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OSCILLATION OF $p(x)$–LAPLACIAN ELLIPTIC INEQUALITIES WITH MIXED VARIABLE EXPONENTS

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Abstract. Oscillation criteria are established for $p(x)$-Laplacian elliptic inequalities with mixed variable nonlinearities of the form

$$u\big[\nabla\cdot(A(x)|\nabla u|^{p(x)-2}\nabla u)+\langle b(x),|\nabla u|^{p(x)-2}\nabla u\rangle-h(x,u)+g(x,u)\big]\leqslant 0,\quad x\in\Omega,$$

where $\beta(x)>p(x)>\gamma(x)>1$, Ω is an exterior domain in $\mathbb{R}^N$, and

$$h(x,u)=\ln|u|\,|\nabla u|^{p(x)-2}\,(A(x)\nabla u)\cdot\nabla p(x),$$
$$g(x,u)=c(x)|u|^{p(x)-2}u+c_1(x)|u|^{\beta(x)-2}u+c_2(x)|u|^{\gamma(x)-2}u+f(x).$$

The function $h(x,u)$ recently introduced in [N. Yoshida, Nonlinear Anal. 74 (2011) 2563–2575] allows employing the Riccati transformation technique commonly used in the oscillation theory of ordinary differential equations.

It should be noted that the results obtained are new for one dimensional case as well. Examples are given to illustrate the results.

1. Introduction

We consider the elliptic inequality with $p(x)$-Laplacian of the form

$$u\big[\nabla\cdot(A(x)|\nabla u|^{p(x)-2}\nabla u)+\langle b(x),|\nabla u|^{p(x)-2}\nabla u\rangle \\ -h(x,u)+g(x,u)\big]\leqslant 0,\quad x\in\Omega, \tag{E}$$

where $\beta(x)>p(x)>\gamma(x)>1$, Ω is an exterior domain in $\mathbb{R}^N$, and

$$h(x,u)=\ln|u|\,|\nabla u|^{p(x)-2}\,(A(x)\nabla u)\cdot\nabla p(x),$$
$$g(x,u)=c(x)|u|^{p(x)-2}u+c_1(x)|u|^{\beta(x)-2}u+c_2(x)|u|^{\gamma(x)-2}u+f(x).$$

For simplicity, we take

$$\Omega=\Omega(r_0):=\{x\in\mathbb{R}^N:|x|>r_0\},$$

where $r_0>0$ is a fixed real number.

It is assumed throughout this paper that

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(i) $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix with $a_{ij} \in C^1(\Omega,\mathbb{R})$,

(ii) $b = (b_i)_{\mathbb{N}\times 1}$ is a real vector with $b_i \in C(\Omega,\mathbb{R})$,

(iii) $c, c_1, c_2, f \in C(\Omega,\mathbb{R})$,

(iv) $p \in C^1(\Omega,(1,\infty))$; $\gamma,\beta \in C(\Omega,(1,\infty))$.

A function $u \in C^1(\Omega,\mathbb{R})$ with property that $A(x)|\nabla u|^{p(x)-2}\nabla u \in C^1(\Omega,\mathbb{R})$ is said to be a solution of (E) in Ω provided that $u(x)$ satisfies (E) for all $x \in \Omega$. A solution is called oscillatory if the set $\{x \in \Omega : u(x) = 0\}$ is unbounded; otherwise it is said to be nonoscillatory. (E) is oscillatory if all solutions are oscillatory. As it is pointed out by Yoshida [1], by defining $u\ln|u| := 0$ when $u = 0$, $u\ln|u|$ becomes a continuous function, and therefore we observe that (E) has no singularity.

The oscillation theory of differential equations dates back to 1836, when Sturm introduced his oscillation and comparison theorems. Since then there has been a great deal of works concerning mostly on the oscillation of ordinary differential equations. To the best of our knowledge, the first study on elliptic differential equations involving a Sturmian comparison theorem for self-adjoint second order linear elliptic equations was performed by Hartman and Wintner [3]. Later, several authors have investigated the oscillation of partial differential equations in various forms by making use of comparison methods, Riccati transformations, and Picone type identities. For a sample of works we refer to [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references cited therein. Among them we choose to mention the following works.

Usami [7] derived oscillation criteria for half-linear elliptic partial differential equations with p-Laplacian

$$-\nabla\cdot\big(a(x)|\nabla u|^{p-2}\nabla u\big) + c(x)|u|^{p-2}u = 0, \quad p > 1$$

by using Riccati method.

Marik [8], by using a radialization method for

$$\nabla\cdot(a(x)|\nabla u|^{p-2}\nabla u) + \langle b(x), |\nabla u|^{p-2}\nabla u\rangle + c(x)|u|^{p-2}u = 0$$

obtained via an ordinary differential equation several oscillation criteria.

Yoshida [9] has studied the oscillation of super- and sub- half-linear elliptic equations with damping of the form

$$\nabla\cdot(a(x)|\nabla u|^{\alpha-1}\nabla u) + (\alpha+1)B(x)(|\nabla y|^{\alpha-1}\nabla u) + C(x)|u|^{\beta-1}y + D(x)|u|^{\gamma-1}u = f(x)$$

when $0 < \gamma < \alpha < \beta$ by utilizing a Picone-type inequality.

Recently, there is also an increasing interest in studying equations with $p(x)$-Laplacian $-\nabla\cdot\Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big)$, since such equations arise quite naturally in applied problems [16, 17, 18]. Indeed, the existence of weak solutions for

$$-\nabla\cdot\Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big) + c(x)|u|^{p(x)-2}u = f(x,u), \quad x \in \mathbb{R}^N.$$

has been already established [19, 20, 21, 22]. For the existence and as well as the oscillation theory of nonlinear elliptic differential equations, we refer the reader in particular to the articles [23, 24, 25] and the monographs [26, 27, 28].

As far as the oscillation of $p(x)$-Laplacian type equations are concerned, there are only a few works in the literature, see [1, 29] and the references cited therein. The main reason seems to be the fact that the $p(x)$-Laplacian equation

$$-\nabla\cdot\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)+c(x)|u|^{p(x)-2}u=0$$

is not half-linear, i.e a constant multiple of a solution is not a solution anymore unless p is a constant function. However, the related elliptic inequality

$$u\Big[\nabla\cdot\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)-a(x)\ln|u|\,|\nabla u|^{p(x)-2}\nabla p(x)\cdot\nabla u \\ +|\nabla u|^{p(x)-2}b(x)\cdot\nabla u+c(x)|u|^{p(x)-2}u\Big]\leqslant 0, \qquad (1.1)$$

is half-linear as easily checked. This crucial observation made by Yoshida [1] is very important, since it allows one to study such inequalities via Riccati type inequalities with variable exponents [1, Proposition 2.1] as in linear case. This approach nicely applied to (1.1) in [1] to derive several new oscillation criteria of integral averaging type. See also [2] for Picone type identities with applications to Sturmian comparison theory for half-linear elliptic operators with $p(x)$-Laplacians. We should note that Noussair and Swanson [30] were the first to consider the oscillation of semilinear elliptic inequalities of the fom

$$\nabla\cdot(a(x)\nabla u)+p(x)f(u)\leqslant 0 \qquad (1.2)$$

by making use of vector Riccati type transformation

$$w(x)=-\frac{\alpha(|x|)}{f(u(x))}(a\nabla u)(x)$$

where $\alpha\in C^2(0,\infty)$ is an arbitrary positive function.

Motivated by the work of Yoshida [1], we establish new oscillation criteria for (E) by using the arguments developed in [14] and Riccati transformation technique as in [1]. It is clear that (E) contains (1.1) as a special case by taking $c_1(x)=c_2(x)=f(x)\equiv 0$ and $A(x)=a(x)I$. As opposed to most oscillation criteria in the literature including the ones in [1] our theorems do not require information on the whole exterior domain Ω but rather on a sequence of bounded domains in Ω. Moreover, we are not confined ourselves to annular domains $\Omega(r_0)$ only.

2. Preliminaries

Let $A^{-1}(x)$ be the inverse of $A(x)$ and $\lambda_{\min}(x)$ denote the smallest eigenvalue of the matrix $A(x)$. As usual, by $|A(x)|$ we mean the matrix norm induced by any vector norm in $\mathbb{R}^N$.

We will make use of the following four lemmas. The first one is the generalized Young inequality [15, p. 17] and the next three are some Riccati type inequalities related to elliptic inequality (E).

LEMMA 2.1. *Let $n \geqslant 2$ be an integer. If the numbers $q_i > 1$, $i = 1, 2, \dots, n$, satisfy*

$$\sum_{i=1}^{n} \frac{1}{q_i} = 1,$$

then for any real numbers $u_1, u_2, \dots, u_n$, the inequality

$$\prod_{i=1}^{n} |u_i| \leqslant \sum_{i=1}^{n} \frac{|u_i|^{q_i}}{q_i}. \tag{2.1}$$

holds.

LEMMA 2.2. *Let $\Omega_* \subset \Omega$. Suppose that c_1 and c_2 are nonnegative on Ω_*. Choose $\eta_0 : \Omega \to (0,1)$ so that*

$$(\beta(x) - 1)\eta_0(x) < \beta(x) - p(x),$$

and define

$$\eta_1(x) = \frac{p(x) - \gamma(x) + \eta_0(x)(\gamma(x) - 1)}{\beta(x) - \gamma(x)},$$

and

$$\eta_2(x) = \frac{\beta(x) - p(x) - \eta_0(x)(\beta(x) - 1)}{\beta(x) - \gamma(x)}.$$

If u is a solution of (E) without any zero on Ω_* and $u(x)f(x) \geqslant 0$ with $f \not\equiv 0$ on Ω_*, then the vector function w defined by

$$w(x) = \frac{A(x)|\nabla u|^{p(x)-2}\nabla u}{|u|^{p(x)-2}u}, \quad x \in \Omega_*. \tag{2.2}$$

satisfies the Riccati inequality

$$\nabla \cdot w \leqslant -C_f(x) - b^T(x)A^{-1}(x)w - \frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}|w|^{q(x)}, \quad x \in \Omega_*, \tag{2.3}$$

where q is the conjugate of p, i.e.,

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$$

and

$$C_f(x) = c(x) + (|f(x)|/\eta_0(x))^{\eta_0(x)}(c_1(x)/\eta_1(x))^{\eta_1(x)}(c_2(x)/\eta_2(x))^{\eta_2(x)}. \tag{2.4}$$

Proof. We first note that (E) can be written as

$$uQ(u) \leqslant 0, \quad x \in \Omega, \tag{2.5}$$

where

$$Q(u) = \nabla \cdot \left(A(x)|\nabla u|^{p(x)-2}\nabla u\right) - \ln|u||\nabla u|^{p(x)-2}(A(x)\nabla u)\cdot\nabla p(x)$$
$$+\langle b(x), |\nabla u|^{p(x)-2}\nabla u\rangle + g(x,u)$$

By a direct calculation we have

$$\begin{aligned}\nabla\cdot w(x) &= \frac{1}{|u|^{p(x)-2}u}\nabla\cdot(A(x)|\nabla u|^{p(x)-2}\nabla u)\\ &\quad +(A(x)|\nabla u|^{p(x)-2}\nabla u)\cdot\left[\frac{1-p(x)}{|u|^{p(x)}}\nabla u - \frac{\ln|u|}{|u|^{p(x)-2}u}\nabla p(x)\right]\\ &= \frac{uQ[u]}{|u|^{p(x)}} - \frac{p(x)-1}{|u|^{p(x)}}\langle A(x)|\nabla u|^{p(x)-2}\nabla u, \nabla u\rangle\\ &\quad -\left\langle b(x), \frac{|\nabla u|^{p(x)-2}\nabla u}{|u|^{p(x)-2}u}\right\rangle - \frac{g(x,u)}{|u|^{p(x)-2}u},\end{aligned}$$

which in view of (2.2) and (2.5) leads to

$$\nabla\cdot w \leqslant -\frac{p(x)-1}{|u|^{p(x)}}\langle A(x)|\nabla u|^{p(x)-2}\nabla u, \nabla u\rangle - \langle b(x), A^{-1}(x)w\rangle - \frac{g(x,u)}{|u|^{p(x)-2}u}.$$

By making use of the inequalities

$$(\nabla u)^T A(x)\nabla u \geqslant \lambda_{\min}(x)|\nabla u|^2,$$

and

$$\frac{|\nabla u|^{p(x)-1}}{|u|^{p(x)-1}} \geqslant \frac{|w(x)|}{|A(x)|},$$

we thus have

$$\nabla\cdot w \leqslant -b^T(x)A^{-1}(x)w - \frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}|w|^{q(x)} - \frac{g(x,u)}{|u|^{p(x)-2}u}. \tag{2.6}$$

Let $x \in \Omega_*$. In view of $u(x)f(x) \geqslant 0$ we may write

$$\frac{g(x,u)}{|u|^{p(x)-2}u} = c(x) + |f(x)||u|^{1-p(x)} + c_1(x)|u|^{\beta(x)-p(x)} + c_2(x)|u|^{\gamma(x)-p(x)}.$$

Since $\eta_i > 0$ for $i = 0,1,2$ and $\eta_0+\eta_1+\eta_2 = 1$, we can apply Lemma 2.1 with

$$u_1 = (c_1(x)q_1(x))^{1/q_1(x)}|u|^{\frac{\beta(x)-p(x)}{q_1(x)}}, \quad q_1 = 1/\eta_1$$

$$u_2 = (c_2(x)q_2(x))^{1/q_2(x)}|u|^{\frac{\gamma(x)-p(x)}{q_2(x)}}, \quad q_2 = 1/\eta_2$$

and

$$u_3 = (|f(x)|q_3(x))^{1/q_3(x)}|u|^{\frac{1-p(x)}{q_3(x)}}, \quad q_3 = 1/\eta_0$$

to get

$$\begin{aligned}\frac{g(x,u)}{|u|^{p(x)-2}u} - c(x) &= \frac{u_1(x)^{q_1(x)}}{q_1(x)} + \frac{u_2(x)^{q_2(x)}}{q_2(x)} + \frac{u_3(x)^{q_3(x)}}{q_3(x)} \\ &\geqslant u_1(x)u_2(x)u_3(x) \\ &= (c_1(x)/\eta_1(x))^{\eta_1(x)}(c_2(x)/\eta_2(x))^{\eta_2(x)}(|f(x)|/\eta_0(x))^{\eta_0(x)}.\end{aligned}$$

Using this inequality in (2.6) completes the proof. □

In case $f \equiv 0$ we can similarly prove the following lemma. In fact, making use of the convention that $0^0 = 1$ and taking $\eta_0 = 0$, it coincides with the previous lemma.

LEMMA 2.3. *Let $\Omega_* \subset \Omega$ and $f \equiv 0$. Suppose that c_1 and c_2 are nonnegative on Ω_* Define*

$$\hat{\eta}_1(x) = \frac{p(x)-\gamma(x)}{\beta(x)-\gamma(x)}, \quad \hat{\eta}_2(x) = \frac{\beta(x)-p(x)}{\beta(x)-\gamma(x)}.$$

If u is a solution of (E) without any zero on Ω_*, then the vector function w defined by (2.2) satisfies the Riccati inequality

$$\nabla \cdot w \leqslant -C_0(x) - b^T(x)A^{-1}(x)w - \frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}|w|^{q(x)}, \quad x \in \Omega_*, \tag{2.7}$$

where q is the conjugate of p and

$$C_0(x) = c(x) + (c_1(x)/\hat{\eta}_1(x))^{\hat{\eta}_1(x)}(c_2(x)/\hat{\eta}_2(x))^{\hat{\eta}_2(x)}. \tag{2.8}$$

Finally, we give a lemma which will enable us to relax the nonnegativity condition imposed on $c_2(x)$. This can however be made possible only when the function $f(x)$ does not vanish on $\Omega[a,b]$.

LEMMA 2.4. *Let $\Omega_* \subset \Omega$. Suppose that c_1 is nonnegative on Ω_*. Let δ_1 and δ_2 be positive real numbers such that $\delta_1 + \delta_2 = 1$.*

If u is a solution of (E) without any zero and $u(x)f(x) > 0$ on Ω_*, then the vector function w defined by (2.2) satisfies the Riccati inequality

$$\nabla \cdot w \leqslant -C(x) - b^T(x)A^{-1}(x)w - \frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}|w|^{q(x)}, \quad x \in \Omega_*, \tag{2.9}$$

where q is the conjugate of p and

$$C(x)=c(x)+(\beta(x)-1)\Big(\frac{\delta_1|f(x)|}{\beta(x)-p(x)}\Big)^{\frac{\beta(x)-p(x)}{\beta(x)-1}}\Big(\frac{c_1(x)}{p(x)-1}\Big)^{\frac{p(x)-1}{\beta(x)-1}}$$
$$-(\gamma(x)-1)\Big(\frac{\delta_2|f(x)|}{p(x)-\gamma(x)}\Big)^{\frac{\gamma(x)-p(x)}{\gamma(x)-1}}\Big(\frac{c_2^-(x)}{p(x)-1}\Big)^{\frac{p(x)-1}{\gamma(x)-1}} \tag{2.10}$$

with

$$c_2^-(x)=-\min\{c_2(x),0\}.$$

Proof. Proceeding as in the proof of Lemma 2.2, we have (2.6). We may write

$$\frac{g(x,u)}{|u|^{p(x)-2}u}=c(x)+S_1(x)+S_2(x), \tag{2.11}$$

where

$$S_1(x)=c_1(x)|u|^{\beta(x)-p(x)}+\delta_1|f(x)||u|^{1-p(x)}$$

and

$$S_2(x)=\delta_2|f(x)||u|^{1-p(x)}+c_2(x)|u|^{\gamma(x)-p(x)}$$

Applying Lemma 2.1 with

$$u_1=(c_1(x)q_1(x))^{1/q_1(x)}|u|^{\frac{\beta(x)-p(x)}{q_1(x)}},\quad q_1(x)=\frac{\beta(x)-1}{p(x)-1}$$

and

$$u_2=(\delta_1|f(x)|q_2(x))^{1/q_2(x)}|u|^{\frac{1-p(x)}{q_2(x)}},\quad q_2(x)=\frac{\beta(x)-1}{\beta(x)-p(x)}$$

we have

$$S_1(x)=\frac{u_1(x)^{q_1(x)}}{q_1(x)}+\frac{u_2(x)^{q_2(x)}}{q_2(x)}\geqslant u_1(x)u_2(x)$$
$$=(\beta(x)-1)\Big(\frac{\delta_1|f(x)|}{\beta(x)-p(x)}\Big)^{\frac{\beta(x)-p(x)}{\beta(x)-1}}\Big(\frac{c_1(x)}{p(x)-1}\Big)^{\frac{p(x)-1}{\beta(x)-1}}$$

Similarly, with

$$u_1=(\delta_2|f(x)|q_1(x))^{1/q_1(x)}|u|^{\frac{1-p(x)}{q_1(x)}},\quad q_1(x)=\frac{p(x)-1}{p(x)-\gamma(x)}$$
$$u_2=c_2^-(x)(q_1(x)\delta_2|f(x)|)^{-1/q_1(x)}|u|^{\gamma(x)-p(x)-\frac{1-p(x)}{q_1(x)}},\quad q_2(x)=\frac{p(x)-1}{\gamma(x)-1}$$

we obtain

$$\begin{aligned} S_2(x) &\geqslant \delta_2|f(x)||u|^{1-p(x)} - c_2^-(x)|u|^{\gamma(x)-p(x)} \\ &= \frac{u_1(x)^{q_1(x)}}{q_1(x)} - u_1(x)u_2(x) \geqslant -\frac{u_2(x)^{q_2(x)}}{q_2(x)} \\ &= -(\gamma(x)-1)\left(\frac{\delta_2|f(x)|}{p(x)-\gamma(x)}\right)^{\frac{\gamma(x)-p(x)}{\gamma(x)-1}} \left(\frac{c_2^-(x)}{p(x)-1}\right)^{\frac{p(x)-1}{\gamma(x)-1}} \end{aligned}$$

Thus, from (2.11) we get

$$\frac{g(x,u)}{|u|^{p(x)-2}u} \geqslant C(x).$$

This completes the proof. □

3. The main results

Let

$$D_k = \{H \in C^1(\bar{\Omega}_k, \mathbb{R}) : H(x) > 0,\ x \in \Omega_k;\ \ H(x) = 0,\ x \in \Gamma_k\},$$

where Ω_k is an open bounded subset of Ω with piecewise smooth boundary Γ_k, and define

$$K(x) = H(x)\ln H(x)\nabla p(x) + p(x)\nabla H(x) - b^T(x)A^{-1}(x)H(x), \quad x \in \bar{\Omega}_*. \tag{3.1}$$

Note that since $\lim_{H\to 0} H\ln H = 0$, the function K can be made continuous on $\bar{\Omega}_*$ by defining $(H\ln H)(0) = 0$.

THEOREM 3.1. *Suppose that for any given $r \geqslant r_0$ there exist $\Omega_1, \Omega_2 \subset \Omega(r)$ such that*

$$c_i(x) \geqslant 0, \quad x \in \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad (i = 1,2) \tag{3.2}$$

and

$$(-1)^k f(x) \leqslant 0\ (\not\equiv 0), \quad x \in \bar{\Omega}_k, \quad (k = 1,2). \tag{3.3}$$

If there exists a function $H \in D_k$ for $k = 1,2$ such that

$$\int_{\Omega_k} \left[C_f(x)[H(x)]^{p(x)} - \frac{|A(x)|^{p(x)}}{[p(x)]^{p(x)}\lambda_{\min}^{p(x)-1}}|K(x)|^{p(x)} \right] dx > 0, \tag{3.4}$$

where the function C_f is as defined in (2.4) and K is as in (3.1), then (E) is oscillatory.

Proof. Suppose on the contrary that there is a nonoscillatory solution u of (E). We may assume that $u(x)$ is positive for all $x \in \Omega(a_0)$ for some $a_0 \geqslant r_0$ sufficiently large. Taking $r = a_0$, we can find Ω_1 so that $c_1(x) \geqslant 0$, $c_2(x) \geqslant 0$ and $f(x) \geqslant 0 (\not\equiv 0)$ for all $x \in \bar{\Omega}_1$.

By Lemma 2.2, the inequality (2.3) holds. Multiplying (2.3) by $[H(x)]^{p(x)}$ and integrating over the domain Ω_1, we get

$$\begin{aligned}\int_{\Omega_1} C_f(x)[H(x)]^{p(x)}dx \leqslant &-\int_{\Omega_1}[H(x)]^{p(x)}\nabla\cdot w(x)dx\\ &-\int_{\Omega_1}[H(x)]^{p(x)}b^T(x)A^{-1}(x)w(x)dx\\ &-\int_{\Omega_1}\frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}[H(x)]^{p(x)}|w(x)|^{q(x)}dx.\end{aligned}$$

Employing the higher dimensional equivalent of integration by parts formula derived from the divergence theorem with $H|_{\Gamma_1}=0$, we have

$$\begin{aligned}\int_{\Omega_1}[H(x)]^{p(x)}\nabla\cdot w(x)dx &= -\int_{\Omega_1}\nabla[H(x)]^{p(x)}\cdot w(x)dx\\ &= -\int_{\Omega_1}\left[\ln H(x)\nabla p(x)+p(x)\frac{\nabla H(x)}{H(x)}\right][H(x)]^{p(x)}\cdot w(x)dx\end{aligned}$$

Therefore,

$$\int_{\Omega_1} C_f(x)[H(x)]^{p(x)}dx \leqslant \int_{\Omega_1}[u_1(x)\cdot w(x)-u_2(x)|w(x)|^{q(x)}]dx. \tag{3.5}$$

where

$$u_1(x)=[H(x)]^{p(x)}\left(\ln H(x)\nabla p(x)+p(x)\frac{\nabla H(x)}{H(x)}-[A^T(x)]^{-1}b(x)\right)$$

and

$$u_2(x)=\frac{(p(x)-1)\lambda_{\min}(x)}{|A(x)|^{q(x)}}[H(x)]^{p(x)}.$$

Young inequality with $n=2$ leads to

$$u_1\cdot w\leqslant (u_2q)^{-1/q}(u_2q)^{1/q}|u_1||w|\leqslant \frac{|u_1|^p\,(u_2q)^{1-p}}{p}+u_2\,|w|^q \tag{3.6}$$

From (3.5) and (3.6), we have

$$\int_{\Omega_1} C_f(x)[H(x)]^{p(x)}dx \leqslant \int_{\Omega_1}\frac{|A(x)|^{p(x)}}{[p(x)]^{p(x)}\lambda_{\min}^{p(x)-1}}\left|K(x)\right|^{p(x)}dx.$$

This last inequality contradicts (3.4) when $k=1$. The proof when $u(x)<0$ eventually is similar by working with Ω_2. In fact, $v(x)=-u(x)>0$ solves (E) with f replaced by $-f$, and we have $-f(x)\geqslant 0$ on $\bar{\Omega}_2$ by our assumption, and so Lemma 2.2 is applicable with $-f(x)v(x)\geqslant 0$. $\square$

If $f(x)\equiv 0$, we obtain the following theorem. The proof is in fact a simpler version of the proof of Theorem 3.1. It suffices to take $f(x)\equiv 0$ and $\eta_0(x)\equiv 0$ and employ Lemma 2.3 in this special case.

THEOREM 3.2. *Suppose that for any given $r \geqslant r_0$ there exists an $\Omega_1 \subset \Omega(r)$ such that*

$$c_i(x) \geqslant 0, \quad x \in \bar{\Omega}_1, \quad (i = 1,2). \tag{3.7}$$

If there exists a function $H \in D_1(a,b)$ such that

$$\int_{\Omega_1} \left[C_0(x)[H(x)]^{p(x)} - \frac{|A(x)|^{p(x)}}{p(x)^{p(x)} \lambda_{\min}^{p(x)-1}} |K(x)|^{p(x)} \right] dx > 0, \tag{3.8}$$

where the function C_0 is as defined in (2.8) and K is as in (3.1), then (E) with $f(x) \equiv 0$ is oscillatory.

In our last theorem we remove the sign condition on $c_2(x)$ by requiring that $f(x)$ never vanishes in the domain of interest.

THEOREM 3.3. *Suppose that for any given $r \geqslant r_0$ there exist $\Omega_1, \Omega_2 \subset \Omega(r)$ such that*

$$c_1(x) \geqslant 0, \quad x \in \Omega_1 \cup \bar{\Omega}_2 \tag{3.9}$$

and

$$(-1)^k f(x) < 0, \quad x \in \bar{\Omega}_k, \quad (k = 1,2). \tag{3.10}$$

If there exist a function $H \in D_k$ and positive numbers δ_1 and δ_2 with $\delta_1 + \delta_2 = 1$ such that

$$\int_{\Omega_k} \left[C(x)[H(x)]^{p(x)} - \frac{|A(x)|^{p(x)}}{p(x)^{p(x)} \lambda_{\min}^{p(x)-1}} |K(x)|^{p(x)} \right] dx > 0, \tag{3.11}$$

for $k = 1,2$, where the function C is as defined in (2.10) and K is as in (3.1), then (E) is oscillatory.

Proof. We proceed exactly as in the proof of Theorem 3.1, except that we employ Lemma 2.4 instead of Lemma 2.2. □

REMARK 1. The domains Ω_k, $k = 1,2$, could be quite complicated in general. For many cases, it suffices to take

$$\Omega_k = \Omega(a_k, b_k) = \{x \in \mathbb{R}^N : a_k < |x| < b_k\},$$

where a_k and b_k are real numbers such that $r_0 < a_k < b_k$. See the examples in the last section.

4. One dimensional case

Let $N = 1$ and $\mathbb{R}_+ = [0,\infty)$; $a,p \in C^1(\mathbb{R}_+,\mathbb{R})$, $a(x) > 0$; $\beta,\gamma,b,c,c_1,c_2,f \in C(\mathbb{R}_+,\mathbb{R})$, and $\Phi_*(y) = |y|^{*-2}y$. Then (E) reduces to ordinary differential inequality

$$y\Big[\Big(a(t)\Phi_{p(t)}(y')\Big)' + \big(b(t) - a(t)p'(t)\ln|y|\big)\Phi_{p(t)}(y') + c(t)\Phi_{p(t)}(y) + c_1(t)\Phi_{\beta(t)}(y) + c_2(t)\Phi_{\gamma(t)}(y) + f(t)\Big] \leqslant 0 \qquad (E_1)$$

where

$$\beta(t) > p(t) > \gamma(t) > 1.$$

When p,β, and γ are constant (functions) and $p = 2$, the inequality (E_1) in the equality case with $b \equiv 0$ is studied by Sun and Wong in [31]. The following corollaries extending to variable exponents the interval oscillation criteria obtained in [31] are the direct consequences of Theorem 3.1, Theorem 3.2, and Theorem 3.3, respectively.

Let

$$D(a,b) = \{H \in C^1([a,b],\mathbb{R}) : H(t) > 0,\ t \in (a,b);\ \ H(a) = H(b) = 0.$$

COROLLARY 4.1. *Suppose for any given $T \geqslant 0$ there exist a_1,b_1,a_2,b_2 such that $T \leqslant a_1 < b_1$, $T \leqslant a_2 < b_2$ and that*

$$c_i(t) \geqslant 0, \quad t \in [a_1,b_1] \cup [a_2,b_2],\ (i = 1,2)$$

and

$$(-1)^k f(t) \leqslant 0\ (\not\equiv 0), \quad t \in [a_k,b_k], \quad (k = 1,2).$$

If there exists a function $H \in D(a_k,b_k)$ for $k = 1,2$ such that

$$\int_{a_k}^{b_k} \Big[C_f(t)[H(t)]^{p(t)} - \frac{|H(t)\ln H(t)a(t)p'(t) + a(t)p(t)H'(t) - b(t)H(t)|^{p(t)}}{(a(t))^{p(t)-1}(p(t))^{p(t)}}\Big]dt > 0,$$

where the function C_f is as defined in Lemma 2.2 (with x replaced by t), then (E_1) is oscillatory.

COROLLARY 4.2. *Suppose that for any given $T \geqslant$ there exist a,b such that $T \leqslant a < b$ and that*

$$c_i(t) \geqslant 0, \quad t \in [a,b], \quad (i = 1,2).$$

If there exists a function $H \in D(a,b)$ such that

$$\int_a^b \Big[C_0(t)[H(t)]^{p(t)} - \frac{|H(t)\ln H(t)a(t)p'(t) + a(t)p(t)H'(t) - b(t)H(t)|^{p(t)}}{(a(t))^{p(t)-1}(p(t))^{p(t)}}\Big]dt > 0,$$

where the function C_0 is as defined in Lemma 2.3, then (E_1) with $f \equiv 0$ is oscillatory.

COROLLARY 4.3. *Suppose that for any given* $T \geqslant$ *there exist* a_1, b_1, a_2, b_2 *such that* $T \leqslant a_1 < b_1$, $T \leqslant a_2 < b_2$ *and that*

$$c_1(t) \geqslant 0, \quad t \in [a_1, b_1] \cup [a_2, b_2]$$

and

$$(-1)^k f(t) < 0, \quad t \in [a_k, b_k], \quad (k = 1, 2).$$

If there exist a function $H \in D(a_k, b_k)$ *and positive numbers* δ_1 *and* δ_2 *with* $\delta_1 + \delta_2 = 1$ *such that*

$$\int_{a_k}^{b_k} \Big[C(t)[H(t)]^{p(t)} - \frac{|H(t)\ln H(t)a(t)p'(t) + a(t)p(t)H'(t) - b(t)H(t)|^{p(t)}}{(a(t))^{p(t)-1}(p(t))^{p(t)}}\Big] dt > 0,$$

for $k = 1, 2$, *where the function* C *is as defined in Lemma* 2.4, *then* (E_1) *is oscillatory.*

5. Examples

Two examples are given to illustrate the results. We should note that no oscillation criterion in the literature is applicable for these cases.

EXAMPLE 5.1. Consider the nonlinear partial differential inequality (E) with $N = 2$,

$$p(x) = 4 + 3k^2(|x|)), \ \beta(x) = 5 + 4k^2(|x|), \ \gamma(x) = 3 + 2k^2(|x|), \ k(|x|) = \sin^2 2|x|,$$

$$A(x) = I \text{ (Identity matrix)}, \ b(x) = \Big[\tfrac{12k(|x|)}{|x|^{1/4}} \ln\Big(\tfrac{k(|x|)}{|x|^{1/4}}\Big) \tfrac{\sin 4|x|}{|x|^{3/4}} - \tfrac{(4+3k^2(|x|))}{4|x|^2}\Big] x, \ c(x) \equiv 0,$$

and

$$c_1(x) = m_1 |x|^{1/2} \sin^{1/7}|x|, \quad c_2(x) = m_2 |x|^{3/2} \sin^9 |x|, \quad f(x) = -|x|^{5/4} \cos^5 |x|.$$

With the choice of $\eta_0 = 1/5$, we have $\eta_1 = 7/10$, $\eta_2 = 1/10$. It is easy to see that

$$C_f(x) = M|x|^{3/4} \big|\cos|x|\big| \sin|x|, \quad M = \frac{10 m_1^{7/10} m_2^{1/10}}{2^{1/5} 7^{7/10}}.$$

If we take $H(x) = \frac{k(|x|)}{|x|^{1/4}}$, then we calculate

$$K(x) = 2(4 + 3\sin^4(2|x|)) \frac{\sin 4|x|}{|x|^{5/4}} x.$$

Let $a_1 = 2i\pi + \pi/2$, $b_1 = 2i\pi + \pi$, $a_2 = 2(i+1)\pi$, $b_2 = 2(i+1)\pi + \pi/2$ for $i \in \mathbb{N}$.

Since $4 \leqslant p(x) \leqslant 7$, we have

$$C_f(x)[H(x)]^{p(x)} \geqslant M|\cos|x|| \sin|x| \frac{\sin^{14} 2|x|}{|x|}$$

and

$$\frac{|K(x)|^{p(x)}}{p(x)^{p(x)}} \leqslant 2^7 \frac{\sin^4 4|x|}{|x|}.$$

$$\begin{aligned}
&\int_{\Omega(a_1,b_1)} \Big[C_f(x)[H(x)]^{p(x)} - \frac{|K(x)|^{p(x)}}{p(x)^{p(x)}}\Big]dx \\
&\qquad \geqslant \int_{\Omega(a_1,b_1)} \Big[M|\cos|x||\sin|x|\frac{\sin^{14} 2|x|}{|x|} - 2^7\frac{\sin^4 4|x|}{|x|}\Big]dx \\
&\qquad \geqslant 2\pi \int_{\pi/2}^{\pi} \big[M(-\cos r)\sin r \sin^{14} 2r - 2^7 \sin^4 4r\big]dr \\
&\qquad = \frac{2^{11} M\pi}{6435} - 2^5(\pi)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\Omega(a_2,b_2)} \Big[C_f(x)H(x)^{p(x)} - \frac{|K(x)|^{p(x)}}{p(x)^{p(x)}}\Big]dx \\
&\qquad \geqslant 2\pi \int_0^{\pi/2} \big[M\cos r \sin r \sin^{14} 2r - 2^7 \sin^4 4r\big]dr \\
&\qquad = \frac{2^{11} M\pi}{6435} - 2^5(\pi)^2.
\end{aligned}$$

It follows that (3.4) is satisfied when $m_1^7 m_2 > 3.26 \times 10^{21}$, and so we conclude that if this condition is met, then (E) is oscillatory by Theorem 3.1.

EXAMPLE 5.2. Consider the nonlinear partial differential inequality (E) with $N = 2,\ A(x) = I$ (Identity matrix), $c(x) \equiv 0,\ c_1(x) = m_1^2|x|^{1/2}\big|\cos|x|\big|\sin|x|\ \ (m_1 > 0)$, $c_2(x) = m_2^2|x|\big|\cos|x|\big|\sin|x|\ \ (m_2 > 0),\ \ f(x) \equiv 0$. The functions b, k, p, β, γ are the same as in Example 5.1.

We take $\eta_1 = \eta_2 = 1/2$, $a = 2i\pi + \pi/2$, and $b = 2i\pi + \pi$. It is not difficult to see that

$$\begin{aligned}
&\int_{\Omega(a,b)} \Big[C_0 H(x)^{p(x)} - \frac{|K(x)|^{p(x)}}{p(x)^{p(x)}}\Big]dx \\
&\qquad \geqslant 2\pi \int_{\pi/2}^{\pi} \big[2m_1m_2|\cos r|\sin r \sin^{14} 2r - 2^7 \sin^4 4r\big]dr \\
&\qquad = \frac{2^{12} m_1 m_2 \pi}{6435} - 2^5(\pi)^2.
\end{aligned}$$

We see that condition (3.8) is satisfied when $m_1 m_2 > 157.94$, and so in this case the inequality (E) with $f(x) \equiv 0$ is oscillatory by Theorem 3.2.

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