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CVAR–BASED FORMULATION AND APPROXIMATION METHOD FOR A CLASS OF STOCHASTIC VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we consider CVaR-based formulation and approximation method proposed by Chen and Lin [5] for a class of stochastic variational inequality problems (for short, SVIP). Different from the work mentioned above, we regard the regularized gap function for SVIP as a loss function for SVIPs and obtain a restrained deterministic minimization reformulation for SVIPs. We show that the reformulation is a convex program for a wider class of SVIPs than that in [5]. Furthermore, by using the smoothing techniques and Monte Carlo method, we get an approximation problem of the minimization reformulation and consider the convergence of optimal solutions and stationary points of the approximation problems. Finally we apply our proposed model to solve the migration equilibrium problem under uncertainty.

1. Introduction

Given a mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ and a nonempty closed convex set $S \subset \mathbb{R}^n$, the classical variational inequality problem (for short, VIP) is to find a vector $x^* \in S$ such that

$$(x-x^*)^T f(x^*) \geqslant 0, \quad \forall x \in S.$$

It is well known that the VIP has been used widely since its origins in the 1960s (see, for example, [12, 15, 18, 40] and the references therein). Its applications to the modeling of economic equilibrium, optimization and control, transportation and regional science generate great interest (see [12, 15]). If $S = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geqslant 0\}$, then the VIP reduces to the nonlinear complementarity problem (for short, NCP): find an $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad f(x) \geqslant 0, \quad x^T f(x) = 0.$$

For more details about the basic theory, effective algorithms and important applications of the VIP and the NCP, we refer to [9, 12, 14, 18, 30] and the references therein.

In some important practical instances, the data in f often involves some stochastic factors. In order to take the uncertainty into account, the stochastic variational

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inequality problem (for short, SVIP) and stochastic complementarity problems (for short, SCP) have been receiving more and more attention in the recent literature (see [1, 4, 5, 7, 13, 17, 20, 21, 22, 23, 24, 25, 26, 27, 36, 37, 38, 39, 40] and the references therein). Let $(\Omega, \mathscr{F}, P)$ be a probability space. The SVIP is to find an $x^* \in S$ such that

$$P\{\omega \in \Omega : \quad (x - x^*)^T f(x^*, \omega) \geqslant 0, \quad \forall x \in S\} = 1,$$

or equivalently,

$$(x - x^*)^T f(x^*, \omega) \geqslant 0, \qquad \forall x \in S, \quad a.s. \quad \omega \in \Omega, \tag{1}$$

and the SCP is to find an $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \qquad f(x, \omega) \geqslant 0, \qquad x^T f(x, \omega) = 0, \quad a.s. \quad \omega \in \Omega,$$

where $f : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is a mapping and $a.s.$ is abbreviation for "almost surely" under the given probability measure P.

Because of the existence of a random element ω, we can not generally find a vector $x^* \in S$ such that (1) holds almost surely. That is, (1) is not well defined if we think of solving (1) before knowing the realization ω. Therefore, in order to get a reasonable resolution, an appropriate deterministic reformulation of the SVIP becomes an important issue in the study of the considered problem.

In 2005, Chen and Fukushima [4] employed the expected residual minimization method to solve stochastic linear complementarity problems and got a sufficient condition for the existence of a solution to the expected residual minimization problem. Furthermore, Fang et al. [13] got a necessary and sufficient condition for the solution set of the expected residual minimization problem to be nonempty and bounded.

In 2009, Luo and Lin [26] employed the expected residual minimization method to solve the SVIP. By employing the regularized gap function

$$g_a(x, \omega) = \max_{y \in S} \left\{ (x - y)^T f(x, \omega) - \frac{a}{2} \|x - y\|^2 \right\}, \tag{2}$$

they presented the following deterministic optimization problem:

$$\min_{x \in S} E[g_a(x, \omega)], \tag{3}$$

where E denotes the expectation operator and a is a given positive number, and regarded the solution of (3) as the resolution of (1).

On the other hand, in the view of portfolio optimization, Chen and Lin [5] regarded the so called D-gap function $\theta_{ab}(x, \omega) = g_a(x, \omega) - g_b(x, \omega)$ as "loss function" and by employing the risk measure, conditional value-at-risk(CVaR), proposed a new deterministic reformulation for (1) as follows:

$$\min_{x \in \mathbb{R}^n} CVaR_\alpha(x), \tag{4}$$

where $CVaR_\alpha(x)$ is defined as the conditional expectation of the loss associated with x. For a fixed confidence level $\alpha \in (0,1)$, the value-at-risk (VaR) for the loss associated with x is defined as

$$VaR_\alpha(x) = \min\{u | P[\theta_{ab}(x, \omega) \leqslant u] \geqslant \alpha\},$$

which is the lowest value such that, with probability α, the loss will not exceed the amount. Thought VaR is a popular risk measure, it is generally nonconvex and computationally nontractable which makes the resulting VaR optimization problem hard to solve. CVaR, a best convex approximation to VaR, is the expectation of the loss associated with x in the conditional distribution of its upper α-tail, that is,

$$CVaR_\alpha(x) = E_{\alpha-\text{tail}}[\theta_{ab}(x,\omega)],$$

where the α-tail cumulative distribution function of $\theta_{ab}(x,\omega)$ is given by

$$F_\alpha(x,z) = \begin{cases} 0 & \text{for} \quad z < VaR_\alpha(x), \\ (1-\alpha)^{-1}(F(x,z)-\alpha) & \text{for} \quad z \geqslant VaR_\alpha(x), \end{cases}$$

and $F(x,z)$ denotes the distribution of the random variable $z(\omega) = \theta_{ab}(x,\omega)$.

By Theorem 14 of [34], problem (4) is equivalent to

$$\min_{(x,u)\in\mathbb{R}^{n+1}} \Theta(x,u) = u + (1-\alpha)^{-1}E[\theta_{ab}(x,\omega)-u]_+, \tag{5}$$

where $[t]_+ = \max\{t,0\}$ for any $t \in \mathbb{R}$, in the sense that (x^*,u^*) solves (5) if and only if x^* solves (4) and u^* solves $\min_{u\in\mathbb{R}} \Theta(x^*,u)$.

In order to guarantee that problem (5) is a convex program, some conditions of $f(x,\omega)$ are needed. However, the conditions given by Chen and Lin [5] may be too strong and will not be satisfied in some situations (see Example 2.1 for more details). Motivated by the work mentioned above, we regard the regularized gap function $g_a(x,\omega)$ as the "loss function". In a similar manner to [5], we define the deterministic reformulation for (1) as follows:

$$\min_{(x,u)\in S\times\mathbb{R}} \overline{\Theta}(x,u) = \mathbb{E}[\vartheta(x,u,\omega)] = u + (1-\alpha)^{-1}E[g_a(x,\omega)-u]_+, \tag{6}$$

where $\vartheta(x,u,\omega) = u + (1-\alpha)^{-1}[g_a(x,\omega)-u]_+$. We show that (6) is a convex program under suitable conditions and that reformulation (6) is able to solve a wider class of SVIPs than reformulation (5).

The rest of this paper is organized as follows. In Section 2, we show that problem (6) is a convex program and the level sets of $\overline{\Theta}$ are bounded under some suitable conditions. Then, smoothing techniques and Monte Carlo method for solving (6) are presented in Section 3, the convergence results of optimal solutions and stationary points are also given. Finally, in Section 4, we discuss a migration equilibrium problem under uncertainty and use the results in this paper to find the equilibrium pattern.

2. Properties of the objective function

It is well known that $g_a(x,\omega)$ is a merit function for (1), that is, $g_a(x,\omega)$ is nonnegative-valued on S and the variational inequality problem (1) is equivalent to $g_a(x^*,\omega) = 0$. We refer the readers to [10, 11] for more details. It has been shown in

[11] that, for any fixed $\omega \in \Omega$, $g_a(x,\omega)$ is well defined and continuously differentiable everywhere if $f(\cdot,\omega)$ is. In particular, the gradient is given by

$$\nabla_x g_a(x,\omega) = f(x,\omega) - (\nabla_x f(x,\omega) - aI)(y_a(x,\omega) - x),$$

where I denotes the $n\times n$ identity matrix and $y_a(x,\omega) = Proj_S[x - a^{-1}f(x,\omega)]$ denotes the projection of $x - a^{-1}f(x,\omega)$ onto S and the unique solution of the optimization problem on the right hand-side of (2).

LEMMA 1. (see Theorem 2.1 in [5]) *For any positive number a, the regularized gap function $g_a(x,\omega)$ and its gradient $\nabla_x g_a(x,\omega)$ are measurable in ω for every $x \in S$.*

THEOREM 1. *Suppose that $f(x,\omega) = M(\omega)x + Q(\omega)$, where $M : \Omega \to \mathbb{R}^{n\times n}$ and $Q : \Omega \to \mathbb{R}^n$, and $0 < \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)^T M(\omega))$, where Ω_0 is a null subset of Ω, $\lambda_{min}(G)$ stands for the smallest eigenvalue of a symmetric matrix G. We have the following statements:*

(i) If $0 < a \leqslant \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)+M(\omega)^T)$, the regularized gap function $g_a(\cdot,\omega)$ is convex for almost every $\omega \in \Omega$ and hence problem (6) is a convex program.

(ii) Furthermore, if $0 < a < \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)+M(\omega)^T)$, the function $g_a(\cdot,\omega)$ is strongly convex for almost every $\omega \in \Omega$ and the level set $\ell(\tau) = \{(x,u) \in S\times\mathbb{R} | \overline{\Theta}(x,u) \leqslant \tau\}$ is bounded for any given positive scalar τ.

Proof. For any positive semidefinite matrix $A \in \mathbb{R}^{n\times n}$, we have

$$\sqrt{\lambda_{min}(A^T A)} \leqslant \lambda_{min}\left(\frac{A^T + A}{2}\right).$$

From the condition $0 < \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)^T M(\omega))$, we know that

$$0 < \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)+M(\omega)^T)$$

and a is well defined.

For any fixed $y \in S$, we define the function

$$h(x,y,\omega) = (x-y)^T(M(\omega)x + Q(\omega)) - \frac{a}{2}\parallel x-y \parallel^2,$$

and get that $\nabla_x^2 h(x,y,\omega) = 2M(\omega) - aI$. For any vector $z \in \mathbb{R}^n$, we have

$$\begin{aligned} z^T\nabla_x^2 h(x,y,\omega)z &= 2z^T M(\omega)z - a\parallel z\parallel^2 \\ &= z^T[M(\omega)+M(\omega)^T - aI]z \\ &\geqslant [\lambda_{min}(M(\omega)+M(\omega)^T) - a]\parallel z\parallel^2 . \end{aligned} \tag{7}$$

Since $a \leqslant \inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)+M(\omega)^T)$, we have from (7) that $z^T\nabla_x^2 h(x,y,\omega)z \geqslant 0$ holds for any $z \in \mathbb{R}^n$ and almost every $\omega \in \Omega$, which means that the Hessen matrix

$\nabla_x^2 h(x,y,\omega)$ of $h(x,y,\omega)$ is positive semidefinite and hence $h(x,y,\omega)$ is convex in x for any $y \in S$ and almost every $\omega \in \Omega$. Thus, by (2), the regularized gap function $g_a(x,\omega)$ is a convex function in x for almost every $\omega \in \Omega$. It then follows from Corollary 11 of [34] that problem (6) is a convex program.

Furthermore, if

$$0 < a < \inf_{\omega \in \Omega \setminus \Omega_0} \lambda_{min}(M(\omega) + M(\omega)^T),$$

then for any $z \in \mathbb{R}^n$ and almost every $\omega \in \Omega$,

$$z^T \nabla_x^2 h(x,y,\omega) z \geqslant \mu \parallel z \parallel^2$$

holds, where

$$\mu = \inf_{\omega \in \Omega \setminus \Omega_0} \lambda_{min}(M(\omega) + M(\omega)^T) - a$$

and $\mu > 0$. This means the function $h(x,y,\omega)$ is strongly convex in x with modulus μ for any $y \in S$ and almost every $\omega \in \Omega$ and hence, by (2), the regularized gap function $g_a(\cdot,\omega)$ is strongly convex with modulus μ for almost every $\omega \in \Omega$. As a result, the function $E[g_a(x,\omega)]$ is also strongly convex and

$$\lim_{\|x\| \to \infty} E[g_a(x,\omega)] = +\infty.$$

Next, we will assume that there exists a number τ^* such that $\ell(\tau^*)$ is unbounded, that is, there exists a sequence $\{(x^k,u^k)\} \subseteq \ell(\tau^*)$ and $\lim_{k\to\infty} \parallel (x^k,u^k) \parallel = +\infty$.

By the definition of the level set $\ell(\tau)$ and the function $\overline{\Theta}$, for any k, we get

$$\tau^* \geqslant \overline{\Theta}(x^k,u^k) \geqslant u^k$$

and

$$\begin{aligned}
\tau^* \geqslant \overline{\Theta}(x^k,u^k) &= u^k + (1-\alpha)^{-1} E[g_a(x^k,\omega) - u^k]_+ \\
&\geqslant u^k + (1-\alpha)^{-1} E[g_a(x^k,\omega) - u^k] \\
&\geqslant [1-(1-\alpha)^{-1}]u^k.
\end{aligned}$$

Since $\alpha \in (0,1)$, we have $-\alpha^{-1}(1-\alpha)\tau^* \leqslant u^k \leqslant \tau^*$ for each k, that means the sequence $\{u^k\}$ is bounded, and thus $\lim_{k\to\infty} \parallel x^k \parallel = +\infty$. Hence, we obtain that

$$\begin{aligned}
\tau^* &\geqslant u^k + (1-\alpha)^{-1} E[g_a(x^k,\omega)] - (1-\alpha)^{-1} u^k \\
&= [1-(1-\alpha)^{-1}]u^k + E[g_a(x^k,\omega)] \\
&\to +\infty.
\end{aligned}$$

This is a contradiction and we know that the level set $\ell(\tau)$ is bounded for any given positive scalar τ. This completes the proof. $\square$

REMARK 1. In this section, we have proved that problem (6) is a convex program under some suitable conditions. In order to ensure the convexity of problem (5), Chen and Lin [5] assumed that

$$0 < \inf_{\omega\in\Omega\backslash\Omega_0} \lambda_{min}(M(\omega)^T M(\omega))$$

and

$$\sup_{\omega\in\Omega\backslash\Omega_0} \lambda_{max}(M(\omega)^T M(\omega)) < +\infty. \tag{8}$$

Meanwhile, they assumed that

$$S = \left\{ x \in \mathbb{R}^n | \frac{1}{2} x^T Q_i x + q_i^T x + r_i \leqslant 0, i = 1, \cdots, l \right\}, \tag{9}$$

where $Q_i \ (i = 1,2,\cdots,l)$ are $n \times n$ symmetric positive semidefinite matrices, $q_i \in \mathbb{R}^n$ and $r_i \in \mathbb{R}$ are constant vectors or numbers. However, we only need the condition

$$0 < \inf_{\omega\in\Omega\backslash\Omega_0} \lambda_{min}(M(\omega)^T M(\omega))$$

and assume that the constraint set S is nonempty, closed and convex, which is more general than the form given by (9).

The following example shows that all the conditions of Theorem 1 are satisfied but condition (8) fails.

EXAMPLE 1. Let

$$M(\omega) = \begin{pmatrix} 1+\xi(\omega) & 0\ 0\ 0 \\ 0 & 1\ 0\ 0 \\ 0 & 0\ 1\ 0 \\ 0 & 0\ 0\ 1 \end{pmatrix},$$

where $\xi(\omega)$ is an exponential random variable with intensity λ. It is easy to get that

$$\lambda_{min}(M(\omega) + M(\omega)^T) = 2,$$

$$\lambda_{min}(M(\omega)^T M(\omega)) = 1$$

and

$$\lambda_{max}(M(\omega)^T M(\omega)) = [1+\xi(\omega)]^2.$$

Thus, we have

$$\inf_{\omega\in\Omega\backslash\Omega_0} \lambda_{min}(M(\omega)^T M(\omega)) = 1, \qquad \inf_{\omega\in\Omega\backslash\Omega_0} \lambda_{min}(M(\omega) + M(\omega)^T) = 2$$

and so all the conditions of Theorem 1 are satisfied. However, it is easy to see that

$$\sup_{\omega\in\Omega\backslash\Omega_0} \lambda_{max}(M(\omega)^T M(\omega)) = \sup_{\omega\in\Omega\backslash\Omega_0} [1+\xi(\omega)]^2 = +\infty,$$

which implies that condition (8) is not satisfied.

3. Convergence of solutions and stationary points

In this section, we will give the approximation methods for (6). Note that the objective function $\overline{\Theta}(x,u)$ is nonsmooth and contains mathematical expectation. In order to deal with the objective function, we will use smoothing techniques and the Monte carlo sampling techniques.

The smoothing technique is used to deal with the nonsmoothness of the objective function. As an approximation of the function $[\cdot]_+$, the smoothing class of Chen and Mangasarian [6] is as follows: given a small scalar $\mu > 0$, we define

$$[t]_\mu = \begin{cases} t, & t > \mu, \\ \frac{1}{4\mu}(t+\mu)^2, & -\mu \leqslant t \leqslant \mu, \\ 0, & t < -\mu. \end{cases} \tag{10}$$

It is easy to verify that

$$[t]'_\mu = \begin{cases} 1, & t > \mu, \\ \frac{1}{2\mu}(t+\mu), & -\mu \leqslant t \leqslant \mu, \\ 0, & t < -\mu, \end{cases}$$

and the following result.

LEMMA 2. *For any real numbers t and s, we have*

$$|[t]_\mu - [s]_\mu| \leqslant |t-s|, \quad |[t]_\mu - [s]_+| \leqslant |t-s| + \frac{\mu}{4}, \quad |[t]_+ - [s]_+| \leqslant |t-s|.$$

We next tackle the mathematical expectation in the objective function. Throughout this paper, we assume that $E[g_a(x,\omega)-u]_+$ cannot be calculated in a closed form so that we will have to approximate it through discretization. One of the most well known discretization approaches is Monte Carlo method. In general, for an integrable function $\phi : \Omega \to \mathbb{R}$, we approximate the expected value $E[\phi(\omega)]$ with sample average $\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\phi(\omega_i)$, where $\omega_1,\cdots,\omega_{N_k}$ are independently and identically distributed random samples of ω and $\Omega_k = \{\omega_1,\cdots,\omega_{N_k}\}$. By the strong law of large numbers, we know that $\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\phi(\omega_i) = E[\phi(\omega)]$ holds with probability one (abbreviated by"w.p.1").

Let $\hat{\vartheta}(x,u,\omega,\mu) = u + (1-\alpha)^{-1}[g_a(x,\omega)-u]_\mu$. Applying the above techniques, we get the following smooth approximation of (6):

$$\begin{aligned} \min_{(x,u)\in S\times\mathbb{R}} \overline{\Theta}_k(x,u) &= \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\hat{\vartheta}(x,u,\omega_i,\mu_k) \\ &= u + (1-\alpha)^{-1}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x,\omega_i)-u]_{\mu_k}, \end{aligned} \tag{11}$$

where $\mu_k \downarrow 0$ and $N_k \uparrow \infty$ as $k\to\infty$. Note that if the function $f(x,\omega) = M(\omega)x + Q(\omega)$ and satisfies the condition (i) of the Theorem 1, then problem (11) becomes a convex program. We next will investigate the limiting behavior of the optimal solutions and stationary points of (11). Consider the case $f(x,\omega) = M(\omega)x + Q(\omega)$, where $M : \Omega \to \mathbb{R}^{n\times n}$ and $Q : \Omega \to \mathbb{R}^n$ are measurable functions such that

$$\mathbb{E}[\| M(\omega) \|^2] < +\infty, \quad \mathbb{E}[\| Q(\omega) \|^2] < +\infty. \tag{12}$$

This condition implies that

$$\mathbb{E}[M(\omega)] < +\infty, \quad \mathbb{E}[Q(\omega)] < +\infty, \quad \mathbb{E}[\| M(\omega) \| \| Q(\omega) \|] < +\infty \tag{13}$$

and, for any scalar c,

$$\mathbb{E}[\| M(\omega) - cI \|^2] < +\infty. \tag{14}$$

THEOREM 2. *Let $\{(x^k,u^k)\}$ be a sequence of optimal solutions of problem (11). Then, any accumulation point of $\{(x^k,u^k)\}$ is an optimal solution of problem (6).*

Proof. Let (x^*,u^*) be an accumulation point of $\{(x^k,u^k)\}$. Without loss of generality, we assume that (x^k,u^k) itself converges to (x^*,u^*) as k tends to infinity. It is obvious that $(x^*,u^*) \in S\times\mathbb{R}$. At first, we will show that, for any $(x,u) \in S\times\mathbb{R}$,

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_{\mu_k} = \mathbb{E}[g_a(x,\omega) - u]_+. \tag{15}$$

It follows from Lemma 2 that

$$\begin{aligned}
&\left| \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_{\mu_k} - \mathbb{E}[g_a(x,\omega) - u]_+ \right| \\
\leqslant &\left| \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_{\mu_k} - \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_+ \right| \\
&+ \left| \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_+ - \mathbb{E}[g_a(x,\omega) - u]_+ \right| \\
\leqslant &\frac{1}{N_k} \sum_{\omega_i\in\Omega_k} \left| [g_a(x,\omega_i) - u]_{\mu_k} - [g_a(x,\omega_i) - u]_+ \right| \\
&+ \left| \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_+ - \mathbb{E}[g_a(x,\omega) - u]_+ \right| \\
\leqslant &\frac{\mu_k}{4} + \left| \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_+ - \mathbb{E}[g_a(x,\omega) - u]_+ \right|.
\end{aligned}$$

Since

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{\omega_i\in\Omega_k} [g_a(x,\omega_i) - u]_+ = \mathbb{E}[g_a(x,\omega) - u]_+$$

and $\lim_{k\to\infty}\mu_k = 0$, we obtain that the right-hand side of the above inequality converges to 0 as k tends to infinity, and so (15) holds for any $(x,u) \in S\times\mathbb{R}$.

We next show that

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_{\mu_k} = \mathbb{E}[g_a(x^*,\omega)-u^*]_+. \tag{16}$$

In order to get that, we need the following result:

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+ = \mathbb{E}[g_a(x^*,\omega)-u^*]_+. \tag{17}$$

Noting that $\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^*,\omega_i)-u^*]_+ = \mathbb{E}[g_a(x^*,\omega)-u^*]_+$, we just need to verify that

$$\lim_{k\to\infty}\left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+ - \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^*,\omega_i)-u^*]_+\right| = 0. \tag{18}$$

From mean-value theorem, we have

$$\begin{aligned}
&\left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+ - \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^*,\omega_i)-u^*]_+\right| \\
&\leqslant \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left|[g_a(x^k,\omega_i)-u^k]_+ - [g_a(x^*,\omega_i)-u^*]_+\right| \\
&\leqslant \left|u^k-u^*\right| + \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left|g_a(x^k,\omega_i)-g_a(x^*,\omega_i)\right| \\
&\leqslant \left|u^k-u^*\right| + \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\parallel\nabla_x g_a(y^{ki},\omega_i)\parallel\cdot\parallel x^k-x^*\parallel,
\end{aligned} \tag{19}$$

where $y^{ki} = \lambda_{ki}x^k + (1-\lambda_{ki})x^*$ with $\lambda_{ki}\in[0,1]$.

On the other hand, from the nonexpansivity of the projection operator, for any $y\in S$ and $\omega\in\Omega$, we have

$$\begin{aligned}
&\parallel\nabla_x g_a(y,\omega)-\nabla_x g_a(x^*,\omega)\parallel \\
&= \parallel M(\omega)y+Q(\omega)-(M(\omega)-aI)(Proj_S(y-a^{-1}(M(\omega)y+Q(\omega)))-y)- \\
&\quad [M(\omega)x^*+Q(\omega)-(M(\omega)-aI)(Proj_S(x^*-a^{-1}(M(\omega)x^*+Q(\omega)))-x^*)]\parallel \\
&= \parallel(2M(\omega)-aI)(y-x^*)+(M(\omega)-aI)(Proj_S(x^*-a^{-1}(M(\omega)x^*+Q(\omega))) \\
&\quad -Proj_S(y-a^{-1}(M(\omega)y+Q(\omega))))\parallel \\
&\leqslant \|2M(\omega)-aI\|\cdot\|y-x^*\| + \|M(\omega)-aI\|[\|y-x^*\|+a^{-1}\|M(\omega)\|\cdot\|y-x^*\|] \\
&= \left\{\|2M(\omega)-aI\|+\|M(\omega)-aI\|+a^{-1}\|M(\omega)\|\cdot\|M(\omega)-aI\|\right\}\|y-x^*\|
\end{aligned} \tag{20}$$

and thus,

$$
\begin{aligned}
&\left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(y^{ki}, \omega_i) \| - \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| \right| \\
\leqslant & \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(y^{ki}, \omega_i) \| - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(x^*, \omega_i) \| \right| \\
& + \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(x^*, \omega_i) \| - \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| \right| \\
\leqslant & \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(y^{ki}, \omega_i) - \nabla_x g_a(x^*, \omega_i) \| \\
& + \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(x^*, \omega_i) \| - \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| \right| \\
\leqslant & \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \{ \| 2M(\omega_i) - aI \| + \| M(\omega_i) - aI \| \\
& + a^{-1} \| M(\omega_i) \| \cdot \| M(\omega_i) - aI \| \} \| y^{ki} - x^* \| \\
& + \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(x^*, \omega_i) \| - \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| \right| \\
\leqslant & \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \{ \| 2M(\omega_i) - aI \| + \| M(\omega_i) - aI \| \\
& + a^{-1} \| M(\omega_i) \| \cdot \| M(\omega_i) - aI \| \} \| x^k - x^* \| \\
& + \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(x^*, \omega_i) \| - \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| \right|.
\end{aligned} \tag{21}
$$

From (13) and (14), we know that the right-hand side of (21) converges to 0 as k tends to infinity and hence,

$$
\lim_{k \to \infty} \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \| \nabla_x g_a(y^{ki}, \omega_i) \| = \mathbb{E} \| \nabla_x g_a(x^*, \omega) \| . \tag{22}
$$

Since $\lim_{k \to \infty}(x^k, u^k) = (x^*, u^*)$, the right-hand side of (19) converges to 0 and (18) holds, that implies (17) holds also.

From Lemma 2, we have

$$
\begin{aligned}
&\left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} [g_a(x^k, \omega_i) - u^k]_{\mu_k} - \mathbb{E}[g_a(x^*, \omega) - u^*]_+ \right| \\
\leqslant & \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} [g_a(x^k, \omega_i) - u^k]_{\mu_k} - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} [g_a(x^k, \omega_i) - u^k]_+ \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+-\mathbb{E}[g_a(x^*,\omega)-u^*]_+\right| \\
&\leqslant \frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\left|[g_a(x^k,\omega_i)-u^k]_{\mu_k}-[g_a(x^k,\omega_i)-u^k]_+\right| \\
&\quad+\left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+-\mathbb{E}[g_a(x^*,\omega)-u^*]_+\right| \\
&\leqslant \frac{\mu_k}{4}+\left|\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_+-\mathbb{E}[g_a(x^*,\omega)-u^*]_+\right|
\end{aligned}
$$

with the right-hand side converges to 0 as k tends to infinity, which means that (16) holds.

Since (x^k,u^k) is an optimal solution of problem (11) for each k, we have that, for any $(x,u)\in S\times\mathbb{R}$,

$$
u^k+(1-\alpha)^{-1}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x^k,\omega_i)-u^k]_{\mu_k}\leqslant u+(1-\alpha)^{-1}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}[g_a(x,\omega_i)-u]_{\mu_k}.
$$

Letting $k\to\infty$ above, we get from (15) and (16) that

$$
u^*+(1-\alpha)^{-1}\mathbb{E}[g_a(x^*,\omega)-u^*]_+\leqslant u+(1-\alpha)^{-1}\mathbb{E}[g_a(x,\omega_i)-u]_+,
$$

which means (x^*,u^*) is an optimal solution of problem (6). This completes the proof. □

In general, it is difficult to obtain a global optimal solution of problem (11), whereas computation of stationary points is relatively easy. Therefore, it is important to study the limiting behavior of stationary points of problem (11).

DEFINITION 1. (i) (x^k,u^k) is said to be stationary to problem (11) if

$$
0\in\nabla_{(x,u)}\overline{\Theta}_k(x^k,u^k)+\mathcal{N}_{S\times\mathbb{R}}(x^k,u^k),
$$

where $\mathcal{N}_{S\times\mathbb{R}}(x^k,u^k)$ denotes the normal cone (see e.g. [3]) of $S\times\mathbb{R}$ at (x^k,u^k) and

$$
\mathcal{N}_{S\times\mathbb{R}}(x^k,u^k)=\{(x,u)\in\mathbb{R}^{n+1}:(x,u)^T((x',u')-(x^k,u^k))\leqslant 0,\quad\forall(x',u')\in S\times\mathbb{R}\};
$$

(ii) (x^*,u^*) is said to be stationary to problem (6) if

$$
0\in\partial_{(x,u)}\overline{\Theta}(x^*,u^*)+\mathcal{N}_{S\times\mathbb{R}}(x^*,u^*),
$$

where $\partial_{(x,u)}\overline{\Theta}(x^*,u^*)$ is the Clarke generalized gradient (see [8]) of $\overline{\Theta}(x,u)$ with respect to (x,u) at (x^*,u^*) and

$$
\partial_{(x,u)}\overline{\Theta}(x^*,u^*)=\operatorname{conv}\{\lim_{(x,u)\in D_{\overline{\Theta}(\cdot)},(x,u)\to(x^*,u^*)}\nabla_{(x,u)}\overline{\Theta}(x,u)\}
$$

with $D_{\overline{\Theta}(\cdot)}$ representing the set of points near (x^*,u^*) where $\overline{\Theta}$ is Frechét differentiable, $\nabla_{(x,u)}\overline{\Theta}(x,u)$ denoting the usual gradient of $\overline{\Theta}$ and "conv" meaning the convex hull of a set.

In order to get our desirable result, we need the following lemma.

LEMMA 3. (see Theorem 4.4 in [39]) *Let $\{(x^k,u^k)\}$ be a sequence of stationary points of problem (11) and let (x^*,u^*) be an accumulation point of $\{(x^k,u^k)\}$. Assume that $\mathscr{C}$ is a compact subset of $S\times\mathbb{R}$ which contains a neighborhood of (x^*,u^*) w.p.1 and there exists a small positive constant $\mu_0>0$ and a measurable function $\kappa(\omega)$ such that*

$$\sup_{(x,u)\in\mathscr{C},\mu\in[0,\mu_0]} \| \partial_{(x,u)}\hat{\vartheta}(x,u,\omega,\mu) \| \leqslant \kappa(\omega)$$

holds. Then w.p.1 (x^*,u^*) satisfies

$$0\in\mathbb{E}[\partial_{(x,u)}\hat{\vartheta}(x^*,u^*,\omega,0)]+\mathscr{N}_{S\times\mathbb{R}}(x^*,u^*).$$

If, in addition, $\hat{\vartheta}$ satisfies the gradient consistency, that is,

$$\partial_{(x,u)}\hat{\vartheta}(x^*,u^*,\omega,0)\subset\partial_{(x,u)}\vartheta(x^*,u^*,\omega),$$

then (x^*,u^*) is a weak stationary point of (6), that is,

$$0\in\mathbb{E}[\partial_{(x,u)}\vartheta(x^*,u^*,\omega)]+\mathscr{N}_{S\times\mathbb{R}}(x^*,u^*).$$

THEOREM 3. *Let (x^k,u^k) be stationary to problem (11) for each k. Then any accumulation point (x^*,u^*) of $\{(x^k,u^k)\}$ is a stationary point of problem (6).*

Proof. Without loss of generality, we assume that $\{(x^k,u^k)\}$ itself converges to (x^*,u^*). At first, we show that, for any $(\bar{x},\bar{u})\in S\times\mathbb{R}$,

$$\mathbb{E}[\partial_{(x,u)}\vartheta(\bar{x},\bar{u},\omega)]\subseteq\partial_{(x,u)}\overline{\Theta}(\bar{x},\bar{u}). \tag{23}$$

In fact, for any $(\dot{x},\dot{u})$ and $(\ddot{x},\ddot{u})$ in the unit ball $\bar{B}$ with the center at $(\bar{x},\bar{u})$, we have from the mean-value theorem that

$$\begin{aligned}
&\big|\vartheta(\dot{x},\dot{u},\omega)-\vartheta(\ddot{x},\ddot{u},\omega)\big|\\
\leqslant{}& \big|\dot{u}-\ddot{u}\big|+(1-\alpha)^{-1}\big|[g_a(\dot{x},\omega)-\dot{u}]_+-[g_a(\ddot{x},\omega)-\ddot{u}]_+\big|\\
\leqslant{}& \big|\dot{u}-\ddot{u}\big|+(1-\alpha)^{-1}\big|g_a(\dot{x},\omega)-\dot{u}-g_a(\ddot{x},\omega)+\ddot{u}\big|\\
\leqslant{}& \frac{2-\alpha}{1-\alpha}\big|\dot{u}-\ddot{u}\big|+(1-\alpha)^{-1}\|\nabla_x g_a(\tilde{x}_\omega,\omega)\|\cdot\|\dot{x}-\ddot{x}\|,
\end{aligned} \tag{24}$$

where $\tilde{x}_\omega=\lambda_\omega\dot{x}+(1-\lambda_\omega)\ddot{x}$ with $\lambda_\omega\in[0,1]$. Note that there exists a constant $C>0$ such that $\| z\|\leqslant C$ and $\| Proj_S[z]-z\|\leqslant C$ hold for any $z\in\bar{B}$. It follows that

$$
\begin{aligned}
& \| \nabla_x g_a(\tilde{x}_\omega, \omega) \| \\
\leqslant & \| M(\omega) \| \cdot \| \tilde{x}_\omega \| + \| Q(\omega) \| \\
& + \| M(\omega) - aI \| \cdot \| Proj_S[\tilde{x}_\omega - a^{-1}(M(\omega)\tilde{x}_\omega + Q(\omega))] - \tilde{x}_\omega \| \\
\leqslant & C \| M(\omega) \| + \| Q(\omega) \| \\
& + \| M(\omega) - aI \| \cdot \| Proj_S[\tilde{x}_\omega - a^{-1}(M(\omega)\tilde{x}_\omega + Q(\omega))] - Proj_S[\tilde{x}_\omega] \| \\
& + \| M(\omega) - aI \| \cdot \| Proj_S[\tilde{x}_\omega] - \tilde{x}_\omega \| \\
\leqslant & C(\| M(\omega) \| + \| M(\omega) - aI \|) + \| Q(\omega) \| \\
& + a^{-1} \| M(\omega) - aI \| (C \| M(\omega) \| + \| Q(\omega) \|) \\
= & C(\| M(\omega) \| + \| M(\omega) - aI \|) + \| Q(\omega) \| \\
& + a^{-1} C \| M(\omega) - aI \| \cdot \| M(\omega) \| + a^{-1} \| M(\omega) - aI \| \cdot \| Q(\omega) \| . \qquad (25)
\end{aligned}
$$

Thus, from (12)–(14) and (24)–(25), we know that there exists an integrable function $\kappa : \Omega \to \mathbb{R}_+$ such that

$$\left|\vartheta(\dot{x}, \dot{u}, \omega) - \vartheta(\ddot{x}, \ddot{u}, \omega)\right| \leqslant \frac{2-\alpha}{1-\alpha}\left|\dot{u} - \ddot{u}\right| + \| \kappa(\omega) \| \| \dot{x} - \ddot{x} \|$$

holds for any $(\dot{x}, \dot{u})$ and $(\ddot{x}, \ddot{u})$ in $\bar{B}$. Hence, (23) follows from Theorem 9 of Chapter 2 in [35] immediately.

We next show that $0 \in \mathbb{E}[\partial_{(x,u)}\vartheta(x^*, u^*, \omega)] + \mathcal{N}_{S\times\mathbb{R}}(x^*, u^*)$. Noting that $0 \leqslant [t]'_\mu \leqslant 1$ holds for any $t \in \mathbb{R}$ and small scalar $\mu > 0$, and

$$\nabla_{(x,u)}\hat{\vartheta}(x, u, \omega, \mu) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (1-\alpha)^{-1}[g_a(x, \omega) - u]'_\mu \begin{pmatrix} \nabla_x g_a(x, \omega) \\ -1 \end{pmatrix}, \qquad (26)$$

we have from (25) that, for any compact set $\mathscr{C}$ containing a neighborhood of (x^*, u^*) and small scalar μ_0, there exists an integrable function $\kappa' : \Omega \to \mathbb{R}$ such that

$$\sup_{(x,u)\in\mathscr{C}, \mu\in[0,\mu_0]} \| \partial_{(x,u)}\hat{\vartheta}(x, u, \omega, \mu) \| \leqslant \kappa'(\omega)$$

and so

$$0 \in \mathbb{E}[\partial_{(x,u)}\hat{\vartheta}(x^*, u^*, \omega, 0)] + \mathcal{N}_{S\times\mathbb{R}}(x^*, u^*)$$

holds from Lemma 3. It is well known that the function $[t]_\mu$ satisfies the gradient consistency at t (see [32, 39]) and so

$$\partial_{(x,u)}\hat{\vartheta}(x^*, u^*, \omega, 0) \subset \partial_{(x,u)}\vartheta(x^*, u^*, \omega),$$

which implies

$$0 \in \mathbb{E}[\partial_{(x,u)}\vartheta(x^*, u^*, \omega)] + \mathcal{N}_{S\times\mathbb{R}}(x^*, u^*).$$

It follows from (23) that

$$0 \in \partial_{(x,u)}\overline{\Theta}(x^*, u^*) + \mathcal{N}_{S\times\mathbb{R}}(x^*, u^*).$$

This completes the proof. □

REMARK 2. In this section, we investigate the limiting behavior of the optimal solutions and stationary points of (11). Comparing with the proof of Theorem 3.6 in [5], we give a different approach to get the convergence of the stationary points. In fact, Chen and Lin [5] constructed a set-valued mapping and applied Lemma 3.4 of [5] to get the convergence of the stationary points. However, we first show that the accumulation point of the stationary points of (11) is a weak stationary point of (6) and then employ the relation (23) to get the desired result.

4. An application

In this section, we investigate the migration equilibrium problem under uncertainty by using the results in previous sections. We first introduce the migration equilibrium problem under uncertainty and formulate this problem as a SVIP. Then we present some computational results.

We consider a closed economy consisting of n locations, typically denoted by i, and J classes, typically denoted by k. Assume that the attractiveness of any location i to class k is represented by a utility u_i^k. Denote by p_i^k the population of class k at location i and by $\bar{p}^k$ the total population of class k which is fixed and known. Group the utilities into a vector $u \in \mathbb{R}^{Jn}$ and the populations to a vector $p \in \mathbb{R}^{Jn}$, respectively. In general, the utility u can be a function of multiclass population vector p. We assume that there are no migration costs between locations and there are no births and no deaths in this economy.

The population of each class k must be conserved in the economy, that is, the following equations

$$\bar{p}^k = \sum_{i=1}^{n} p_i^k, \qquad k = 1, \dots, J \tag{27}$$

must be satisfied. Let

$$S = \{p \in \mathbb{R}^{Jn} : \ p \geqslant 0 \text{ and satisfy } (27)\}.$$

Then the migration equilibrium problem is to find a multiclass population vector $p* \in S$ such that there is no individual of any class having any incentive to move since a unilateral decision will no longer increase his/her utility. The migration equilibrium problem can be represented as the following variational inequality problem (see Theorem 5.1 in [28]): find a vector $p^* \in S$ such that

$$-u(p^*)^T(p - p^*) \geqslant 0, \qquad \forall p \in S.$$

Let Ω denote the sample space of factors contributing to the uncertainty in the migration problem, such as, weather, environment, public policy, etc. Let $(\Omega, \mathscr{F}, P)$ be a probability space. The utility will be affected by the uncertainty factors, that is, $u = u(p, \omega)$.

The migration equilibrium problem under uncertainty can be written as the following stochastic variational inequality problem: find a vector p^* such that

$$-u(p^*, \omega)^T(p - p^*) \geqslant 0, \qquad \forall p \in S, \quad a.s. \quad \omega \in \Omega.$$

Now we give an example for the migration equilibrium problem with uncertainty.

EXAMPLE 2. Assume that there are two classes and two locations in the economy. Denote by $\xi(\omega)$ a standard normal random variable on the probability space $(\Omega, \mathscr{F}, P)$. The utility functions are given by

$$u_1^1 = -p_1^1 + 5, \quad u_1^2 = -(1+e^{\xi(\omega)})p_1^2 - 0.5p_1^1 + 20,$$
$$u_2^1 = -p_2^1 + 15, \, u_2^2 = -p_2^2 + 10.$$

Let

$$f(p,\omega) = M(\omega)p + Q$$

with

$$M(\omega) = \begin{pmatrix} 1 & 0 & 0 \; 0 \\ 0.5 & 1+e^{\xi(\omega)} & 0 \; 0 \\ 0 & 0 & 1 \; 0 \\ 0 & 0 & 0 \; 1 \end{pmatrix}, \quad p = \begin{pmatrix} p_1^1 \\ p_1^2 \\ p_2^1 \\ p_2^2 \end{pmatrix}, \quad Q = \begin{pmatrix} -5 \\ -20 \\ -15 \\ -10 \end{pmatrix}.$$

Then it is easy to verify that

$$2, \quad 2, \quad 2 + e^{\xi(\omega)} + \sqrt{e^{2\xi(\omega)} + \frac{1}{4}} \quad \text{and} \quad 2 + e^{\xi(\omega)} - \sqrt{e^{2\xi(\omega)} + \frac{1}{4}}$$

are eigenvalues of $M(\omega) + M(\omega)^T$, respectively. It is also easy to see that

$$1, \quad 1, \quad \frac{1}{2}\left(1.25 + x(\omega)^2 + \sqrt{(x(\omega)^2 - 0.75)^2 + 1}\right)$$
$$\text{and} \quad \frac{1}{2}\left(1.25 + x(\omega)^2 - \sqrt{(x(\omega)^2 - 0.75)^2 + 1}\right)$$

are eigenvalues of $M(\omega)^T M(\omega)$, respectively, where $x(\omega) = 1 + e^{\xi(\omega)} \in (1, +\infty)$. Consider the function $\varphi(y) = y - \sqrt{y^2 + b}$ with $b > 0$. It is obvious that $\varphi(y)$ is an increasing function and that $\lim_{y \to +\infty} \varphi(y) = 0$ holds. Taking $y = x(\omega)^2 - 0.75$ and $b = 1$, we get that, for any $\omega \in \Omega$,

$$\frac{1}{2}\left(1.25 + x(\omega)^2 - \sqrt{(x(\omega)^2 - 0.75)^2 + 1}\right) \in \left[\frac{9 - \sqrt{17}}{8}, 1\right].$$

Letting $y = e^{\xi(\omega)}$ and $b = \frac{1}{4}$, for any $\omega \in \Omega$, we have

$$2 + e^{\xi(\omega)} - \sqrt{e^{2\xi(\omega)} + \frac{1}{4}} \in \left[\frac{3}{2}, 2\right].$$

Thus,

$$\begin{aligned} \inf_{\omega \in \Omega \setminus \Omega_0} \lambda_{min}(M(\omega)^T M(\omega)) &= \inf_{\omega \in \Omega \setminus \Omega_0} \frac{1}{2}\left(1.25 + x(\omega)^2 - \sqrt{(x(\omega)^2 - 0.75)^2 + 1}\right) \\ &= \frac{9 - \sqrt{17}}{8} > 0 \end{aligned}$$

and

$$\inf_{\omega\in\Omega\setminus\Omega_0} \lambda_{min}(M(\omega)+M(\omega)^T) = \inf_{\omega\in\Omega\setminus\Omega_0} \left\{ 2+e^{\xi(\omega)} - \sqrt{e^{2\xi(\omega)}+\frac{1}{4}} \right\} = \frac{3}{2}.$$

It follows that all the conditions of Theorem 1 are satisfied. Set $a=1$ and assume that the total populations are give by $\bar{p}^1=2$ and $\bar{p}^2=8$. Let

$$S=\left\{ p \,\middle|\, p\geqslant 0, \begin{pmatrix} 1\ 0\ 1\ 0 \\ 0\ 1\ 0\ 1 \end{pmatrix} p = \begin{pmatrix} 2 \\ 8 \end{pmatrix} \right\}.$$

We solve (6) by using the solver *fmincon* in the optimization Toolbox of Matlab and compute $g_a(x,\omega)$ for each $\omega\in\Omega$ by employing the quadratic programming solver *quadprog* of Matlab. The numerical results are presented in Table 1.

Table 1: *equilibrium patterns with different sizes of samples*

n	p_1^1	p_1^2	p_2^1	p_2^2
10^2	0	3.2776	2.0000	4.7224
5×10^2	0	3.0184	2.0000	4.9816
10^3	0	2.9491	2.0000	5.0509

Here n denotes the size of samples. In Table 1, we take the smoothing parameter $\mu=10^{-3}$, 10^{-4}, 10^{-5} corresponding to the size of sampling data $n=10^2$, 5×10^2, 10^3, respectively. As shown in Table 1, the final population of class 1 at location 1 is $p_1^1=0$. This is realistic due to the fact that the utility u_1^1 of location 1 to class 1 is always less than the utility u_2^1 of location 2 to class 1. Thus, all the individual of class 1 has incentive to move to location 2.

Example 2 reveals that reformulation (6) is efficient and can be used to solve some practical problems.

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