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CONVERGENCE THEOREMS FOR k–DIMEICONTACTIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we prove weak and strong convergence theorems for Moudafi's iterative scheme of two k-demicontractive mappings in Hilbert spaces. Our results improve and extend the recent results of Kim and some others.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the fixed point set of a mapping $T : C \to C$ by $F(T)$. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx-Ty\| \leqslant \|x-y\|$$

for all $x,y \in C$. A mapping $T : C \to C$ is said to be *quasi-nonexpansive* if the set of fixed points of T is nonempty and

$$\|Tx-y\| \leqslant \|x-y\|$$

for all $x \in C$ and $y \in F(T)$. If $T : C \to C$ is nonexpansive and $F(T)$ is nonempty, then T is quasi-nonexpansive. A mapping $T : C \to C$ is said to be *demicontractive* (or k*-demicontractive*) if there exists $k \in [0,1)$ such that

$$\|Tx-p\|^2 \leqslant \|x-p\|^2 + k\|Tx-x\|^2$$

for all $x \in C$ and $p \in F(T)$. The concept of a demicontractive mapping was introduced by Hicks and Kubicek [6] and this class includes the well-known classes of quasi-nonexpansive mappings (see Remark 2.3) and strictly pseudocontractive mappings of Browder-Petryshn type with the nonempty fixed point sets of the given nonlinear mappings. Furthermore, a mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx-Fy\|^2 \leqslant \langle x-y, Fx-Fy\rangle$$

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for all $x,y \in C$ (see [2, 4, 5, 17]). It is known that a mapping $F: C \to C$ is firmly nonexpansive if and only if

$$\|Fx - Fy\|^2 + \|(I-F)x - (I-F)y\|^2 \leqslant \|x-y\|^2$$

for all $x,y \in C$, where I is the identity mapping on H. Also, it is known that every firmly nonexpansive mapping is nonexpansive (see Remark 2.3) and of the from $F = 1/2(I+T)$ with a nonexpansive mapping T (see [4, 5] for instance). In 2008, Kohsaka and Takahashi [11] studied the existence and approximation of fixed points of the mappings of firmly nonexpansive type in Banach spaces. Kohsaka and Takahashi [12] also introduced the class of nonspreading mappings in Banach spaces and proved some convergence theorems for nonspreading mappings.

Let E be a real smooth, strictly convex and reflexive Banach space and let j denote the duality mapping of E. Let C be a nonempty closed convex subset of a Banach space E. A mapping $T: C \to C$ is said to be *nonspreading* if

$$\phi(Tx,Ty) + \phi(Ty,Tx) \leqslant \phi(Tx,y) + \phi(Ty,x)$$

for all $x,y \in C$, where $\phi(x,y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$ for all $x,y \in E$. In the case when E is a Hilbert space, we know that $\phi(x,y) = \|x-y\|^2$ for all $x,y \in E$. So, a nonspreading mapping $S: C \to C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leqslant \|Sx - y\|^2 + \|x - Sy\|^2 \tag{1}$$

for all $x,y \in C$. In [7], it is proved that (1) is equivalent to the following:

$$\|Sx - Sy\|^2 \leqslant \|x-y\|^2 + 2\langle x - Tx, y - Ty\rangle \tag{2}$$

for all $x,y \in C$. We know that, in Hilbert spaces, every firmly nonexpansive mapping is nonspreading (see Remark 2.3) and, if the set of fixed points of a nonspreading mapping is nonempty, then the nonspreading mapping is quasi-nonexpansive ([12]).

On the other hand, weak convergence theorems for two nonexpansive mappings T_1, T_2 of C into itself were discussed by Takahashi and Tamura in [20]. They considered the following iterative procedure:

$$\begin{cases} x_1 \in C \ \textit{chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1(\beta_n T_2 x_n + (1-\beta_n)x_n) \end{cases} \tag{3}$$

for all $n \geqslant 1$, where $F(T_1) \cap F(T_2)$ is nonempty. In 2007, Moudafi [14] considered another iterative procedure for two nonexpansive mappings T_1, T_2 of C into itself as follows:

$$\begin{cases} x_1 \in C \ \textit{chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n T_1 x_n + (1-\beta_n)T_2 x_n) \end{cases} \tag{4}$$

for all $n \geqslant 1$, where $F(T_1)$ and $F(T_2)$ are nonempty. In 2009, Iemoto and Takahashi [7] extended the result of [14] by approximating of common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space by using Moudafi's iterative scheme as follows:

$$\begin{cases} x_1 \in C \ \textit{chosen arbitrary}, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\beta_n S x_n + (1-\beta_n)T x_n) \end{cases} \tag{5}$$

for all $n \geqslant 1$, where S is a nonspreading mapping, T is a nonexpansive mapping and $F(S) \cap F(S)$ is nonempty. Recently, Kim [9] generalized the result of [7] by approximating of common fixed points of two quasi-nonexpansive mappings in a Hilbert space by using Moudafi's iterative scheme.

In this paper, we study the approximation of common fixed points for two k-demicontractive mappings in a Hilbert space by using Moudafi's iterative scheme. The result of this paper extend and generalize the corresponding results given by Kim and some others in the literature.

2. Preliminaries

Throughout this paper, we denote $\mathbb{N}$ by the set of positive integers and $\mathbb{R}$ by the set of real numbers. Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$.

First, we start with a brief recollection of basic concepts and facts in a Hilbert space H and the following results are very important for our consideration in next sections.

In a Hilbert space H, it is well known that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y \rangle. \tag{6}$$

Further, in a Hilbert space H, we have

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{7}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see, for instance, [19]). We know that a Hilbert space H satisfies *Opial's condition* ([15]), that is, for any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{i\to\infty} \|x_n - y\| \tag{8}$$

for all $y \in H$ with $y \neq x$, where "$\rightharpoonup$" stands for the weak convergence of the sequence $\{x_n\}$. A mapping $T : C \to H$ is said to be *demiclosed* at $y \in H$ ([3]) if, for any sequence $\{x_n\}$ in C with $x_n \rightharpoonup x$ and $Tx_n \to y$, it follows that $x \in C$ and $Tx = y$. If $I - T$ is demiclosed at zero, i.e., for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply $x = Tx$, where "$\to$" denotes the strong convergence of the sequence $\{x_n\}$. We say that a mapping $T : C \to C$ satisfies the *condition* $(\mathbf{A})$ ([18]) if there exists a nondecreasing $f : [0,\infty] \to [0,\infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$f(d(x,F(T)) \leqslant \|x - Tx\|$$

for all $x \in C$, where $d(x,F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

In 2005, Khan and Fukhar-ud-din [8] modified this condition for two mappings as follows: Two mappings $S, T : C \to C$ satisfy the *condition* $(\mathbf{A}')$ if there exists a nondecreasing $f : [0,\infty] \to [0,\infty]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$f(d(x,\mathscr{F})) \leqslant \frac{1}{2}(\|x - Tx\| + (\|x - Sx\|)$$

for all $x \in C$, where $d(x,\mathscr{F}) = \inf\{\|x - x^*\| : x^* \in \mathscr{F}\}$ and $\mathscr{F} := F(T) \cap F(S)$.

LEMMA 1. ([22]) *Suppose that* $\{s_n\}$ *and* $\{e_n\}$ *are two sequences of nonnegative real numbers such that* $s_{n+1} \leqslant s_n + e_n$ *for all* $n \in \mathbb{N}$. *If* $\sum_1^\infty e_n < \infty$, *then* $\lim_{n\to\infty} s_n$ *exists.*

THEOREM 1. ([21]) *Let* H *be a Hilbert space and let* $\{x_n\}$ *be a bounded sequence in* H. *Then* $\{x_n\}$ *is weakly convergent if and only if each weakly convergent subsequence of* $\{x_n\}$ *has the same weak limit, that is, for any* $x \in H$,

$$x_n \rightharpoonup x \quad \Longleftrightarrow \quad (x_{n_i} \rightharpoonup y \Longrightarrow x = y).$$

Now, we give some relations among a k-demicontractive mapping, a quasi-nonexpansive mapping, a nonspreading mapping and a nonexpansive mapping and their examples.

REMARK 1. (1) In Hilbert spaces, every firmly nonexpansive mapping is nonexpansive and nonspreading.

(2) In Hilbert spaces, every nonspreading mapping with the nonempty fixed point set is quasi-nonexpansive.

(3) Every nonexpansive mapping with the nonempty fixed point set is quasi-nonexpansive.

(4) Every quasi-nonexpansive mapping is k-demicontractive.

EXAMPLE 1. Let H be a Hilbert space. Set $E = \{x \in H : \|x\| \leqslant 1\}$, $D = \{x \in H : \|x\| \leqslant 2\}$ and $C = \{x \in H : \|x\| \leqslant 3\}$. Define a mapping $T : C \to C$ as follows:

$$Tx = \begin{cases} x, & \text{if } x \in D, \\ P_E(x), & \text{if } x \in C\backslash D, \end{cases} \tag{9}$$

where P_E is the metric projection of H onto E. Then T is a nonspreading mapping, but it is not nonexpansive (see [7]).

EXAMPLE 2. Consider $\mathbb{R}$ with the usual norm and $C = [0,2]$. Let $T : C \to C$ be a function defined by

$$Tx = \frac{1}{3}(x^2 + 2)$$

for all $x \in C$, where $x + 2 \leqslant 3$ for all $x \in [0,1]$ and $x + 2 = 3$ for all $x > 1$. Then T is a quasi-nonexpansive mapping, but it is not nonspreading and nonexpansive (see [9]).

EXAMPLE 3. Consider $\mathbb{R}^2$ with the usual norm and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by

$$T(x,y) = (-y,x)$$

for all $(x,y) \in \mathbb{R}^2$. Clearly, T is a nonexpansive mapping, but T is not firmly nonexpansive. In fact, if we take $x^* = (1,2)$, $y^* = (4,6) \in \mathbb{R}^2$, then

$$\|Tx^* - Ty^*\| = \|T(1,2) - T(4,6)\| = 5 = \|x^* - y^*\|$$

and

$$\|(x^* - Tx^*) - (y^* - Ty^*)\| = \|[(1,2) - T(1,2)] - [(4,6) - T(4,6)]\| = 5\sqrt{2}.$$

Thus we have

$$\|Tx^* - Ty^*\|^2 + \|(x^* - Tx^*) - (y^* - Ty^*)\|^2 = 25 + 50 > 25 = \|x^* - y^*\|^2$$

and so T is not firmly nonexpansive.

EXAMPLE 4. Let $\mathbb{R}$ denote the reals with the usual norm and $T : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$Tx = \begin{cases} x, & \text{if } -\infty < x < 0, \\ -3x, & \text{if } 0 \leqslant x < \infty, \end{cases}$$

Observe that $F(T) = (-\infty, 0]$. If $x \in (-\infty, 0)$, then $x \in Fix(T)$. For any $p \in Fix(T)$, we have

$$|Tx - p|^2 = |x - p|^2, \quad |x - Tx|^2 = 0.$$

Thus $|Tx - p|^2 = |x - p|^2 + k|x - Tx|^2$ for all $k \in [0,1)$. If $x \in [0, \infty)$, then, for any $p \in$ Fix(T), we get

$$|Tx - p|^2 = |-3x - p|^2 = 9x^2 + 6xp + p^2$$

and

$$|x - Tx|^2 = |x + 3x|^2 = 16x^2.$$

Hence, since $6xp \leqslant 0$ and $-2xp \geqslant 0$, we have

$$\begin{aligned} |Tx - p|^2 &= 9x^2 + 6xp + p^2 \\ &\leqslant x^2 - 2xp + p^2 + 8x^2 \\ &= |x - p|^2 + \frac{1}{2}(16x^2) \\ &= |x - p|^2 + \frac{1}{2}|x - Tx|^2. \end{aligned}$$

Observe that, for any $x \in (0, \infty)$, $|Tx - 0|^2 = 9|x - 0|^2 \nleq |x - 0|^2$ and so T is not a quasi-nonexpansive mapping. Moreover, T is not a nonexpansive mapping. In fact, for any $x, y \in (0, \infty)$ with $x \neq y$, we have

$$|Tx - Ty|^2 = 9|x - y|^2 \nleq |x - y|^2.$$

Also, T is not a nonspreading mapping. In fact, if we take $x = 0$ and $y = 3$, then we get

$$|T(0) - T(3)|^2 = 81, \quad |T(0) - 3|^2 = 9, \quad |T(3) - 0|^2 = 81$$

and hence

$$2|T(0) - T(3)|^2 = 162 > 9 + 81 = 90.$$

Therefore, T is $\frac{1}{2}$-demicontractive, but it is not quasi-nonexpansive, nonexpansive and nonspreading.

3. Weak convergence theorems

In this section, we prove weak convergence theorems for two k-demicontractive mappings in a Hilbert space H by using Moudafi's iterative scheme.

THEOREM 2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,T : C \to C$ be two k-demicontractive mappings such that $I-S$ is demiclosed at zero with $F(S)\cap F(T) \neq \emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n S x_n + (1-\beta_n) T x_n) \end{cases} \tag{10}$$

for all $n \geqslant 1$, where $\{\alpha_n\} \subset [k,1]$ and $\{\beta_n\} \subset [0,1]$ are the sequences such that

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - k) > 0, \quad \sum_{n=1}^{\infty}(1-\beta_n) < \infty.$$

Then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)$.

Proof. First, we show that $\{x_n\}$ is bounded. Let $U_n = \beta_n S + (I-\beta_n)T$ for all $n \geqslant 1$. From (7), it follows that, for all $x,y \in C$,

$$\begin{aligned} &\|U_n x - U_n y\|^2 \\ &= \|\beta_n(Sx-Sy) + (1-\beta_n)(Tx-Ty)\|^2 \\ &= \beta_n\|Sx-Sy\|^2 + (1-\beta_n)\|Tx-Ty\|^2 - \beta_n(1-\beta_n)\|(Sx-Sy)-(Tx-Ty)\|^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned} \|x-U_n x\|^2 &= \|x-\beta_n Sx - (1-\beta_n)Tx\|^2 \\ &= \|\beta_n(x-Sx) + (1-\beta_n)(x-Tx)\|^2 \\ &= \beta_n\|x-Sx\|^2 + (1-\beta_n)\|Tx-x\|^2 - \beta_n(1-\beta_n)\|Tx-Sx\|^2. \end{aligned} \tag{12}$$

Using (11) and (12), since S and T are k-demicontractive, for any $p \in F(S)\cap F(T)$, we have

$$\begin{aligned} &\|x_{n+1}-p\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|U_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - U_n x_n\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\big[\beta_n\|Sx_n-p\|^2 + (1-\beta_n)\|Tx_n-p\|^2 \\ &\quad -\beta_n(1-\beta_n)\|Sx_n-Tx_n\|^2\big] - \alpha_n(1-\alpha_n)\big[\beta_n\|x_n-Sx_n\|^2 \\ &\quad +(1-\beta_n)\|Tx_n-x_n\|^2 - \beta_n(1-\beta_n)\|Tx_n-Sx_n\|^2\big] \\ &\leqslant \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\big[\|x_n-p\|^2 + k\beta_n\|x_n-Sx_n\|^2 \\ &\quad +k(1-\beta_n)\|Tx_n-x_n\|^2 - \beta_n(1-\beta_n)\|Sx_n-Tx_n\|^2\big] - \alpha_n(1-\alpha_n)\big[\beta_n\|x_n-Sx_n\|^2 \\ &\quad +(1-\beta_n)\|Tx_n-x_n\|^2 - \beta_n(1-\beta_n)\|Tx_n-Sx_n\|^2\big] \end{aligned}$$

$$
\begin{aligned}
&\leqslant \|x_n - p\|^2 - (1-\alpha_n)(\beta_n)(\alpha_n - k)\|x_n - Sx_n\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n)(\alpha_n - k)\|Tx_n - x_n\|^2 - (1-\alpha_n)^2(\beta_n)(1-\beta_n)\|Tx_n - Sx_n\|^2 \\
&\leqslant \|x_n - p\|^2 - (1-\alpha_n)(\beta_n)(\alpha_n - k)\|x_n - Sx_n\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n)(\alpha_n - k)\|Tx_n - x_n\|^2 \\
&\leqslant \|x_n - p\|^2
\end{aligned}
$$

for all $n \geqslant 1$. Thus $\lim_{n\to\infty}\|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. Also, say $c = \lim_{n\to\infty}\|x_n - p\|$. Let $z_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n$ for all $n \geqslant 1$. Then we get

$$
\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= \|\alpha_n x_n + (1-\alpha_n)Ux_n - \alpha_n x_n - (1-\alpha_n)Sx_n\| \\
&= (1-\alpha_n)\|\beta_n Sx_n + (1-\beta_n)Tx_n - Sx_n\| \\
&= (1-\alpha_n)(1-\beta_n)\|Tx_n - Sx_n\| \\
&\leqslant (1-\beta_n)\|Tx_n - Sx_n\|.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0 \tag{13}$$

and hence

$$\lim_{n\to\infty}\|z_n - p\| = \lim_{n\to\infty}\|x_n - p\| = c. \tag{14}$$

Since

$$
\begin{aligned}
\|z_{n+1} - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Sx_n - p\|^2 \\
&= \|\alpha_n(x_n - p) + (1-\alpha_n)(Sx_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Sx_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\
&\leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left[\|x_n - p\|^2 + k\|x_n - Sx_n\|^2\right] \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - Sx_n\|^2 \\
&= \|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - k)\|x_n - Sx_n\|^2,
\end{aligned}
\tag{15}
$$

we have

$$(1-\alpha_n)(\alpha_n - k)\|x_n - Sx_n\|^2 \leqslant \|x_n - p\|^2 - \|z_{n+1} - p\|^2$$

and hence

$$\lim_{n\to\infty}(1-\alpha_n)(\alpha_n - k)\|x_n - Sx_n\|^2 = 0. \tag{16}$$

From (16) and the condition $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - k) > 0$, we can conclude that

$$\lim_{n\to\infty}\|x_n - Sx_n\|^2 = 0.$$

Since $\{x_n\}$ is bounded, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $v \in C$ such that $x_{n_i} \rightharpoonup v$. Since $I - S$ is demiclosed at zero, it follows that $v \in F(S)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ with converges weakly to a point $v^* \in C$. By the same argument as above, we can see that $v^* \in F(S)$.

Now, we show that $v = v^*$. Before proving this, we prove that, for any $z \in F(S)$, $\lim_{n\to\infty}\|x_n - z\|$ exists. As in the proof of the inequality (15), we can show that, for each $z \in F(S)$, we have

$$\|z_{n+1} - z\| \leqslant \|x_n - z\|.$$

Hence we obtain

$$\|z_{n+1} - z\| \leqslant \|x_n - z\| \leqslant \|z_n - z\| + \|x_n - z_n\|.$$

It follows from Lemma 1 that $\lim_{n\to\infty}\|z_n - z\|$ exists and so, by (13), we can show that $\lim_{n\to\infty}\|x_n - z\|$ exists.

Suppose that $v \neq v^*$. Then, by Opial's condition, we get

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - v\| &< \liminf_{i\to\infty}\|x_{n_i} - v^*\| \\
&= \lim_{n\to\infty}\|x_n - v^*\| \\
&= \liminf_{j\to\infty}\|x_{n_j} - v^*\| \\
&< \liminf_{j\to\infty}\|x_{n_j} - v\| \\
&= \lim_{n\to\infty}\|x_n - v\| \\
&= \liminf_{i\to\infty}\|x_{n_i} - v\|,
\end{aligned}$$

which is a contradiction. Therefore, $v = v^*$ and, by Theorem 1, the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)$. This completes the proof. □

Next, we give an example of two k-demicontractive mappings to illustrate our Theorem 2.

EXAMPLE 5. Consider a Hilbert space $H = \mathbb{R}$ with the usual norm. Let $S : \mathbb{R} \to \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Sx = \begin{cases} \dfrac{x}{2}, & \text{if } -\infty < x < 0, \\ -3x, & \text{if } 0 \leqslant x < \infty, \end{cases}$$

$$Tx = \begin{cases} \frac{1}{2}x\sin\left(\frac{1}{x}\right), & \text{if } x \in [-1,0) \cup (0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $F(S) = \{0\}$, $F(T) = \{0\}$ and $F(S) \cap F(T) = \{0\} \neq \emptyset$.

First, we show that S is a $\frac{1}{2}$-demicontractive mapping. If $x \in (-\infty, 0)$, then we have

$$\begin{aligned}
|Sx - 0|^2 &\leqslant \left|\frac{1}{2}x - 0\right|^2 \\
&= \frac{1}{2}|x - 0|^2 \\
&\leqslant |x - 0|^2 + \frac{1}{2}|x - Sx|^2
\end{aligned}$$

If $x \in [0,\infty)$, then $|x - Sx|^2 = |x+3x|^2 = 16x^2$ and

$$|Sx-0|^2 = 9x^2 = x^2 + \frac{1}{2}(16)x^2 = |x-0|^2 + \frac{1}{2}|x-Sx|^2.$$

Therefore, S is a $\frac{1}{2}$-demicontractive mapping.

Now, we prove that T is a k-demicontractive mapping. In case $x \notin [-1,0)\cup(0,1]$, we get

$$|Tx-0|^2 = 0 \leqslant |x-0|^2 + k|x-Tx|^2.$$

Let $x \in [-1,0)\cup(0,1]$. Then we have

$$\begin{aligned} |Tx-0|^2 &= \left|\frac{1}{2}x\sin\left(\frac{1}{x}\right)\right|^2 \\ &\leqslant \left|\frac{1}{2}x\right|^2 \\ &\leqslant |x|^2 \\ &\leqslant |x-0|^2 + k|x-Tx|^2 \end{aligned}$$

for all $k \in [0,1)$. Thus T is k-demicontractive. For any fixed $x_1 \in H$, take the sequence $\{x_n\}$ as in Theorem 2 with $\alpha_n = \frac{3}{4}$ for all $n \in \mathbb{N}$ and $\beta_n = 1 - \frac{1}{2^n}$.

We have to consider five cases to show that $x_n \rightharpoonup 0 \in F(S)$.

Case I. If $x_n = 0$ for all $n \geqslant N$, for some $N \geqslant 1$, then we get

$$\begin{aligned} |x_{n+1}| &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)Sx_n + \frac{1}{2^n}Tx_n\right]\right| \\ &= \frac{3}{4}|x_n|. \end{aligned}$$

Case II. If $x_n \in [-1,0)$ for all $n \geqslant N$, for some $N \geqslant 1$, then we get

$$\begin{aligned} |x_{n+1}| &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)Sx_n + \frac{1}{2^n}Tx_n\right]\right| \\ &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)\left(\frac{1}{2}\right)x_n + \frac{1}{2^n}\left(\frac{1}{2}x_n\sin(\frac{1}{x_n}\right)\right]\right| \\ &= \left|\frac{3}{4}x_n + \frac{1}{8}x_n - \frac{1}{2^{n+3}}x_n + \frac{1}{2^{n+3}}x_n\sin\left(\frac{1}{x_n}\right)\right| \\ &\leqslant \frac{7}{8}|x_n| + \frac{1}{2^{n+3}}|x_n| + \frac{1}{2^{n+3}}\left|x_n\sin\left(\frac{1}{x_n}\right)\right| \\ &\leqslant \frac{7}{8}|x_n| + \frac{1}{2^{n+3}}|x_n| + \frac{1}{2^{n+3}}|x_n| \\ &= \frac{7}{8}|x_n| + \frac{2}{2^{n+3}}|x_n| \\ &= \left(\frac{7}{8} + \frac{1}{2^{n+2}}\right)|x_n|. \end{aligned}$$

Case III. If $x_n \in (0,1]$ for all $n \geqslant N$, for some $N \geqslant 1$, then we get

$$\begin{aligned}
|x_{n+1}| &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)Sx_n + \frac{1}{2^n}Tx_n\right]\right| \\
&= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)(-3x_n) + \frac{1}{2^n}\left(\frac{1}{2}x_n \sin(\frac{1}{x_n}\right)\right]\right| \\
&= \left|\frac{3}{4}x_n - \frac{3}{4}x_n + \frac{3}{2^{n+2}}x_n + \frac{1}{2^{n+3}}x_n \sin\left(\frac{1}{x_n}\right)\right| \\
&\leqslant \frac{3}{2^{n+2}}|x_n| + \frac{1}{2^{n+3}}\left|x_n \sin\left(\frac{1}{x_n}\right)\right| \\
&\leqslant \frac{3}{2^{n+2}}|x_n| + \frac{1}{2^{n+3}}|x_n| \\
&= \left(\frac{3}{2^{n+2}} + \frac{1}{2^{n+3}}\right)|x_n| \\
&= \frac{7}{2^{n+3}}|x_n|.
\end{aligned}$$

Case IV. If $x_n \in (-\infty,-1)$ for all $n \geqslant N$, for some $N \geqslant 1$, then $Tx_n = 0$ for all $n \geqslant N$ and hence

$$\begin{aligned}
|x_{n+1}| &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)Sx_n + \frac{1}{2^n}Tx_n\right]\right| \\
&= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)\left(\frac{1}{2}\right)x_n + 0\right]\right| \\
&= \left|\frac{3}{4}x_n + \frac{1}{8}x_n - \frac{1}{2^{n+3}}x_n\right| \\
&\leqslant \frac{7}{8}|x_n| + \frac{1}{2^{n+3}}|x_n| \\
&= \left(\frac{7}{8} + \frac{1}{2^{n+3}}\right)|x_n|.
\end{aligned}$$

Case V. If $x_n \in (1,\infty)$ for all $n \geqslant N$, for some $N \geqslant 1$, then $Tx_n = 0$ for all $n \geqslant N$ and hence

$$\begin{aligned}
|x_{n+1}| &= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)Sx_n + \frac{1}{2^n}Tx_n\right]\right| \\
&= \left|\frac{3}{4}x_n + \frac{1}{4}\left[\left(1-\frac{1}{2^n}\right)(-3x_n) + 0\right]\right| \\
&= \left|\frac{3}{4}x_n - \frac{3}{4}x_n + \frac{3}{2^{n+2}}x_n\right| \\
&= \frac{3}{2^{n+2}}|x_n|.
\end{aligned}$$

So, in all the cases, we can see that $\{x_n\}$ is bounded and hence $x_n \to 0 \in F(S)$ as $n \to \infty$. Therefore $x_n \rightharpoonup 0$. Moreover, by the similar argument of Example 4, we conclude that S is not a quasi-nonexpansive mapping. Therefore, the results of Kim in [9] can not be applied to this example and our main result, Theorem 2.

COROLLARY 1. ([9]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,T:C\to C$ be two quasi-nonexpansive mappings such that $I-S$ is demiclosed at zero and $F(S)\cap F(T)\neq\emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0$ and $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.*

Proof. Taking $k=0$ in Theorem 2, we obtain the conclusion. □

COROLLARY 2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S:C\to C$ be a k-demicontractive mapping such that $I-S$ is demiclosed at zero and $F(S)\neq\emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1\in C \text{ chosen arbitrary},\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)Sx_n \end{cases} \tag{17}$$

for all $n\geqslant 1$, where $\{\alpha_n\}\subset[k,1]$ is a sequence such that

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n-k)>0.$$

Then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.

Proof. Setting $\beta=1$ for all $n\geqslant 1$ in Theorem 2, we obtain the conclusion. □

COROLLARY 3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S:C\to C$ be a quasi-nonexpansive mapping such that $I-S$ is demiclosed at zero and $F(S)\neq\emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1\in C \text{ chosen arbitrary},\\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)Sx_n \end{cases} \tag{18}$$

for all $n\geqslant 1$, where $\{\alpha_n\}\subset[0,1]$ is a sequence such that

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0.$$

Then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.

Proof. Setting $\beta=1$ for all $n\geqslant 1$ in Corollary 1, we obtain the conclusion. □

If T is a nonspreading mapping (or a nonexpansive mapping), then $I-T$ is demiclosed at zero (see [1, 7, 16]) and so, from Theorem 2, we have the following:

COROLLARY 4. ([7]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S:C\to C$ be a nonspreading mapping and let $T:C\to C$ be a nonexpansive mapping such that $F(S)\cap F(T)\neq\emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0$ and $\sum_{n=1}^{\infty}(1-\beta_n)<\infty$, then the sequence $\{x_n\}$ converges weakly to a point $v\in F(S)$.*

Proof. Since every nonexpansive mapping and nonspreading mapping with a nonempty fixed point set are quasi-nonexpansive, by Corollary 1, we obtain the conclusion. □

COROLLARY 5. ([13]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonspreading mapping such that $F(S) \neq \emptyset$. Define the sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C \ \textit{chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n \end{cases} \tag{19}$$

for all $n \geqslant 1$, where $\{\alpha_n\} \subset [0,1]$ is a sequence such that

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0.$$

Then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)$.

Proof. Setting $\beta_n = 1$ for all $n \geqslant 1$ in Corollary 4, we obtain the conclusion. □

THEOREM 3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,T : C \to C$ be two k-demicontractive mappings such that $I-S$ and $I-T$ are demiclosed at zero and $F(S)\cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C by (10). If the following conditions hold:*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - k) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)\cap F(T)$.

Proof. Observe that the inequality (10) is equal to the following:

$$x_{n+1} = \beta_n[\alpha_n x_n + (1-\alpha_n)Sx_n] + (1-\beta_n)[\alpha_n x_n + (1-\alpha_n)Tx_n]$$

for all $n \geqslant 1$. Putting

$$V_n = \beta_n[\alpha_n I + (1-\alpha_n)S] + (1-\beta_n)[\alpha_n I + +(1-\alpha_n)T]$$

for all $n \geqslant 1$, we have $x_{n+1} = Vx_n$ for all $n \geqslant 1$.

First, we show that the sequence $\{x_n\}$ converges weakly to a point in $F(S)$. Let $u \in F(S)\cap F(T)$ and $x \in C$. Then we have

$$
\begin{aligned}
&\|V_nx_n-u\|^2\\
=\ &\beta_n\|\alpha_nx_n+(1-\alpha_n)Sx_n-u\|^2+(1-\beta_n)\|\alpha_nx_n+(1-\alpha_n)Tx_n-u\|^2\\
&-\beta_n(1-\beta_n)(1-\alpha_n)\|Sx_n-Tx_n\|^2\\
\leqslant\ &\beta_n\Big[\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Sx_n-u\|^2-\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2\Big]\\
&+(1-\beta_n)\Big[\alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Tx_n-u\|^2-\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\Big]\\
\leqslant\ &\beta_n\Big[\|x_n-u\|^2+k(1-\alpha_n)\|Sx_n-x_n\|^2-\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2\Big]\\
&+(1-\beta_n)\Big[\|x_n-u\|^2+k(1-\alpha_n)\|Tx_n-x_n\|^2-\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\Big]\\
\leqslant\ &\beta_n\|x_n-u\|^2+k\beta_n(1-\alpha_n)\|Sx_n-x_n\|^2-\beta_n\alpha_n(1-\alpha_n)\|Sx_n-x_n\|^2\\
&+(1-\beta_n)\|x_n-u\|^2+k(1-\beta_n)(1-\alpha_n)\|Tx_n-x_n\|^2\\
&-(1-\beta_n)\alpha_n(1-\alpha_n)\|Tx_n-x_n\|^2\\
\leqslant\ &\|x_n-u\|^2-\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2-(1-\beta_n)(1-\alpha_n)(\alpha_n-k)\|Tx_n-x_n\|^2\\
\leqslant\ &\|x_n-u\|^2-\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2\\
\leqslant\ &\|x_n-u\|^2
\end{aligned}
\tag{20}
$$

and hence

$$0\leqslant\|x_n-u\|^2-\|V_nx_n-u\|^2=\|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

Taking $n\to\infty$ in the above inequality, by (14), we get

$$\lim_{n\to\infty}\left(\|x_n-u\|^2-\|V_nx_n-u\|^2\right)=0. \tag{21}$$

Thus, from (20), it follow that

$$
\begin{aligned}
0&\leqslant\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2\\
&=\|x_n-u\|^2-\left[\|x_n-u\|^2-\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2\right]\\
&\leqslant\|x_n-u\|^2-\|V_nx_n-u\|^2
\end{aligned}
$$

and so

$$
\begin{aligned}
0&\leqslant(1-\beta_n)\beta_n(1-\alpha_n)(\alpha_n-k)\|x_n-Sx_n\|^2\\
&\leqslant(1-\beta_n)(\|x_n-u\|^2-\|V_nx_n-u\|^2).
\end{aligned}
\tag{22}
$$

Since $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n-k)>0$ and $\liminf_{n\to\infty}(1-\beta_n)(\beta_n)>0$, it follows from (21) and (22) that

$$\lim_{n\to\infty}\|x_n-Sx_n\|^2=0. \tag{23}$$

By the similar argument in the proof of Theorem 2, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a point $v\in F(S)$.

Next, we show that $v \in F(T)$. For any $u \in F(S) \cap F(T)$, again, from (20), it follows that

$$\begin{aligned} 0 &\leqslant \beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2+(1-\beta_n)(1-\alpha_n)(\alpha_n-k)\|Tx_n-x_n\|^2 \\ &= \|x_n-u\|^2-[\|x_n-u\|^2-\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2 \\ &\quad -(1-\beta_n)(1-\alpha_n)(\alpha_n-k)\|Tx_n-x_n\|^2] \\ &\leqslant \|x_n-u\|^2-\|V_nx_n-u\|^2. \end{aligned} \tag{24}$$

Since $k \leqslant \alpha_n < 1$ and $\beta_n \in [0,1]$ for all $n \geqslant 1$, we get

$$\begin{aligned} 0 &\leqslant (1-\beta_n)\beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2 \\ &\quad +\beta_n(1-\beta_n)(1-\alpha_n)(\alpha_n-k)\|Tx_n-x_n\|^2 \\ &\leqslant \beta_n(1-\alpha_n)(\alpha_n-k)\|Sx_n-x_n\|^2+(1-\beta_n)(1-\alpha_n)(\alpha_n-k)\|Tx_n-x_n\|^2 \\ &\leqslant \|x_n-u\|^2-\|V_nx_n-u\|^2. \end{aligned}$$

Thus it follows from $\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n-k)>0$, $\liminf_{n\to\infty}(1-\beta_n)(\beta_n)>0$, (21) and (23) that

$$\lim_{n\to\infty}\|x_n-Tx_n\|^2=0. \tag{25}$$

Since $\{x_{n_i}\}$ converges weakly to $v \in F(S)$, it follows from the demiclosedness at zero of $I-T$ that $v \in F(T)$.

Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ which converges weakly to $v^* \in C$.

Now, we show that $v = v^*$. Suppose the contrary. Then, by Opial's condition, we get

$$\begin{aligned} \liminf_{i\to\infty}\|x_{n_i}-v\| &< \liminf_{i\to\infty}\|x_{n_i}-v^*\| \\ &= \lim_{n\to\infty}\|x_n-v^*\| \\ &= \liminf_{k\to\infty}\|x_{n_k}-v^*\| \\ &< \liminf_{k\to\infty}\|x_{n_k}-v\| \\ &= \lim_{n\to\infty}\|x_n-v\| \\ &= \liminf_{i\to\infty}\|x_{n_i}-v\|, \end{aligned}$$

which is a contradiction and hence $v = v^*$. Therefore, we can conclude that the sequence $\{x_n\}$ converges weakly to a point $v \in F(S) \cap F(T)$. This completes the proof. □

COROLLARY 6. ([9]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,T : C \to C$ be two quasi-nonexpansive mappings such that $I-S$ and $I-T$ are demiclosed at zero and $F(S)\cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If the following conditions hold:*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n)>0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n)>0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S)\cap F(T)$.

COROLLARY 7. ([7]) *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonspreading mapping and let $T : C \to C$ be a nonexpansive mapping such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If the following conditions hold:*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges weakly to a point $v \in F(S) \cap F(T)$.

4. Strong convergence theorems

In this section, we prove strong convergence theorems for two k-demicontractive mappings satisfying the condition (A') in a Hilbert space H by using the Moudafi's iterative scheme.

THEOREM 4. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S, T : C \to C$ be two k-dimicontractive mappings satisfying the condition (A') and $\mathscr{F} := F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If the following conditions hold:*

$$\liminf_{n\to\infty}(1-\alpha_n)(\alpha_n - k) > 0, \quad \liminf_{n\to\infty}(1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges strongly to a point $v \in F(S) \cap F(T)$.

Proof. From the inequalities (23) and (25), we have

$$\lim_{n\to\infty}\|x_n - Sx_n\|^2 = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|^2. \tag{26}$$

Moreover, as in the proof of Theorem 2, we can show that

$$\|x_{n+1} - p\| \leqslant \|x_n - p\| \tag{27}$$

for any $p \in F(S) \cap F(T)$.

On the other hand, by the condition (A') of S and T, we get

$$f(d(x_n, \mathscr{F})) \leqslant \frac{1}{2}\big(\|x_n - Sx_n\| + \|x_n - Tx_n\|\big) \tag{28}$$

for all $n \geqslant 1$. Taking the infimum over $\mathscr{F}$ on both sides of (27), we have $\lim_{n\to\infty} d(x_n, \mathscr{F})$ exists.

Now, we claim that $\lim_{n\to\infty} d(x_n, \mathscr{F}) = 0$. Suppose the contrary. Then we can choose $n_0 \in \mathbb{N}$ such that $0 < \frac{k}{2} < d(x_n, \mathscr{F})$ for all $n \geqslant n_0$. Since f is nondecreasing, it follows from (26) and (28) that

$$0 < f\Big(\frac{k}{2}\Big) \leqslant f(d(x_n, \mathscr{F})) \leqslant \frac{1}{2}\big(\|x_n - Sx_n\| + \|x_n - Tx_n\|\big) \to 0$$

as $n \to \infty$, which is a contradiction. Therefore, we obtain $\lim_{n\to\infty} d(x_n, \mathscr{F}) = 0$ and hence there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, \mathscr{F}) \leqslant \frac{\varepsilon}{2} \tag{29}$$

for all $n \geqslant n_1$. Let $m,n \geqslant n_1$ and $p \in \mathscr{F}$. Then it follows from (27) that

$$\|x_n - x_m\| \leqslant \|x_n - p\| + \|x_m - p\| \leqslant 2\|x_{n_1} - p\|.$$

Taking the infimum over all $\mathscr{F}$ on both sides of the above inequality and using (29), we obtain

$$\|x_n - x_m\| \leqslant 2d(x_{n_1}, \mathscr{F}) < \varepsilon$$

for all $m,n \geqslant n_1$, which implies that $\{x_n\}$ is a Cauchy sequence. Suppose that $\lim_{n\to\infty} x_n = v$ for some $v \in H$. Since $\mathscr{F}$ is closed, we have $v \in \mathscr{F}$. Therefore, the sequence $\{x_n\}$ converges strongly to a point $v \in \mathscr{F}$. This completes the proof. □

COROLLARY 8. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S,T : C \to C$ be two quasi-nonexpansive mappings satisfying the condition (A') and $\mathscr{F} := F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If the following conditions hold:*

$$\liminf_{n\to\infty} (1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty} (1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges strongly to a point $v \in F(S) \cap F(T)$.

COROLLARY 9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonspreading mapping and let $T : C \to C$ be a nonexpansive mapping satisfying the condition (A') and $\mathscr{F} := F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C defined by (10). If the following conditions hold:*

$$\liminf_{n\to\infty} (1-\alpha_n)(\alpha_n) > 0, \quad \liminf_{n\to\infty} (1-\beta_n)(\beta_n) > 0,$$

then the sequence $\{x_n\}$ converges strongly to a point $v \in F(S) \cap F(T)$.

COROLLARY 10. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a k-dimicontractive mapping satisfying the condition (A) and $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 \in C \ \textit{chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S x_n \end{cases} \tag{30}$$

for all $n \geqslant 1$, where $\{\alpha_n\} \subset [k,1]$ is a sequence such that

$$\liminf_{n\to\infty} (1-\alpha_n)(\alpha_n - k) > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $v \in F(S)$.

Proof. Putting $\beta = 1$ for all $n \geqslant 1$ in Theorem 4, the conclusion follows. □

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