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AN EXPERIMENTAL CONJECTURE INVOLVING CLOSED–FORM EVALUATION OF SERIES ASSOCIATED WITH THE ZETA FUNCTIONS

JUNESANG CHOI AND HARI M. SRIVASTAVA

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Abstract. The subject of closed-form summation of series involving the Zeta functions has been remarkably widely investigated. Recently, in the course of his trying to give a closed-form expression for the Dirichlet beta function $\beta(2n)$ $(n \in \mathbb{N})$, Lima [16] posed a very interesting experimental conjecture for a closed-form evaluation of a certain class of series involving the Riemann Zeta function $\zeta(s)$. Here, in the present sequel to Lima's work, we aim at verifying correctness of Lima's conjecture and presenting several general analogues of Lima's conjecture. Our demonstration and derivations are based mainly upon a known formula for series associated with the Zeta functions. Relevant connections of some specialized results of the main identities presented here with those obtained in earlier works are also pointed out.

1. Introduction and preliminaries

A rather classical (over two centuries old) theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782), was revived in 1986 by Shallit and Zikan [20] as the following problem:

$$\sum_{\omega \in \mathscr{S}} (\omega - 1)^{-1} = 1, \tag{1.1}$$

where $\mathscr{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathscr{S} := \left\{ n^k : n, k \in \mathbb{N} \setminus \{1\} \right\}. \tag{1.2}$$

In terms of the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big), \end{cases} \tag{1.3}$$

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Goldbach's theorem (1.1) assumes the elegant form (*cf.* Shallit and Zikan [20, p. 403]):

$$\sum_{k=2}^{\infty} \{\zeta(k)-1\} = 1 \tag{1.4}$$

or, equivalently,

$$\sum_{k=2}^{\infty} \mathscr{F}\big(\zeta(k)\big) = 1, \tag{1.5}$$

where $\mathscr{F}(x) := x-[x]$ denotes the *fractional* part of $x \in \mathbb{R}$. As a matter of fact, it is fairly straightforward to observe also that

$$\sum_{k=2}^{\infty} (-1)^k \mathscr{F}\big(\zeta(k)\big) = \frac{1}{2}, \tag{1.6}$$

$$\sum_{k=1}^{\infty} \mathscr{F}\big(\zeta(2k)\big) = \frac{3}{4} \quad \text{and} \quad \sum_{k=1}^{\infty} \mathscr{F}\left(\zeta(2k+1)\right) = \frac{1}{4}. \tag{1.7}$$

The subject of closed-form summation of series involving the Zeta functions has been remarkably widely investigated (see [8, 9, 10, 11, 13, 21, 22, 23, 24]). Among the various methods and techniques used in the vast literature on the subject, Srivastava and Choi [23, 24] gave reasonably detailed accounts of those using the binomial theorem, generating functions, multiple Gamma functions (see [2, 3, 4, 5, 9, 10, 15, 25]), and hypergeometric identities, presented a rather extensive collection of closed-form sums of series involving the Zeta functions, and showed that many of those summation formulas find their applications in the evaluations of the determinants of the Laplacians for the n-dimensional sphere $\mathbf{S}^n$ with the standard metric (see [7, 8, 9, 14, 15, 17, 18, 19, 23, 24, 25, 26]).

Recently, Lima [16, Eq. (29)] posed an interesting experimental conjecture on closed-form evaluation of series involving the Zeta functions which is recalled here, in a slightly modified form, as follows:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k)_{2n}\cdot 2^{2k}} =& \frac{1}{2\cdot(2n-1)!}\left(\ln \pi - H_{2n-1}\right)\\ &+\sum_{j=1}^{n-1}(-1)^{j+1}\frac{\zeta(2j+1)}{2\pi^{2j}\cdot(2n-1-2j)!} \qquad (n\in\mathbb{N}),\end{aligned} \tag{1.8}$$

where $\mathbb{N}$ denotes the set of positive integers, H_n denotes the harmonic numbers defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N}) \quad \text{and} \quad H_0 := 0, \tag{1.9}$$

and the Pochhammer symbol $(\lambda)_n$ is defined (for $\lambda \in \mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}), \end{cases} \tag{1.10}$$

$\mathbb{C}$ being the set of complex numbers. Here, and in what follows, an empty sum is understood (as usual) to be nil.

Here, in our present sequel to Lima's work, we aim at verifying correctness of Lima's conjecture (1.8) and presenting several general analogues of Lima's conjecture (1.8), by mainly using a known formula for the series associated with the Zeta functions (see [23, p. 149, Theorem 3.1], also see [11]). For this purpose, we begin by recalling the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad (\Re(s) > 1;\, a \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.11}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. It is noted that the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane having simple poles only at $s=1$ with their respective residues 1 at this point.

For their easy use in the next section, we summarize some known formulas as the following lemmas (see, for details, [23, 24]).

LEMMA 1. *The following formulas hold true:*

$$\zeta(s,1) = \zeta(s); \quad \zeta(s,1+a) = \zeta(s,a) - a^{-s}; \quad \zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\,\zeta(s);$$

$$\zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta(-n) = \begin{cases} -\dfrac{1}{2} & (n=0) \\ -\dfrac{B_{n+1}}{n+1} & (n \in \mathbb{N}), \end{cases}$$

where B_n denotes the Bernoulli number of order n defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi);$$

$$\zeta'(-2n) = \begin{cases} -\dfrac{1}{2}\ln(2\pi) & (n=0), \\ (-1)^n \dfrac{(2n)!}{2(2\pi)^{2n}}\,\zeta(2n+1) & (n \in \mathbb{N}); \end{cases}$$

the Psi (or Digamma) function $\psi(z)$ is defined by

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \quad \psi(1) = -\gamma,$$

where Γ is the familiar Gamma function and γ denotes the Euler-Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \cong 0.57721\,56649\,01532\,86060\,6512\cdots.$$

LEMMA 2. (see Adamchik [1, p. 198, Eq. (20)] and Bendersky [6, pp. 273–275]). *The set of constants D_k are defined by*

$$\log D_k := \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^k \log m - p(n,k)\right) \quad (k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\})$$

with $p(n,k)$ given as follows:

$$\begin{aligned} p(n,k) :=& \frac{n^k}{2}\log n + \frac{n^{k+1}}{k+1}\left(\log n - \frac{1}{k+1}\right) \\ &+ k!\sum_{j=1}^{k} \frac{n^{k-j} B_{j+1}}{(j+1)!\,(k-j)!}\left[\log n + (1-\delta_{kj})\sum_{\ell=1}^{j}\frac{1}{k-\ell+1}\right], \end{aligned}$$

where δ_{kj} is the Kronecker symbol. Furthermore, the following relationship holds true:

$$\log D_k = \frac{B_{k+1} H_k}{k+1} - \zeta'(-k) \quad (k \in \mathbb{N}_0).$$

2. Closed-form evaluation of series involving the Zeta functions

We first verify the correctness of Lima's conjecture (1.8) and then present several general analogues of Lima's conjecture (1.8). For this purpose, we begin by recalling a known formula for the series associated the Zeta functions (see [23, p. 149, Theorem 3.1]; see also [11]), which is asserted by the following theorem.

THEOREM 1. *For every nonnegative integer n,*

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{\zeta(k,a)}{(k)_{n+1}} t^{n+k} =& \frac{(-1)^n}{n!}\left[\zeta'(-n,a-t) - \zeta'(-n,a)\right] \\ &+ \sum_{k=1}^{n} \frac{(-1)^{n+k}}{n!}\binom{n}{k}\left[(H_n - H_{n-k})\,\zeta(k-n,a) - \zeta'(k-n,a)\right] t^k \\ &+ \left[H_n + \psi(a)\right]\frac{t^{n+1}}{(n+1)!} \quad (|t| < |a|;\, n \in \mathbb{N}_0), \end{aligned} \tag{2.1}$$

where

$$\zeta'(s,a) = \frac{\partial}{\partial s}\zeta(s,a).$$

Substituting $-t$ for t in (2.1), and considering the cases when $n = 2m-1$ $(m \in \mathbb{N})$ and $n = 2m$ $(m \in \mathbb{N})$, and adding and subtracting the resulting identities, we obtain four general closed-form expressions for certain series involving the Zeta functions as in the following theorem.

THEOREM 2. *Each of the following formulas holds true.*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k+1,a)}{(2k+1)_{2m}}\, t^{2m+2k} &= -\frac{1}{2\cdot(2m-1)!}\\ &\quad\cdot\left[\zeta'(-2m+1,a-t)+\zeta'(-2m+1,a+t)-2\,\zeta'(-2m+1,a)\right]\\ &\quad-\frac{1}{(2m-1)!}\sum_{k=1}^{m-1}\binom{2m-1}{2k}\\ &\quad\cdot\left[(H_{2m-1}-H_{2m-1-2k})\,\zeta(2k-2m+1,a)-\zeta'(2k-2m+1,a)\right]t^{2k}\\ &\quad+\left[H_{2m-1}+\psi(a)\right]\frac{t^{2m}}{(2m)!}\qquad (|t|<|a|;\, m\in\mathbb{N});\end{aligned} \tag{2.2}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k,a)}{(2k)_{2m}}\, t^{2m-1+2k} &= \frac{1}{2\cdot(2m-1)!}\left[\zeta'(-2m+1,a+t)-\zeta'(-2m+1,a-t)\right]\\ &\quad+\frac{1}{(2m-1)!}\sum_{k=1}^{m}\binom{2m-1}{2k-1}\\ &\quad\cdot\left[(H_{2m-1}-H_{2m-2k})\,\zeta(2k-2m,a)-\zeta'(2k-2m,a)\right]t^{2k-1}\\ &\qquad\qquad (|t|<|a|;\, m\in\mathbb{N});\end{aligned} \tag{2.3}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k,a)}{(2k)_{2m+1}}\, t^{2m+2k} &= \frac{1}{2\cdot(2m)!}\left[\zeta'(-2m,a-t)+\zeta'(-2m,a+t)-2\zeta'(-2m,a)\right]\\ &\quad+\frac{1}{(2m)!}\sum_{k=1}^{m}\binom{2m}{2k}\\ &\quad\cdot\left[(H_{2m}-H_{2m-2k})\,\zeta(2k-2m,a)-\zeta'(2k-2m,a)\right]t^{2k}\\ &\qquad\qquad (|t|<|a|;\, m\in\mathbb{N});\end{aligned} \tag{2.4}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k+1,a)}{(2k+1)_{2m+1}}\, t^{2m+2k+1} &= \frac{1}{2\cdot(2m)!}\left[\zeta'(-2m,a-t)-\zeta'(-2m,a+t)\right]\\ &\quad-\frac{1}{(2m)!}\sum_{k=1}^{m}\binom{2m}{2k-1}\\ &\quad\cdot\left[(H_{2m}-H_{2m-2k+1})\,\zeta(2k-1-2m,a)-\zeta'(2k-1-2m,a)\right]t^{2k-1}\\ &\quad+\left[H_{2m}+\psi(a)\right]\frac{t^{2m+1}}{(2m+1)!}\qquad (|t|<|a|;\, m\in\mathbb{N}).\end{aligned} \tag{2.5}$$

Setting $a=1$ in (2.2) to (2.5) and using suitable identities given in Lemma 1, we get the following four closed-form expressions for some series involving the Zeta functions.

COROLLARY 1. *Each of the following formulas holds true.*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)_{2m}}t^{2m+2k}&=-\frac{1}{2\cdot(2m-1)!}\\&\quad\cdot\left[\zeta'(-2m+1,1-t)+\zeta'(-2m+1,1+t)-2\,\zeta'(-2m+1)\right]\\&\quad+\frac{1}{(2m-1)!}\sum_{k=1}^{m-1}\binom{2m-1}{2k}\\&\quad\cdot\left[\frac{B_{2(m-k)}}{2(m-k)}\left(H_{2m-1}-H_{2m-1-2k}\right)+\zeta'(2k-2m+1)\right]t^{2k}\\&\quad+\left(H_{2m-1}-\gamma\right)\frac{t^{2m}}{(2m)!}\qquad(|t|<1;\,m\in\mathbb{N});\end{aligned}\tag{2.6}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k)_{2m}}t^{2m-1+2k}&=\frac{1}{2\cdot(2m-1)!}\\&\quad\cdot\left[\zeta'(-2m+1,1+t)-\zeta'(-2m+1,1-t)+\{\ln(2\pi)-H_{2m-1}\}\,t^{2m-1}\right]\\&\quad+\frac{1}{2}\sum_{j=1}^{m-1}\frac{(-1)^{j+1}}{(2m-2j-1)!}\,\frac{\zeta(2j+1)}{(2\pi)^{2j}}t^{2m-2j-1}\qquad(|t|<1;\,m\in\mathbb{N});\end{aligned}\tag{2.7}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k)_{2m+1}}t^{2m+2k}&=(-1)^{m+1}\frac{\zeta(2m+1)}{2\,(2\pi)^{2m}}+\frac{1}{2\cdot(2m)!}\\&\quad\cdot\left[\zeta'(-2m,1+t)+\zeta'(-2m,1-t)+\{\ln(2\pi)-H_{2m}\}\,t^{2m}\right]\\&\quad+\frac{1}{2}\sum_{j=1}^{m-1}\frac{(-1)^{j+1}}{(2m-2j)!}\,\frac{\zeta(2j+1)}{(2\pi)^{2j}}t^{2m-2j}\qquad(|t|<1;\,m\in\mathbb{N});\end{aligned}\tag{2.8}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)_{2m+1}}t^{2m+2k+1}&=\frac{1}{2\cdot(2m)!}\left[\zeta'(-2m,1-t)-\zeta'(-2m,1+t)\right]\\&\quad+\frac{1}{(2m)!}\sum_{k=1}^{m}\binom{2m}{2k-1}\\&\quad\cdot\left[\frac{B_{2(m-k+1)}}{2(m-k+1)}\left(H_{2m}-H_{2m-2k+1}\right)+\zeta'(2k-1-2m)\right]t^{2k-1}\\&\quad+\left(H_{2m}-\gamma\right)\frac{t^{2m+1}}{(2m+1)!}\qquad(|t|<1;\,m\in\mathbb{N}).\end{aligned}\tag{2.9}$$

Setting $t=\frac{1}{2}$ in (2.6) to (2.9) and using suitable identities given in Lemma 1 and Lemma 2, we find the following more explicit closed-form expressions for several series involving the Zeta functions.

COROLLARY 2. *Each of the following formulas holds true.*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)_{2m}\cdot 2^{2k}} &= \frac{1}{(2m-1)!} \\ &\cdot\left[\left(1+\frac{B_{2m}}{m}\right)\ln 2+2\left(2^{2m}-1\right)\left(\frac{B_{2m}H_{2m-1}}{2m}-\ln D_{2m-1}\right)\right] \\ &+\frac{1}{(2m-1)!}\sum_{k=1}^{m-1}\binom{2m-1}{2k} \\ &\cdot\left[\frac{B_{2(m-k)}}{2(m-k)}H_{2m-1}-\ln D_{2m-2k-1}\right]2^{2m-2k}+\frac{H_{2m-1}-\gamma}{(2m)!} \quad (m\in\mathbb{N});\end{aligned} \tag{2.10}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k)_{2m}\cdot 2^{2k}} &= \frac{1}{2\cdot(2m-1)!}\left(\ln\pi-H_{2m-1}\right) \\ &+\frac{1}{2}\sum_{j=1}^{m-1}\frac{(-1)^{j+1}}{(2m-2j-1)!}\frac{\zeta(2j+1)}{\pi^{2j}} \quad (m\in\mathbb{N});\end{aligned} \tag{2.11}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k)_{2m+1}\cdot 2^{2k}} &= \frac{1}{2\cdot(2m)!}\left[\ln(2\pi)-H_{2m}\right]+(-1)^m\left(\frac{1}{2^{2m+1}}-1\right)\frac{\zeta(2m+1)}{\pi^{2m}} \\ &+\frac{1}{2}\sum_{j=1}^{m-1}\frac{(-1)^{j+1}}{(2m-2j)!}\frac{\zeta(2j+1)}{\pi^{2j}} \quad (m\in\mathbb{N});\end{aligned} \tag{2.12}$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)_{2m+1}\cdot 2^{2k}} &= \frac{\ln 2}{(2m)!}+\frac{H_{2m}-\gamma}{(2m+1)!}+\frac{1}{(2m)!}\sum_{k=1}^{m}\binom{2m}{2k-1} \\ &\cdot\left[\frac{B_{2(m-k+1)}}{2(m-k+1)}H_{2m}-\ln D_{2m-2k+1}\right]2^{2m-2k+2} \quad (m\in\mathbb{N}).\end{aligned} \tag{2.13}$$

We now further specialize our results (2.10) to (2.13) only in the case when $m=1$ (see also [23, p. 128, Problem 3]).

COROLLARY 3. *Each of the following formulas holds true.*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)\cdot 2^{2k}} = 2-\gamma+\frac{7}{3}\ln 2-12\ln A, \tag{2.14}$$

where A is the Glaisher-Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\ln k-\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n+\frac{n^2}{4}\right\}, \tag{2.15}$$

the numerical value of A being given by

$$A \cong 1.282427130\cdots$$

and $D_1 = A$.

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)\cdot 2^{2k}} = \ln\pi-1. \tag{2.16}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)\cdot 2^{2k}} = -\frac{3}{2} + \ln(2\pi) + \frac{7\zeta(3)}{2\pi^2}. \tag{2.17}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)(2k+3)\cdot 2^{2k}} = \frac{3}{2} - \frac{\gamma}{3} + \ln 2 - 8\ln A. \tag{2.18}$$

3. Concluding remarks and observations

We conclude this paper by giving the following comments and observations.

1. Equations (2.14), (2.16), and (2.18) are known identities recorded, for example, in [23, p. 223, Eq. (546)], [23, p. 223, Eq. (545)], and [23, p. 229, Eq. (581)], respectively.

2. Equation (2.17) can be verified by using such known formulas as, for example, [23, p. 162, Eq. (26)], [23, p. 212, Eq. (467)], [23, p. 217, Eq. (506)], and [23, p. 100, Eq. (29)].

3. The Glaisher-Kinkelin constant in (2.15) is a constant naturally arising in the theory of the multiple Gamma functions (see, for more general set of constants, [12]).

4. Equation (2.11) verifies the above-asserted correctness of the Lima's experimental conjecture (1.8) [16, Eq. (29)].

Many other interesting corollaries and consequences of our main results asserted by Theorems 1 and 2 (and Corollaries 1, 2 and 3) can be deduced in an analogous manner.

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J. Choi
Department of Mathematics
Dongguk University
Gyeongju, Korea
e-mail: junesang@mail.dongguk.ac.kr

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
e-mail: harimsri@math.uvic.ca