T is an integer, say $T := n$ where $n \geqslant 3$. Applying the Second Mean Value Theorem gives

$$\tau(n)\log n = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{x(t)}{t} dt$$

$$= \sum_{k=1}^{n-1} \left\{ \frac{1}{k} \int_k^{b_k} x(t)dt + \frac{1}{k+1} \int_{b_k}^{k+1} x(t)dt \right\}$$

$$= \frac{1}{1} \int_1^{b_1} x(t)dt + \sum_{k=1}^{n-1} \frac{1}{k+1} \int_{b_k}^{b_{k+1}} x(t)dt + \frac{1}{n} \int_{b_{n-1}}^{n} x(t)dt, \tag{4.3}$$

where

$$k < b_k < k+1, \quad k = 1,2,\dots,n-1. \tag{4.4}$$

By the definition of the arithmetic average, we may write that

$$\int_b^c x(t)dt = c\sigma(c) - b\sigma(b), \quad 0 < b < c. \tag{4.5}$$

Making use of (4.3) and (4.5), we obtain

$$\tau(n)\log n = \int_1^n \frac{x(t)}{t} dt = \frac{1}{1}\{b_1\sigma(a_1) - \sigma(1)\}$$

$$+ \sum_{k=1}^{n-1} \frac{1}{k+1} \{b_{k+1}\sigma(b_{k+1}) - b_k\sigma(b_k)\} + \frac{1}{n}\{n\sigma(n) - b_{n-1}\sigma(b_{n-1})\}$$

$$= -\sigma(1) + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} b_k \sigma(b_k) + \sigma(n).$$

Hence it follows that

$$\tau(n) = \frac{1}{\log n} \sum_{k=1}^{n-1} \frac{b_k}{k(k+1)} \sigma(b_k) + \frac{1}{\log n} \{\sigma(n) - \sigma(1)\}$$

$$=: \sum_{k=1}^{n-1} a_{n,k} \sigma(b_k) + \frac{1}{\log n} \{\sigma(n) - \sigma(1)\}. \tag{4.6}$$

By (4.4), we have for $k = 1, 2, \ldots, n-1$,

$$0 < a_{n,k} := \frac{1}{\log n} \frac{b_k}{k(k+1)} < \frac{1}{k \log n} \to 0 \quad \text{as} \quad n \to \infty,$$

and it is easy to check that

$$\sum_{k=1}^{n-1} a_{n,k} \to 1 \quad \text{as} \quad n \to \infty.$$

Thus, we may apply the familiar Toeplitz' theorem (see, e.g., [4, p. 74]) to conclude from (4.1) and (4.6) that the limit (4.2) holds in the special case when $T := n \in \mathbb{N}$.

(ii) In the general case when $T > 3$ is a real number, let $n := [T]$, where $[\cdot]$ denotes the integer part. We use equality (4.5) and the Second Mean Value Theorem to obtain

$$\tau(T) \log T - \tau(n) \log n = \int_n^T \frac{x(t)}{t} dt$$

$$= \frac{1}{n} \int_n^b x(t) dt + \frac{1}{T} \int_b^T x(t) dt$$

$$= \left(\frac{1}{n} - \frac{1}{T}\right) b\sigma(b) - \sigma(n) + \sigma(T), \quad \text{where} \quad n < b < T. \tag{4.7}$$

Taking into account that $n := [T]$, it follows from (4.1) and (4.7) that

$$|\tau(T) \log T - \sigma(n) \log n| \leqslant \frac{T-n}{nT} b|\sigma(b)| + |\sigma(T) - \sigma(n)|$$

$$\leqslant \frac{1}{n} |\sigma(b)| + |\sigma(T) - \sigma(n)| \to 0 \quad \text{as} \quad T \to \infty.$$

Thus, we conclude that

$$\lim_{T \to \infty} \tau(T) = \lim_{T \to \infty} \tau(n) \frac{\log n}{\log T} = \xi,$$

which is (4.2) to be proved.

(iii) In order to prove that the converse implication, that is, (4.2) $\Rightarrow$ (4.1) is false, we make a trivial observation. A necessary condition for the existence of the finite limit (4.1) is that for every real number $a > 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_T^{T+a} x(t)dt = 0. \tag{4.8}$$

Indeed, by definition we have

$$\frac{1}{T}\int_T^{T+a} x(t)dt = \frac{T+a}{T}\sigma(T+a) - \sigma(T) \to \xi - \xi = 0 \quad \text{as} \quad T\to\infty.$$

Now, we consider the function $x(t)$ defined by

$$x(t) := \begin{cases} ne^{2^n} & \text{if} \quad t\in[e^{2^n}, e^{2^n}+1),\ n=1,2,\ldots; \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, for $T := e^{2^n}$ we have

$$\frac{1}{T}\int_T^{T+1} x(t)dt = n \not\to 0 \quad \text{as} \quad n\to\infty,$$

that is, condition (4.8) is not satisfied. Thus, the finite limit (4.1) cannot exist for any finite ξ.

On the other hand, if $T \geqslant e^2$, then

$$e^{2^{n-1}} \leqslant T < e^{2^n} \quad \text{for some} \quad n = 2,3\ldots,$$

and we may estimate as follows:

$$\begin{aligned} 0 \leqslant \tau(T) &\leqslant \frac{1}{2^{n-1}}\sum_{k=1}^{n-1}\int_{e^{2^k}}^{e^{2^k}+1}\frac{x(t)}{t}dt \\ &\leqslant \frac{1}{2^{n-1}}\sum_{k=1}^{n-1} k \to 0 \quad \text{as} \quad T\to\infty. \end{aligned}$$

This proves the existence of the limit (4.2) with $\xi = 0$. □

Our last theorem may also be of some interest. It says that (4.2) implies the convergence of the harmonic average of second order to the same limit.

THEOREM 5. *Assume that* $t^{-1}x(t)\in L^1_{\text{loc}}([1,\infty))$ *and* $(t\log t)^{-1}x(t)\in L^1_{\text{loc}}([e,\infty))$. *Then the implication (4.2) $\Rightarrow$ (4.9) holds true, where*

$$\lim_{T\to\infty} \tau_2(T) = \xi. \tag{4.9}$$

The converse implication is not true in general.

Proof. (i) First, we deal with the special case when T is an integer, say $T := n$ where $n \geqslant 6$. Similary to (4.3), applying the Second Mean Value Theorem gives

$$\tau_2(n)\log\log n = \int_e^4 \frac{x(t)}{t\log t}dt + \sum_{k=4}^{n-1}\int_k^{k+1}\frac{x(t)}{t\log t}dt$$

$$
\begin{aligned}
&=: C+\sum_{k=4}^{n-1}\left\{\frac{1}{\log k}\int_{k}^{b_k}\frac{x(t)}{t}dt+\frac{1}{\log(k+1)}\int_{b_k}^{k+1}\frac{x(t)}{t}dt\right\}\\
&= C+\frac{1}{\log 4}\int_{4}^{b_4}\frac{x(t)}{t}dt+\sum_{k=4}^{n-1}\frac{1}{\log(k+1)}\int_{b_k}^{b_{k-1}}\frac{x(t)}{t}dt\\
&\qquad +\frac{1}{\log n}\int_{b_{n-1}}^{n}\frac{x(t)}{t}dt, \qquad (4.10)
\end{aligned}
$$

where C is an absolute constant and for b_k we have (4.4), but this time $k=4,5,\dots,n-1$.

By the definition of the harmonic average, we may write that

$$
\int_{b}^{c}\frac{x(t)}{t}dt=\tau(c)\log c-\tau(b)\log b, \quad 4<b<c. \qquad (4.11)
$$

Making use of (4.10) and (4.11), we obtain

$$
\begin{aligned}
\tau_2(n)\log\log n &= \int_{e}^{n}\frac{x(t)}{t\log t}dt\\
&= C+\frac{1}{\log 4}\{\tau(b_4)\log b_4-\tau(e)\}\\
&\quad +\sum_{k=4}^{n-2}\frac{1}{\log(k+1)}\{\tau(b_{k+1})\log b_{k+1}-\tau(b_k)\log b_k\}\\
&\quad +\frac{1}{\log n}\{\tau(n)\log n-\tau(b_{n-1})\log b_{n-1})\}\\
&= C-\frac{\tau(e)}{\log 4}+\sum_{k=4}^{n-1}\left(\frac{1}{\log k}-\frac{1}{\log(k+1)}\right)\tau(b_k)\log b_k+\tau(n).
\end{aligned}
$$

Hence it follows that

$$
\begin{aligned}
\tau_2(n) &= \frac{1}{\log\log n}\left\{C-\frac{\tau(e)}{\log 4}+\tau(n)\right\}\\
&\quad +\frac{1}{\log\log n}\sum_{k=4}^{n-1}\left(\frac{1}{\log k}-\frac{1}{\log(k+1)}\right)\tau(b_k)\log b_k\\
&=: \sum_{k=4}^{n-1}a_{n,k}\tau(b_k)+\frac{1}{\log\log n}\left\{C-\frac{\tau(e)}{\log 4}+\tau(n)\right\}. \qquad (4.12)
\end{aligned}
$$

By (4.4), it is not difficult to check that for $k=4,5,\dots,n-1$,

$$
0<a_{n,k}:=\left(\frac{1}{\log k}-\frac{1}{\log(k+1)}\right)\frac{\log b_k}{\log\log n}\to 0 \quad \text{as} \quad n\to\infty
$$

and

$$
\sum_{k=4}^{n-1}a_{n,k}\to 1 \quad \text{as} \quad n\to\infty.
$$

Thus, we may apply Toeplitz' theorem to conclude from (4.2) and (4.12) that the limit (4.9) holds in the special case when $T := n \in \mathbb{N}$.

(ii) In the general case when $T > 6$ is a real number, let $n := [T]$. We use equality (4.11) and the Second Mean Value Theorem to obtain

$$\tau_2(T)\log\log T - \tau_2(n)\log\log n = \int_n^T \frac{x(t)}{t\log t}dt$$

$$= \frac{1}{\log n}\int_n^b \frac{x(t)}{t}dt + \frac{1}{\log T}\int_b^T \frac{x(t)}{t}dt$$

$$= \left(\frac{1}{\log n} - \frac{1}{\log T}\right)\tau_2(b)\log b - \tau_2(n) + \tau_2(T), \quad \text{where} \quad n < b < T. \tag{4.13}$$

Taking into account that $n := [T]$, it follows from (4.2) and (4.13) that

$$|\tau_2(T)\log\log T - \tau_2(n)\log\log n|$$

$$\leqslant \frac{1}{\log n}|\tau_2(b)| + |\tau_2(T) - \tau_2(n)| \to 0 \quad \text{as} \quad T \to \infty.$$

Thus, we conclude that

$$\lim_{T\to\infty} \tau_2(T) = \lim_{T\to\infty} \tau_2(n)\frac{\log\log n}{\log\log T} = \xi,$$

which is (4.9) to be proved.

(iii) In order to prove that the converse implication, that is, (4.9) $\Rightarrow$ (4.2) is false, again we make a trivial observation. A necessary condition for the existence of the finite limit (4.2) is that for every real number $a > 1$,

$$\lim_{T\to\infty} \frac{1}{\log T}\int_T^{aT} \frac{x(t)}{t}dt = 0. \tag{4.14}$$

Indeed, by definition we have

$$\frac{1}{\log T}\int_T^{aT} \frac{x(t)}{t}dt = \frac{\log(aT)}{\log T}\tau(aT) - \tau(T) \to \xi - \xi = 0 \quad \text{as} \quad T \to \infty.$$

Now, we consider the function $x(t)$ defined by

$$x(t) := \begin{cases} n\mu_n & \text{if} \quad t \in [e^{\mu_n}, \left(1+\frac{1}{n}\right)e^{\mu_n}), \\ & \text{where} \quad \mu_n := e^{2^n}, n = 1, 2, \ldots; \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, for $t := e^{\mu_n}$ we have

$$\frac{1}{\log T}\int_T^{2T} \frac{x(t)}{t}dt \geqslant n\log\left(1+\frac{1}{n}\right) \to 1 \quad \text{as} \quad n \to \infty,$$

that is, condition (4.14) is not satisfied. Thus, the finite limit (4.2) cannot exist for any ξ.

On the other hand, if $T \geqslant e^{e^2}$, then

$$e^{\mu_{n-1}} \leqslant T < e^{\mu_n} \quad \text{for some} \quad n = 2, 3, \ldots,$$

and we may estimate as follows:

$$\begin{aligned} 0 \leqslant \tau_2(T) &\leqslant \frac{1}{2^{n-1}} \sum_{k=1}^{n-1} \int_{e^{\mu_k}}^{(1+\frac{1}{k})e^{\mu_k}} \frac{x(t)}{t \log t} dt \\ &\leqslant \frac{1}{2^{n-1}} \sum_{k=1}^{n-1} k \log\left(1 + \frac{1}{k}\right) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This proves the existence of the limit (4.9) with $\xi = 0$. □

REMARK. On closing, we note that a trivial necessary condition for the existence of the finite limit (4.9) is that for every real number $a > 1$,

$$\lim_{T \to \infty} \frac{1}{\log \log T} \int_T^{T^a} \frac{x(t)}{t \log t} dt = 0.$$

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