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A NOTE ON THE MOMENT INEQUALITIES FOR STOCHASTIC INTEGRAL

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Abstract. In this paper, we shall apply Itô formula to establish several moment inequalities for stochastic integrals. These will demonstrate the powerfulness of the Itô formula.

1. Introduction

It is well known that the Doob inequality with submartingale process $\{M_t\}_t \geqslant 0$ of the form

$$E\left(\sup_{0\leqslant t\leqslant b} M_t^p\right) \leqslant \left(\frac{p}{p-1}\right)^p EM_b^p$$

plays an important role in characterizing many probability theory, stochastic process, and stochastic differential equation problems. Thus, many authors have studied the generalization, sharpness, application, and similar inequalities of the inequality (see [1], [2], [4], [9]).

One of the most important moment inequality for stochastic integrals is as follows:

THEOREM 1. [5] *If* $p \geqslant 2$, $g \in \mathscr{M}^2([0,T];R^{d\times m})$ *such that*

$$E\int_0^T |g(s)|^p\,\mathrm{d}s < \infty,$$

then

$$E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\,\mathrm{d}s.$$

In particular, for $p = 2$*, there is equality.*

The moment inequalities for stochastic integrals have received considerable attention because of the important applications to a variety of problems in diverse field of stochastic differential and stochastic functional differential equations (see [3], [5]–[8], [10]–[12]). Also a interesting inequality were introduced by Mao [5] by using the Itô's formula as follows:

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THEOREM 2. *If* $p \geqslant 2$, $g \in \mathscr{M}^2([0,T];R^{d\times m})$ *such that*

$$E\int_0^T |g(s)|^p\,\mathrm{d}s < \infty,$$

then

$$E\left(\sup_{0\leqslant t\leqslant T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right) \leqslant \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\,\mathrm{d}s.$$

After this inequality was appeared, Mao and several authors [3], [7], [8], [10]–[12] applied their theorem for stochastic differential equations which includes existence and uniqueness of the solution for stochastic functional differential equations and stochastic functional differential equations (SFDEs) and neutral SFDEs with infinite delay.

In this paper, a new stochastic integral inequality, named as a moment inequality, will be introduced in Section 3, which includes the m-dimensional Brownian motion on complete probability space and $d\times m$-matrix-valued measurable $\mathscr{F}_t$-adapted processes. The main results of this paper can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some stochastic differential equations.

2. Preliminary

Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\mathrm{trace}(A^TA)}$. Let t_0 be a positive constant and $(\Omega,\mathscr{F},P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_{t_0}$ contains all P-null sets). Assume that $B(t)$ is a m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t),B_2(t),\ldots,B_m(t))^T$.

Let $\mathscr{M}^2((0,T];R^{d\times m})$ denote the family of all $d\times m$-matrix-valued measurable $\{\mathscr{F}_t\}_{t\geqslant t_0}$-adapted process $f = \{(f_{ij}(t))_{d\times m}\}_{0\leqslant t\leqslant T}$ such that

$$E\int_0^T |f(t)|^2\,\mathrm{d}t < \infty.$$

And let $\mathscr{L}^p([a,b];R^d)$ denote the family of all R^d-valued measurable $\mathscr{F}_t$-adapted processes $\{f(t)\}_{a\leqslant t\leqslant b}$ such that $\int_a^b |f(t)|^p\,\mathrm{d}t < \infty$ and $L^p(\Omega;R^d)$ denote the family of all R^d-valued random variable f such that $E|f|^p < \infty$.

With all the above preparation, consider the following Doob's martingale inequality:

THEOREM 3. *Let* $\{M_t\}_{t\geqslant 0}$ *be* R^d*-valued martingale. Let* $[a,b]$ *be a bounded interval in* R_+. *If* $p>1$ *and* $M_t \in L^p(\Omega;R^d)$, *then*

$$E\left(\sup_{0\leqslant t\leqslant b}|M_t|^p\right) \leqslant \left(\frac{p}{p-1}\right)^p E|M_b|^p$$

To find out the inequality, we give the definition of d-dimensional Itô process and the theorem of multi-dimensional Itôs formula.

DEFINITION 1. A d-dimensional Itô process is an R^d-value continuous process $x(t) = (x_1(t), \cdots, x_d(t))^T$ on $t \geqslant 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B(s),$$

where $f = (f_1, \cdots, f_d)^T \in \mathscr{L}^1(R_+; R^d)$ and $g = (g_{ij})_{d\times m} \in \mathscr{L}^2(R_+; R^{d\times m})$. We shall say that $x(t)$ has stochastic differential $\mathrm{d}x(t)$ on $t \geqslant 0$ given by

$$\mathrm{d}x(t) = f(t)\,\mathrm{d}t + g(t)\,\mathrm{d}B(t).$$

THEOREM 4. *Let $x(t) =$ be a d-dimensional Itô process on $t \geqslant 0$ with the stochastic differential given by*

$$dx(t) = f(t)\,dt + g(t)\,dB(t),$$

where $f = (f_1, \cdots, f_d)^T \in \mathscr{L}^1(R_+; R^d)$ and $g = (g_{ij})_{d\times m} \in \mathscr{L}^2(R_+; R^{d\times m})$. Let $C^{2,1}(R^d \times R_+; R)$ denote the family of all real-valued functions $V(x,t)$ defined $R^d \times R_+$ such that they are continuously twice differentiable in x and once in t. And let $V \in C^{2,1}(R^d \times R_+; R)$. Then $V(x(t),t)$ ia again an Itô process with the stochastic differential given by

$$\begin{aligned} dV(x(t),t) = {} & [V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}trace(g^T(t))V_{xx}(x(t),t)g(t)]dt \\ & + V_x(x(t),t)g(t)\,dB(t) \quad a.s. \end{aligned}$$

3. Moment inequalities

In this section we shall apply Itô's formula to establish several very important moment inequalities for stochastic integrals as well as the exponential submartingale inequality.

The following lemma is known as a property for Itô integral which will play an important role in this section.

LEMMA 1. [5] *Let $p \geqslant 2$, $g \in \mathscr{M}^2([0,T]; R^{d\times m})$ and let ρ, τ be two stopping times such that $0 \leqslant \rho \leqslant \tau \leqslant T$. Then*

$$E\left(\int_\rho^\tau f(s)\,\mathrm{d}B(s) | \mathscr{F}_\rho \right) = 0.$$

Now we describe the moment inequalities for stochastic integrals.

THEOREM 5. *If $1 < p < 2$, $g \in \mathscr{M}^2([0,T]; R^{d\times m})$ such that*

$$E\left| \int_0^T g(s)\,\mathrm{d}B(s) \right|^p < \infty,$$

then

$$E\int_0^T |g(s)|^p\,\mathrm{d}s \leqslant \left(\frac{p(p-1)}{2}\right)^{-\frac{p}{2}} T^{\frac{2-p}{2}} E\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p. \tag{1}$$

Proof. *For* $0 \leqslant t \leqslant T$, *set*

$$x(t) = \int_0^t g(s)\,\mathrm{d}B(s).$$

By Itô's formula and Lemma 1,

$$\begin{aligned} E|x(t)|^p &= \frac{p(p-1)}{2} E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s + pE\int_0^t |x(s)|^{p-1} g(s)\,\mathrm{d}B(s) \\ &= \frac{p(p-1)}{2} E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s. \end{aligned} \tag{2}$$

Using the Hölder's inequality, we have

$$\begin{aligned} E|x(t)|^p &= \frac{p(p-1)}{2} E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s \\ &\geqslant \frac{p(p-1)}{2}\left(E\int_0^t |x(s)|^p\,\mathrm{d}s\right)^{\frac{p-2}{p}} \left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)^{\frac{2}{p}} \\ &= \frac{p(p-1)}{2}\left(\int_0^t E|x(s)|^p\,\mathrm{d}s\right)^{\frac{p-2}{p}} \left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)^{\frac{2}{p}}. \end{aligned}$$

Note from (2) that $E|x(t)|^p$ *is nondecreasing in* t. *It follows*

$$E|x(t)|^p \geqslant \frac{p(p-1)}{2}\left(tE|x(s)|^p\right)^{\frac{p-2}{p}} \left(E\int_0^t |g(s)|^p\,\mathrm{d}s\right)^{\frac{2}{p}}.$$

This yields

$$E|x(t)|^p \geqslant \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} t^{\frac{p-2}{p}} E\int_0^t |g(s)|^p\,\mathrm{d}s,$$

and the required (1) follows by replacing t *with* T. □

THEOREM 6. *If* $p > 1$, *and* $B(t)$ *is a* m*-dimensional Brownian motion defined on complete probability space such that*

$$E\int_0^T |B(s)|^p\,\mathrm{d}s < \infty,$$

then

$$E\left|\int_0^T B(s)\,\mathrm{d}s\right|^p \leqslant p^p T^{p-1} E\int_0^T |B(s)|^p\,\mathrm{d}s. \tag{3}$$

Proof. *For* $0 \leqslant t \leqslant T$, *set*

$$x(t) = \int_0^t B(s)\,\mathrm{d}s.$$

By Itô's formula and Lemma 1,

$$|x(t)|^p - |x(0)|^p = p\int_0^t |x(s)|^{p-1} B(s)\,\mathrm{d}s, \tag{4}$$

which implies that

$$E|x(t)|^p \leqslant pE\int_0^t |x(s)|^{p-1}|B(s)|\,\mathrm{d}s.$$

Using the Hölder's inequality, we have

$$E|x(t)|^p \leqslant p\left(\int_0^t E|x(s)|^p\,\mathrm{d}s\right)^{\frac{p-1}{p}}\left(E\int_0^t |B(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Note from (4) that $E|x(t)|^p$ *is nondecreasing in* t. *It follows*

$$E|x(t)|^p \leqslant p\left(tE|x(s)|^p\right)^{\frac{p-1}{p}}\left(E\int_0^t |B(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

This yields

$$E|x(t)|^p \leqslant p^p t^{p-1} E\int_0^t |B(s)|^p\,\mathrm{d}s,$$

and the required inequality follows by replacing t *with* T. □

THEOREM 7. *If* $p > 1$, $g \in \mathscr{M}^2([0,T];R^{d\times m})$ *such that*

$$E\int_0^T |g(s)|^{2p}\,\mathrm{d}s < \infty,$$

then

$$E\left|1 + \int_0^T g(s)\,\mathrm{d}B(s) + \int_0^T |g(s)|^2\,\mathrm{d}s\right|^p \leqslant 1 + \left(p + \frac{p(p-1)}{2}\right)^p T^{p-1} E\int_0^T |g(s)|^{2p}\,\mathrm{d}s.$$

Proof. *For* $0 \leqslant t \leqslant T$, *set*

$$x(t) = 1 + \int_0^T g(s)\,\mathrm{d}B(s) + \int_0^T |g(s)|^2\,\mathrm{d}s.$$

By Itô's formula and Lemma 1,

$$|x(t)|^p - |x(0)|^p = p\int_0^t |x(s)|^{p-1}|g(s)|^2\,\mathrm{d}s + \frac{p(p-1)}{2}\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s, \tag{5}$$

which implies that

$$E|x(t)|^p = 1 + pE\int_0^t |x(s)|^{p-1}|g(s)|^2\,\mathrm{d}s + \frac{p(p-1)}{2}E\int_0^t |x(s)|^{p-2}|g(s)|^2\,\mathrm{d}s, \qquad (6)$$

By the equation (5) and Hölder's inequality, we have

$$E|x(t)|^p \leqslant 1 + \left(p + \frac{p(p-1)}{2}\right)\left(\int_0^t E|x(s)|^p\,\mathrm{d}s\right)^{\frac{p-1}{p}}\left(E\int_0^t |g(s)|^{2p}\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

Note from (6) that $E|x(t)|^p$ *is nondecreasing in* t. *It follows*

$$E|x(t)|^p \leqslant 1 + \left(p + \frac{p(p-1)}{2}\right)\left(tE|x(s)|^p\right)^{\frac{p-1}{p}}\left(E\int_0^t |g(s)|^{2p}\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

This yields

$$E|x(t)|^p \leqslant 1 + \left(p + \frac{p(p-1)}{2}\right)t^{p-1}E\int_0^t |g(s)|^{2p}\,\mathrm{d}s,$$

and the required inequality follows by replacing t *with* T. □

THEOREM 8. *If* $p > 1$, $g \in \mathscr{M}^2([0,T];R^{d\times m})$ *such that*

$$E\int_0^T |g(s)|^{2p}\,\mathrm{d}s < \infty,$$

then

$$E\left(\sup_{0\leqslant t\leqslant T}\left|1 + \int_0^t g(s)\,\mathrm{d}B(s) + \int_0^t |g(s)|^2\,\mathrm{d}s\right|^p\right) \leqslant \left(\frac{p^2(p+1)}{2(p-1)}\right)^{\frac{p}{2}} T^{p-1}E\int_0^T |g(s)|^{2p}\,\mathrm{d}s.$$

Proof. *We see that the stochastic integral* $1 + \int_0^T g(s)\,\mathrm{d}B(s) + \int_0^T |g(s)|^2\,\mathrm{d}s$ *is and* R^d *-valued continuous matingale. Hence, by Doob,s martingale inequality, we have*

$$\begin{aligned} &E\left(\sup_{0\leqslant t\leqslant T}\left|1 + \int_0^t g(s)\,\mathrm{d}B(s) + \int_0^t |g(s)|^2\,\mathrm{d}s\right|^p\right) \\ &\leqslant \frac{p}{p-1}E\left(\left|1 + \int_0^T g(s)\,\mathrm{d}B(s) + \int_0^T |g(s)|^2\,\mathrm{d}s\right|^p\right). \end{aligned}$$

In view of Theorem 7, implies that

$$E\left(\sup_{0\leqslant t\leqslant T}\left|1 + \int_0^t g(s)\,\mathrm{d}B(s) + \int_0^t |g(s)|^2\,\mathrm{d}s\right|^p\right) \leqslant \left(\frac{p^2(p+1)}{2(p-1)}\right)^{\frac{p}{2}} T^{p-1}E\int_0^T |g(s)|^{2p}\,\mathrm{d}s.$$

□

REMARK 1. In the Theorem 5-8, we established several moment inequalities for stochastic integrals with m-dimensional Brownian motion defined on complete probability space. Moreover, our new inequalities in this paper is completed by Itô formula.

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