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REMARKS ON THE NUMBER OF PRIME DIVISORS OF INTEGERS

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Abstract. In this paper, we obtain explicit bounds for sums $\sum_{k\leqslant n}\omega(k)$ and $\sum_{k\leqslant n}\Omega(k)-\omega(k)$, where $\omega(k)$ denotes the number of distinct prime divisors of k, and $\Omega(k)$ denotes the total number of its prime divisors. Moreover, we give some better explicit bounds for the sum $\sum_{k\leqslant n}\omega(k)$ under assumption of the Riemann hypothesis.

1. Introduction

As usual, we let $\omega(k)=\sum_{p|k}1$ be the number of distinct prime divisors of the positive integer k. Also, we let $\Omega(k)=\sum_{p^\alpha\|k}\alpha$ be the total number of prime divisors of k. In 1917, Hardy and Ramanujan [3] proved the following average results

$$\sum_{k\leqslant n}\omega(k)=n\log\log n+Mn+O\Big(\frac{n}{\log n}\Big), \tag{1.1}$$

and

$$\sum_{k\leqslant n}\Omega(k)=n\log\log n+M'n+O\Big(\frac{n}{\log n}\Big). \tag{1.2}$$

As a consequence, they obtained

$$\sum_{k\leqslant n}\Omega(k)-\omega(k)=(M'-M)n+O\Big(\frac{n}{\log n}\Big)=n\sum_p\frac{1}{p(p-1)}+O\Big(\frac{n}{\log n}\Big). \tag{1.3}$$

Here, M and M' are constants defined by

$$M=\gamma+\sum_p\left(\log\left(1-p^{-1}\right)+p^{-1}\right)=0.2614972128\cdots,$$

and

$$M'=\gamma+\sum_p\left(\log\left(1-p^{-1}\right)+(p-1)^{-1}\right)=1.0346538818\cdots,$$

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and γ refers to Euler's constant. The constant M is known as the Meissel-Mertens constant [2]. To compute M and M' (and consequently $M'-M$) we may apply rapidly converging series

$$M=\gamma+\sum_{k=2}^{\infty}\frac{\mu(k)\log\zeta(k)}{k}, \qquad \text{and} \qquad M'=\gamma+\sum_{k=2}^{\infty}\frac{\varphi(k)\log\zeta(k)}{k}. \tag{1.4}$$

Recently, we obtained an explicit approximation for the sum of prime powers in the factorization of $n!$ into prime numbers [4], from which we could imply an explicit version of (1.2), as follows.

PROPOSITION 1.1. *For $n\geqslant 2$ we have*

$$n\log\log n+M'n-\frac{814921n}{\log n}<\sum_{k\leqslant n}\Omega(k)<n\log\log n+M'n+\frac{n}{\log^2 n}. \tag{1.5}$$

Our aim in this paper is to obtain similar explicit versions of (1.1) and (1.3). More precisely, we prove the following.

THEOREM 1.2. *For $n\geqslant 2$ we have*

$$n\log\log n+Mn-\frac{3.8854n}{\log n}<\sum_{k\leqslant n}\omega(k)<n\log\log n+Mn+\frac{n}{\log^2 n}. \tag{1.6}$$

As a corollary of this theorem and Proposition 1.1, we find some explicit bounds for $\sum_{k\leqslant n}\Omega(k)-\omega(k)$.

COROLLARY 1.3. *For $n\geqslant 2$ we have*

$$(M'-M)n-\frac{814922.4427n}{\log n}<\sum_{k\leqslant n}\Omega(k)-\omega(k)<(M'-M)n+\frac{5.3281n}{\log n}. \tag{1.7}$$

Finally, we find some bounds for $\sum_{k\leqslant n}\omega(k)$ under assumption of validity of the Riemann hypothesis.

THEOREM 1.4. *Assume that the Riemann hypothesis is true. For $n\geqslant 7$ we have*

$$\sum_{k\leqslant n}\omega(k)<n\log\log n+Mn+0.18\sqrt{n}\log n. \tag{1.8}$$

THEOREM 1.5. *Assume that the Riemann hypothesis is true. For every real number $\eta>1$ there exists sufficiently large integer $n_\eta>0$ such that*

$$\sum_{k\leqslant n}\omega(k)>n\log\log n+Mn-\frac{\eta n}{\log n} \qquad (\text{for } n\geqslant n_\eta). \tag{1.9}$$

More precisely, we have

$$n_\eta=\min_{\substack{n\in\mathbb{Z}\\ n\geqslant x}}\max_{x\in\mathbb{R}}\left\{\mathrm{Li}(x)+\frac{1}{2\pi}\sqrt{x}\log(ex)=\frac{\eta x}{\log x}\right\},$$

where $\mathrm{Li}(x)$ denotes the logarithmic integral function.

In Section 5, we recall definition of the function $\mathrm{Li}(x)$, as more as, we list some values for η and related values of n_η.

Before starting proofs, we recall a generalization of relations (1.1) and (1.2) due to Duncan [1], where for positive real number δ he defines

$$\Omega_\delta(k) = \sum_{p^\alpha \| k} \alpha^\delta, \qquad \text{and} \qquad \mathscr{S}_\delta(n) = \sum_{k \leqslant 1} \Omega_\delta(k).$$

We note that $\Omega_0(k) = \omega(k)$ and $\Omega_1(k) = \Omega(k)$. Duncan proves that for each $\delta \geqslant 0$, there exists an absolute constant M_δ, such that

$$\mathscr{S}_\delta(n) = n\log\log n + M_\delta n + O\Big(\frac{n}{\log n}\Big).$$

In fact, our results (1.5) and (1.6) give some explicit bounds for $\mathscr{S}_1(n)$ and $\mathscr{S}_0(n)$, respectively. A natural question arising to mind is obtaining similar explicit bounds for the summation $\mathscr{S}_\delta(n)$.

Also, we note that $M_0 = M$ and $M' = M_1$, and we have rapidly converging series representations for M and M' as stated in (1.4). Another question is obtaining a similar rapidly converging series for M_δ. However, we have not more evidences than equalities in (1.4), but we guess that for given $\delta \geqslant 0$ there exists an arithmetical function f_δ such that $M_\delta = \gamma + \sum_{k=2}^{\infty}(f_\delta(k)\log\zeta(k))/k$.

2. Proof of Theorem 1.2

Let us denote integer part and fractional part of the real number x by $[x]$ and $\{x\}$, respectively. We set $\mathscr{A}(n) := \sum_{p\leqslant n}\frac{1}{p}$. Then, we have

$$\sum_{k\leqslant n}\omega(k) = \sum_{k\leqslant n}\sum_{p|k}1 = \sum_{p\leqslant n}\sum_{\substack{k\leqslant n\\ p|k}}1 = \sum_{p\leqslant n}\left[\frac{n}{p}\right] = \sum_{p\leqslant n}\left(\frac{n}{p} - \left\{\frac{n}{p}\right\}\right) = n\mathscr{A}(n) - \mathscr{R}(n),$$

say. Since $0 \leqslant \{\frac{n}{p}\} < 1$, we have $0 \leqslant \mathscr{R}(n) < \pi(n)$, where as usual, $\pi(n)$ denotes the number of prime numbers not exceeding n. Thus, we obtain

$$n\mathscr{A}(n) - \pi(n) < \sum_{k\leqslant n}\omega(k) \leqslant n\mathscr{A}(n), \qquad (\text{for } n \geqslant 1). \tag{2.1}$$

Theorem 1 from [5], gives the bound

$$\pi(n) < \frac{n}{\log n}\left(1 + \frac{3}{2\log n}\right)$$

for $n \geqslant 2$. Thus, for $n \geqslant 2$ we have $-\frac{n}{\log n}(1+\frac{3}{2\log n}) < -\mathscr{R}(n) \leqslant 0$. Also, Theorem 5 from [5] implies validity of

$$\log\log n + M - \frac{1}{2\log^2 n} < \mathscr{A}(n) < \log\log n + M + \frac{1}{\log^2 n},$$

for every $n \geqslant 2$. If we combine these inequalities, on one hand we obtain the right hand side of (1.6), and on the other hand, we imply

$$\sum_{k \leqslant n} \omega(k) > n \log\log n + Mn - \frac{n}{\log n} - \frac{2n}{\log^2 n} \qquad (\text{for } n \geqslant 2). \tag{2.2}$$

To remove the term including square of log, we note that the inequality $-\frac{2n}{\log^2 n} \geqslant -\frac{\beta n}{\log n}$ is equivalent to $n \geqslant e^{\frac{2}{\beta}}$. So, we take $e^{\frac{2}{\beta}} = 2$ or equivalently $\beta = \frac{2}{\log 2}$. Thus, we obtain

$$\sum_{k \leqslant n} \omega(k) > n \log\log n + Mn - (1+\beta)\frac{n}{\log n} \qquad (\text{for } n \geqslant 2). \tag{2.3}$$

But, we have $1+\beta < 3.8854$. This, completes the proof of Theorem 1.2.

3. Proof of Corollary 1.3

We consider both sides of (1.5), the right hand side of (1.6), and also the inequality (2.2) to obtain

$$-\left(\frac{814921n}{\log n} + \frac{n}{\log^2 n}\right) < \left(\sum_{k \leqslant n} \Omega(k) - \omega(k)\right) - (M' - M)n < \left(\frac{n}{\log n} + \frac{3n}{\log^2 n}\right),$$

for $n \geqslant 2$. For $\lambda_1 > 1$, the inequality

$$\frac{n}{\log n} + \frac{3n}{\log^2 n} \leqslant \frac{\lambda_1 n}{\log n},$$

is equivalent to $n \geqslant e^{\frac{3}{\lambda_1 - 1}}$. By taking $e^{\frac{3}{\lambda_1 - 1}} = 2$ we get $\lambda_1 = 1 + \frac{3}{\log 2} < 5.3281$. This completes implication of the right hand side of (1.7). To prove the left hand side of (1.7), we set $c := 814921$, and we note that for $\lambda_2 > c$, the inequality

$$-\left(\frac{814921n}{\log n} + \frac{n}{\log^2 n}\right) \geqslant -\frac{\lambda_2 n}{\log n},$$

is equivalent to $n \geqslant e^{\frac{1}{\lambda_2 - c}}$. By letting $e^{\frac{1}{\lambda_2 - c}} = 2$ we obtain $\lambda_2 = c + \frac{1}{\log 2} < 814922.4427$, and this completes the proof of the left hand side of (1.7).

4. Proof of Theorem 1.4

Corollary 2 of [6] asserts that under validity of the Riemann hypothesis, we have

$$\log\log n + M - \mathscr{B}(n) < \mathscr{A}(n) < \log\log n + M + \mathscr{B}(n), \tag{4.1}$$

for $n \geqslant 14$, where $\mathscr{B}(n) = (3\log n + 4)/(8\pi\sqrt{n})$. By considering the right hand sides of (2.1) and (4.1), we imply

$$\sum_{k \leqslant n} \omega(k) < n \log\log n + Mn + \frac{3}{8\pi}\sqrt{n}\log n + \frac{1}{2\pi}\sqrt{n}, \qquad (\text{for } n \geqslant 14).$$

Now, we note that the inequality $\frac{1}{2\pi}\sqrt{n} \leqslant \eta_1\sqrt{n}\log n$ is equivalent to $n \geqslant e^{\frac{1}{2\pi\eta_1}}$. So, by taking $e^{\frac{1}{2\pi\eta_1}} = 14$ we get $\eta_1 = \frac{1}{2\pi\log 14}$. Thus, for $n \geqslant 14$ we obtain

$$\sum_{k\leqslant n}\omega(k) < n\log\log n + Mn + \left(\frac{3}{8\pi} + \frac{1}{2\pi\log 14}\right)\sqrt{n}\log n.$$

The right hand side of last inequality, is less than $\mathscr{U}(n) := n\log\log n + Mn + 0.18\sqrt{n}\log n$. Therefore, we get the validity of (1.8) for $n \geqslant 14$. In the following table, we compare the values of $S(n) := \sum_{k\leqslant n}\omega(k)$ and $\mathscr{U}(n)$ for $2 \leqslant n \leqslant 13$, from which we observe that (1.8) is valid for $7 \leqslant n \leqslant 13$, too.

n	2	3	4	5	6	7	8	9	10	11	12	13
$\omega(n)$	1	1	1	1	2	1	1	1	2	1	2	1
$S(n)$	1	2	3	4	6	7	8	9	11	12	14	15
$\mathscr{U}(n) >$	-0.04	1.4	2.8	4.3	5.8	7.4	9.0	10.6	12.2	13.9	15.6	17.3

REMARK 4.1. Meanwhile, above table shows that all solutions of the equation

$$\sum_{k\leqslant n}\omega(k) = n,$$

are integers $n = 6,7,8,9$. More generally, the relation (1.1) guarantees that for every positive integer $m \geqslant 1$, the equation $\sum_{k\leqslant n}\omega(k) = mn$ (in n) has only a finite number of solutions. Theorem 1.2 gives an upper bound $F(m)$ for the maximum value of such solutions, which is trivially an upper bound for the number of them, too. Indeed, by considering (2.3) we may take $F(m)$ to be the largest solution of the following equation (in n)

$$\log\log n + M - (1+\beta)\frac{1}{\log n} = m.$$

It in not very hard to verify that this solution is $F(m) = e^{g(m)}$, with

$$g(m) = \frac{1+\beta}{W((1+\beta)e^{M-m})},$$

where $W(x)$ denotes the Lambert W function, which is defined by $W(x)e^{W(x)} = x$ for $x \geqslant -e^{-1}$. The following table shows some few values of $g(m)$ and its very fast growth.

m	$g(m) \approx$	m	$g(m) \approx$	m	$g(m) \approx$
1	4.75	6	314.46	11	46100.87
2	8.83	7	848.17	12	125308.48
3	18.98	8	2298.92	13	340617.08
4	45.76	9	6242.43	14	925886.55
5	118.09	10	16962.02	15	2516813.90

5. Proof of Theorem 1.5

To deduce (1.9), we recall Corollary 1 of [6], which asserts that if the Riemann hypothesis is true then we have

$$\pi(n) < \mathrm{Li}(n) + \frac{\sqrt{n}\log n}{8\pi}, \qquad (\text{for } n \geqslant 2), \tag{5.1}$$

where

$$\mathrm{Li}(x) = \lim_{\varepsilon \to 0^+}\left(\int_0^{1+\varepsilon} + \int_{1-\varepsilon}^{x}\right)\frac{dt}{\log t}.$$

By using the left hand sides of (2.1) and (4.1), and also (5.1), for $n \geqslant 14$ we obtain

$$\sum_{k \leqslant n} \omega(k) > n\log\log n + Mn - \mathscr{E}(n),$$

where

$$\mathscr{E}(n) = \mathrm{Li}(n) + \frac{1}{2\pi}\sqrt{n}\log(en).$$

Since $\mathscr{E}(x) = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$, for every real number $\eta > 1$ there exists sufficiently large integer $n_\eta > 0$ such that the inequality $\mathscr{E}(x) \leqslant \frac{\eta x}{\log x}$ is valid for $x \geqslant n_\eta$, and consequently, the inequality

$$\sum_{k \leqslant n} \omega(k) > n\log\log n + Mn - \frac{\eta n}{\log n},$$

is valid for $n \geqslant n_\eta$. In the following tables, we use Maple software to compute and list some values of η and related n_η.

η	n_η	η	n_η	η	n_η	η	n_η
3.5	2	1.9	18	1.5	1017	1.1	3095941
3	2	1.8	99	1.4	2721	1.05	6205177052
2.5	2	1.7	208	1.3	9623	1.04	638938158064
2	2	1.6	442	1.2	61747	1.03	2371092536596259

η	n_η
1.02	39943378574915806977191
1.015	683911233799360864269003256290
1.014	79826081593981598534310117227201
1.013	19384674289582672571718437522866834
1.012	11762019144998810390357072004888542360
1.011	22891804692656934125657683659627630294530
1.01	202723511741684485961583051232880823886730710

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