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A NOTE ON THE TRIANGLE INEQUALITY FOR THE C^*–VALUED NORM ON A HILBERT C^*–MODULE

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Abstract. The C^*-valued norm is defined on a Hilbert C^*-module by its standard inner product. In this short note, we will prove that $|x+y| \leqslant |x|+|y|$ holds for all x, $y \in E$ which is a Hilbert $\mathscr{A}$–module if and only if $\mathscr{J} = \overline{\langle E,E\rangle}$, the closed two-sided ideal in $\mathscr{A}$, is a commutative C^*-algebra.

1. Introduction

Let $\mathscr{A}$ be a C^*-algebra and let a, b be in $\mathscr{A}$. Set $|a| = (a^*a)^{1/2}$, $|b| = (b^*b)^{1/2}$ and $|a+b| = ((a+b)^*(a+b))^{1/2}$. Does the following triangle inequality

$$|a+b| \leqslant |a|+|b| \tag{1.1}$$

hold ? R. Harte (cf. [4]) shows that for certain a and b, (1.1) does not hold. But if $\mathscr{A}$ has a unit e, then for every a, $b \in \mathscr{A}$ and $\varepsilon > 0$, there are unitaries u, $v \in \mathscr{A}$ such that $|a+b| \leqslant u|a|u^* + v|b|v^* + \varepsilon e$ (cf. [1]).

Recall that a (right) $\mathscr{A}$–module E is called the Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ if there is an $\mathscr{A}$–valued mapping (a $\mathscr{A}$–valued “inner product”) $\langle\cdot,\cdot\rangle$ on $E\times E$ such that

(1) $\langle x, \alpha y+\beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$, $\forall x,y,z \in E$ and $\forall \alpha$, $\beta \in \mathbb{C}$,

(2) $\langle x,ya\rangle = \langle x,y\rangle a$ and $\langle x,y\rangle = \langle y,x\rangle^*$, $\forall x,y \in E$ and $\forall a \in \mathscr{A}$,

(3) $\langle x,x\rangle \geqslant 0$, $\forall x \in E$; if $\langle x,x\rangle = 0$, then $x = 0$,

and E is completed with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$, $\forall x \in E$. For $x \in E$, put $|x| = (\langle x,x\rangle)^{1/2}$ (cf. [6]). Lj. Arambašić and R. Rajić(cf. [2]) proved that if $\mathscr{A}$ is a C^*-algebra with a unit e and E is a Hilbert C^*-module over $\mathscr{A}$, then for every x, $y \in E$ and $\varepsilon > 0$, there are a, $b \in \mathscr{A}$ satisfying $\|a\| \leqslant 1$ and $\|b\| \leqslant 1$ such that $|x+y| \leqslant a|x|a^* + b|y|b^* + \varepsilon e$. Now we consider the following problem: when does

$$|x+y| \leqslant |x|+|y| \tag{1.2}$$

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hold for $x, y \in E$? B. Kolarec (cf. [5]) proves that if $x,\ y \in E$ and $|x|,\ |y| \in Z(\mathscr{A})$, then (1.2) holds, where $Z(\mathscr{A})$ is the center of $\mathscr{A}$.

We say that E has the property of C^*-valued triangle inequality, denoted as (TIP) if (1.2) holds for all $x, y \in E$.

In this short note we will prove that E has (TIP) if and only if $\overline{\langle E,E\rangle} = \overline{span}\{\langle x,y\rangle \mid x,y \in E\}$ is a commutative C^*-subalgebra of $\mathscr{A}$. Throughout the paper, $\mathscr{A}$ is always a C^*-algebra, $\mathscr{A}_{sa} = \{a \in \mathscr{A} | a^* = a\}$ and $\mathscr{A}_+ = \{a \in \mathscr{A} | a \geqslant 0\}$.

2. The triangle inequality on a C^*-algebra

Given a C^*-algebra $\mathscr{A}$, $\mathscr{A}''$ is the enveloping Von Neumann algebra of $\mathscr{A}$. For convenience, we assume that $\mathscr{A}$ is the C^*-subalgebra of $B(H)$ for certain complex Hilbert space H such that $\{a\xi \,|\, a \in \mathscr{A}, \xi \in H\}$ is dense in H, where $B(H)$ is the C^*-algebra consisting of all bounded linear operators from H to H. Then $\mathscr{A}$ is strongly dense in $\mathscr{A}''$ (cf. [7, Lemma 4.1.4]). For $T \in B(H)$, let $\mathrm{Ran}(T)$ (resp. $\mathrm{Ker}\,T$) denote the range (resp. null space) of T.

We say that the net $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq B(H)$ converges $*$–strongly to an operator $A \in B(H)$, if $\{A_\lambda\}_{\lambda\in\Lambda}$ converges strongly to A and $\{A_\lambda^*\}_{\lambda\in\Lambda}$ converges strongly to A^*.

Recall that a continuous function $f: \mathbb{R} \to \mathbb{C}$ is called to be strongly continuous if for every Hilbert space H and each net $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq B(H)_{sa}$ converging strongly to an operator $A \in B(H)_{sa}$, we have $\{f(A_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to $f(A)$.

The following lemma comes from [7, Theorem 4.3.2].

LEMMA 2.1. *If $f: \mathbb{R} \to \mathbb{C}$ is a continuous bounded function, then f is strongly continuous.*

LEMMA 2.2. *Suppose that $|a+b| \leqslant |a| + |b|$, $\forall a, b \in \mathscr{A}$. Then for any $a, b \in \mathscr{A}''$, $|a+b| \leqslant |a| + |b|$.*

Proof. Let $a, b \in \mathscr{A}''$. By Kaplansky's density theorem (cf. [7, Theorem 4.3.3]), there are four nets $\{a_\lambda^{'}\}_{\lambda\in\Lambda}$, $\{a_\lambda^{''}\}_{\lambda\in\Lambda}$, $\{b_\lambda^{'}\}_{\lambda\in\Lambda}$ and $\{b_\lambda^{''}\}_{\lambda\in\Lambda}$ in $\mathscr{A}_{sa}$ with $\|a_\lambda^{'}\|$, $\|a_\lambda^{''}\| \leqslant \|a\|$ and $\|b_\lambda^{'}\|$, $\|b_\lambda^{''}\| \leqslant \|b\|$, $\forall \lambda \in \Lambda$ such that $a_\lambda^{'}$ converges strongly to $Re(a)$; $a_\lambda^{''}$ converges strongly to $Im(a)$; $b_\lambda^{'}$ converges strongly to $Re(b)$ and $b_\lambda^{''}$ converges strongly to $Im(b)$. Set $a_\lambda = a_\lambda^{'} + ia_\lambda^{''}$ and $b_\lambda = b_\lambda^{'} + ib_\lambda^{''}$. Thus a_λ converges $*$–strongly to a and b_λ converges $*$–strongly to b. So $\{a_\lambda^* a_\lambda\}_{\lambda\in\Lambda}$, $\{b_\lambda^* b_\lambda\}_{\lambda\in\Lambda}$ and $\{(a_\lambda^* + b_\lambda^*)(a_\lambda + b_\lambda)\}_{\lambda\in\Lambda}$ are bounded nets of $\mathscr{A}$ with $s - \lim_\lambda a_\lambda^* a_\lambda = a^*a$, $s - \lim_\lambda b_\lambda^* b_\lambda = b^*b$ and

$$(a^* + b^*)(a+b) = s - \lim_\lambda (a_\lambda^* + b_\lambda^*)(a_\lambda + b_\lambda).$$

Set $M = 100(\|a\|^2 + \|b\|^2)^{1/2}$ and define a continuous real-valued function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x \leqslant 0 \\ \sqrt{x} & 0 < x \leqslant M^2 \\ M & x > M^2 \end{cases}.$$

Since $0 \leqslant \|a\|,\ \|b\|,\ \|a+b\| \leqslant M$ and $0 \leqslant \|a_\lambda\|,\ \|b_\lambda\|,\ \|a_\lambda+b_\lambda\| \leqslant M$, $\forall \lambda \in \Lambda$, we have

$$f(a^*a) = |a|,\ f(b^*b) = |b|,\ f(a_\lambda^* a_\lambda) = |a_\lambda|,\ f(b_\lambda^* b_\lambda) = |b_\lambda|, \tag{2.1}$$

$$f((a^*+b^*)(a+b)) = |a+b|,\ f((a_\lambda^*+b_\lambda^*)(a_\lambda+b_\lambda)) = |a_\lambda+b_\lambda|. \tag{2.2}$$

Note that $f(a_\lambda^* a_\lambda)$ converges strongly to $f(a^*a)$, $f(b_\lambda^* b_\lambda)$ converges strongly to $f(b^*b)$ and $f((a_\lambda^*+b_\lambda^*)(a_\lambda+b_\lambda))$ converges strongly to $f((a^*+b^*)(a+b))$ by Lemma 2.1 and $|a_\lambda+b_\lambda| \leqslant |a_\lambda|+|b_\lambda|$, $\forall \lambda \in \Lambda$. So $|a+b| \leqslant |a|+|b|$ by using (2.1) and (2.2). □

PROPOSITION 2.3. *For the C^*-algebra $\mathscr{A}$, the following statements are equivalent:*

(1) *$\mathscr{A}$ is not commutative;*

(2) *$\mathscr{A}''$ is not commutative;*

(3) *There exists a C^*-subalgebra $\mathscr{B}$ in $\mathscr{A}''$ such that $\mathscr{B}$ is $*$–isomorphic to $\mathrm{M}_2(\mathbb{C})$.*

Proof. The implications (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (2) are obvious. We now prove (2) $\Rightarrow$ (3).

Since $\mathscr{A}''$ is not commutative, it follows from [3, 2.12.21] that $\mathscr{A}''$ contains a non–zero nilpotent element a (i.e., $a^2 = 0 \neq a$). Thus, $\mathrm{Ran}(a) \subset \mathrm{Ker}\, a$.

Let $a = u(a^*a)^{\frac{1}{2}}$ be the polar decomposition of a in $\mathscr{A}''$, where $u \in \mathscr{A}''$ is a partial isometry with $\mathrm{Ran}(u) = \overline{\mathrm{Ran}(a)}$ and $\mathrm{Ker}\, u = \mathrm{Ker}\, a$ (cf. [7, Theorem 4.1.10]). Let $\mathscr{B}$ be the C^*-algebra generated by u and u^*. Put $p_1 = u^*u$ and $p_2 = uu^*$. From $\mathrm{Ran}(a) \subset \mathrm{Ker}\, a$, we have $u^2 = 0$ and hence $p_1p_2 = 0$. Note that $p_1u = 0$, $up_1 = u$, $p_2u = u$ and $up_2 = 0$. So $\mathscr{B}$ has the form $\mathscr{B} = \mathrm{span}\{u, u^*, p_1, p_2\}$.

Define a mapping $\psi \colon \mathscr{B} \to \mathrm{M}_2(\mathbb{C})$ by

$$\psi(\lambda_1 u + \lambda_2 u^* + \lambda_3 p_1 + \lambda_4 p_2) = \begin{pmatrix} \lambda_3 & \lambda_2 \\ \lambda_1 & \lambda_4 \end{pmatrix}.$$

It is easy to check that ψ is a $*$–isomorphism. □

As a consequence of Proposition 2.3 and Lemma 2.2, we have the following result.

COROLLARY 2.4. *If for all $a, b \in \mathscr{A}$, $|a+b| \leqslant |a|+|b|$, then $\mathscr{A}$ must be a commutative C^*-algebra.*

Proof. By Lemma 2.2, $|a+b| \leqslant |a|+|b|$, $\forall a, b \in \mathscr{A}''$. So if $\mathscr{A}$ is not commutative, then $\mathscr{A}''$ has a C^*-subalgebra $\mathscr{B}$ which is $*$–isomorphic to $\mathrm{M}_2(\mathbb{C})$ by Proposition 2.3. It implies that

$$|a+b| \leqslant |a|+|b|, \quad \forall a, b \in \mathrm{M}_2(\mathbb{C}).$$

But it is not true. Set $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We may obtain

$$|a| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ |b| = b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ |a+b| = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

and $|a|+|b|-|a+b| = \begin{pmatrix} 1-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} \end{pmatrix}$ is not a positive matrix. □

3. The C^*-valued triangle inequality in a C^*-module

If E is a Hilbert $\mathscr{B}$–module and $\mathscr{A}$ is a C^*-algebra containing $\overline{\langle E,E\rangle}$ as an ideal, then there is a way to make E into a Hilbert $\mathscr{A}$–module without changing the "inner product". Namely, let $\{\mu_i\}$ be an approximate unit for $\overline{\langle E,E\rangle}$. If $a \in \mathscr{A}$, $e \in E$, then the identity

$$\begin{aligned}\langle e\mu_j a - e\mu_i a, e\mu_j a - e\mu_i a\rangle = a^*\mu_j\langle e,e\rangle\mu_j a + a^*\mu_i\langle e,e\rangle\mu_i a \\ - a^*\mu_j\langle e,e\rangle\mu_i a - a^*\mu_i\langle e,e\rangle\mu_j a\end{aligned}$$

shows that $\{e\mu_i a\}$ converges in E. We can define $ea = \lim_i e\mu_i a$, and it is straightforward to check that this makes E into a Hilbert $\mathscr{A}$–module. (In fact, it is the only possible way to make E into a Hilbert $\mathscr{A}$–module with the same "inner product".) So every Hilbert $\mathscr{A}$–module is also a Hilbert $\widetilde{\mathscr{A}}$–module, where $\widetilde{\mathscr{A}}$ is the unitization of $\mathscr{A}$.

LEMMA 3.1. *Let E be a Hilbert $\mathscr{A}$–module with* (TIP). Put $\mathscr{I} = \overline{\langle E,E\rangle}$. Then for each nonzero element e in E, $\overline{|e|\mathscr{I}|e|}$ is a commutative C^*-algebra.

Proof. For convenience, we assume $\mathscr{A}$ has the unit 1. (Otherwise, E can be regarded as a Hilbert $\widetilde{\mathscr{A}}$–module.)

Set $x_n = e(|e|^2+n^{-2})^{-\frac{1}{2}}$ $(n \in \mathbb{N})$. Then

$$c_n = \langle x_n, x_n\rangle = |e|^2(|e|^2+n^{-2})^{-1} \leqslant 1$$

and

$$(1-c_n)\mid e\mid^2(1-c_n) = n^{-4}\mid e\mid^2(\mid e\mid^2+n^{-2})^{-2} \leqslant \frac{1}{4}n^{-2}. \tag{3.1}$$

From (3.1), we get that $\lim_{n\to\infty}\|(1-c_n)|e|\| = 0$. Hence for every $a \in \overline{\mid e\mid \mathscr{I} \mid e\mid}$, $\lim_{n\to\infty} a^*(1-c_n)a = 0$. Since $\sqrt{x}$ is a continuous mapping on $\mathscr{A}_+$, we have $\lim_{n\to\infty}|x_n a| = |a|$, $\forall a \in \overline{\mid e\mid \mathscr{I} \mid e\mid}$.

Since E has (TIP), it follows that

$$|x_n(a+b)| \leqslant |x_n a| + |x_n b|, \quad \forall a, b \in \overline{\mid e\mid \mathscr{I} \mid e\mid}. \tag{3.2}$$

Note that $\lim_{n\to\infty}|x_n a| = |a|$ and $\lim_{n\to\infty}|x_n b| = |b|$. So we have $|a+b| \leqslant |a|+|b|$ by letting $n\to\infty$ in (3.2), $\forall a,b \in \overline{|e|\mathscr{I}|e|}$.

Therefore, $\overline{|e|\mathscr{I}|e|}$ is a commutative C^*-algebra by Corollary 2.4. □

Let E be a Hilbert $\mathscr{A}$–module and $f\colon \mathscr{A}\to\mathscr{B}$ be a surjective $*$–homomorphism of C^*-algebras. Define a Hilbert submodule N_f of E by $N_f = \{x\in E | f(\langle x,x\rangle)=0\}$. Set $E_f' = E/N_f$ and let $q\colon E\to E_f'$ denote the quotient mapping. Then E_f' is a right $\mathscr{B}$–module by defining $q(x)f(a) = q(xa)$, $x\in E$, $a\in\mathscr{A}$. With a $\mathscr{B}$–valued "inner product" defined by

$$\langle q(x),q(y)\rangle = f(\langle x,y\rangle), \quad x,y\in E,$$

E_f' becomes a pre–Hilbert $\mathscr{B}$–module. Let E_f denote the Hilbert $\mathscr{B}$–module obtained from E_f' by completion. The next lemma is clear.

LEMMA 3.2. *Let E be a Hilbert $\mathscr{A}$–module and let $f\colon\mathscr{A}\to\mathscr{B}$ be a surjective $*$–homomorphism. If E has* (TIP), then E_f must have (TIP).

LEMMA 3.3. *Let $\mathscr{A}$ be a C^*-algebra. Then the following conditions are equivalent:*

(1) *$\mathscr{A}$ is commutative.*

(2) *Every nonzero irreducible representation of $\mathscr{A}$ is one–dimensional.*

Proof. (1) $\Rightarrow$ (2). Let (H,π) be a nonzero irreducible representation of $\mathscr{A}$, and let ξ be a nonzero element of H. So $\mathscr{B}=\pi(\mathscr{A})$ is a commutative C^*-subalgebra of $B(H)$. $\mathscr{B}'$, the set of all elements of $B(H)$ that commute with all the elements of $\mathscr{B}$, must be $\mathbb{C}I$, where I is the unit of $B(H)$. But $\mathscr{B}\subseteq\mathscr{B}'=\mathbb{C}I$, and $\mathscr{B}\xi = H$ which follows from [7, Theorem 5.1.5]. So we have $H = span\{\xi\}$, H is one–dimensional.

(2) $\Rightarrow$ (1). Let a, b be two nonzero elements of $\mathscr{A}$. Then by [7, Theorem 5.1.12], there is an irreducible representation (H,π) of $\mathscr{A}$ such that $\|ab-ba\| = \|\pi(ab-ba)\| = \|\pi(a)\pi(b)-\pi(b)\pi(a)\|$. If (H,π) is zero, then $ab=ba$. Otherwise H is one–dimensional, which means π is a $*$–homomorphism from $\mathscr{A}$ to $\mathbb{C}$, and $\|\pi(a)\pi(b)-\pi(b)\pi(a)\|=0$. Consequently, $ab=ba$. □

THEOREM 3.4. *Let E be a Hilbert $\mathscr{A}$–module. If E has* (TIP), then $\mathscr{I}=\overline{\langle E,E\rangle}$ is a commutative C^*-algebra.

Proof. It is obvious that every Hilbert $\mathscr{A}$–module is also a Hilbert $\mathscr{I}$–module. So for convenience, we assume $\mathscr{A}=\mathscr{I}$.

Let (H,π) be a nonzero irreducible representation of $\mathscr{I}$ and put $\mathscr{B}=\pi(\mathscr{I})$. Then we get the Hilbert $\mathscr{B}$–module E_π. Let $q\colon E\to E_\pi'$ denote the quotient mapping. By polarization identity, there is $x\in E$ such that $\pi(\langle x,x\rangle)\neq 0$ (i.e. $q(x)\neq 0$). For convenience, we assume that $\|q(x)\|=1$. Since $D=\overline{|x|\mathscr{I}|x|}$ is a hereditary C^*-subalgebra of $\mathscr{I}$, it follows from [7, Theorem 5.5.2] that the restriction $(H,\pi)_D$ of the representation (H,π) to D is an irreducible representation of D. Note that D is a commutative C^*-algebra by Lemma 3.1. So $K=\overline{\pi(D)H}$ must be one–dimensional by Lemma 3.3 and $\langle q(x),q(x)\rangle=\pi(\langle x,x\rangle)$ is a rank–one projection of H.

We now show that $\dim H=1$. From the above, $0\neq\pi(\langle x,x\rangle)\in\mathscr{B}\cap K(H)$, so $\mathscr{B}\cap K(H)\neq\{0\}$, where $K(H)$ is the C^*-algebra consisting of all compact linear operators from H to H. $\mathscr{B}$ acts irreducibly on H, so $\mathscr{B}\supseteq K(H)$ by [7, Theorem 2.4.9]. If $\dim H\geqslant 2$, we can find a rank–one projection $p\in K(H)$ such that $p|q(x)|=0$. Then there is a non-zero nilpotent element $v\in K(H)\subseteq\mathscr{B}$ such that $v^*v=|q(x)|$, $vv^*=p$. Set $a=q(x)$, $b=q(x)v^*\in E_\pi$. By Lemma 3.2,

$$|q(x)+q(x)v^*|=|a+b|\leqslant|a|+|b|=|q(x)|+|q(x)v^*|. \tag{3.3}$$

Since $|q(x)|+|q(x)v^*|=|q(x)|+p$ is a projection and

$$\frac{1}{2}|q(x)+q(x)v^*|^2=\frac{1}{2}(|q(x)|+p+v+v^*)$$

is also a projection, we have $\||a+b|\|=\sqrt{2}>1=\||a|+|b|\|$, which contradicts to (3.3). So $\dim H=1$.

We conclude from above that every nonzero irreducible representation of $\mathscr{I}$ is one–dimensional and consequently, $\mathscr{I}$ is a commutative C^*-algebra by Lemma 3.3. □

LEMMA 3.5. *Let E be a Hilbert $\mathscr{A}$–module, and $x,y\in E$. Then there is an element $c\in\overline{\langle E,E\rangle}''$ with $\|c\|\leqslant 1$ such that $\langle x,y\rangle=|x|c|y|$.*

Proof. Assume that $\langle E,E\rangle''$ acts on a Hilbert space H. For x, $y\in E$, set

$$c_n=(|x|+n^{-1})^{-1}\langle x,y\rangle(|y|+n^{-1})^{-1}(n\in\mathbb{N}).$$

Then for any $n\geqslant 1$,

$$\begin{aligned}
\|c_n\|^2&=\|c_n^*c_n\|\\
&=\|\langle y(|y|+n^{-1})^{-1},x(|x|+n^{-1})^{-1}\rangle\langle x(|x|+n^{-1})^{-1},y(|y|+n^{-1})^{-1}\rangle\|\\
&\leqslant\|x(|x|+n^{-1})^{-1}\|^2\|y(|y|+n^{-1})^{-1}\|^2\\
&=\||x|(|x|+n^{-1})^{-1}\|^2\||y|(|y|+n^{-1})^{-1}\|^2\\
&\leqslant 1,
\end{aligned}$$

that is, $\{c_n\}$ is bounded. Thus, it has a weakly limit point in the unit ball of $B(H)$, say, c. Since $|x|=\lim_{n\to\infty}(|x|+n^{-1})$, $|y|=\lim_{n\to\infty}(|y|+n^{-1})$ in norm and $(|x|+n^{-1}))c_n(|y|+n^{-1})=\langle x,y\rangle$, it follows that $|x|c|y|=\langle x,y\rangle$. □

Finally, we characterize (TIP) in Hilbert C^*-modules as follows.

THEOREM 3.6. *Let E be a Hilbert $\mathscr{A}$–module. Then the following conditions are equivalent:*

(1) *E has* (TIP).

(2) $\mathscr{I} = \overline{\langle E, E\rangle}$ is a commutative C^*-algebra.

Proof. (1) $\Rightarrow$ (2) is Theorem 3.4. We now prove (2) $\Rightarrow$ (1).

Condition (2) implies that $\mathscr{I}''$ is commutative by Proposition 2.3. Thus, by Lemma 3.5, for $x, y \in E$ we get

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 + \langle x, y\rangle + \langle y, x\rangle \\ &= |x|^2 + |y|^2 + |x|c|y| + |y|c^*|x| \\ &= |x|^2 + |y|^2 + c|x||y| + c^*|x||y| \\ &\leqslant |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2 \end{aligned}$$

and consequently, $|x+y| \leqslant |x| + |y|$. □

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