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REGULARITY FOR SOLUTIONS OF NONLINEAR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the existence and uniqueness of solutions for the nonlinear functional differential equations with time delay. The regularity and a variation of constant formula for solutions of the given equations are also studied.

1. Introduction

Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H. The subject of this paper is to investigate the regularity for a solution of the following nonlinear functional differential equation on H:

$$\begin{cases} x'(t)+Ax(t)=\int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds)+k(t), & 0<t\leqslant T, \\ x(0)=g^0, \quad x(s)=g^1(s) & s\in[-h,0). \end{cases} \tag{NE}$$

Let the principal operator A be given a single valued, monotone operator, which is hemicontinuous and coercive from V to V^*. Here V^* stands for the dual space of V.

If the nonlinear integral term and the forcing term k belong to $L^2(0,T;V^*)$, the basic assumption made in these investigations is taken from the regularity result for the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in [3]):

$$\begin{cases} x'(t)+Ax(t)=k(t), & 0<t\leqslant T, \\ x(0)=g^0. \end{cases}$$

The regular problems of semilinear differential equations with the linear operator A were studied by Vrabie [7] and Jeong et al. [6]. The existence of solutions for a class of nonlinear evolution equations in the case in which A is nonlinear were developed in many references [1, 3-5]. Ahmed and Xiang [1] gave some existence results for the initial value problem in case where the nonlinear term is not monotone, which improve Hirano's result [5].

In this paper, we will establish the existence and regularity for solutions of the equation (NE) with a nonlinear operator A on $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$ under some

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general condition of the Lipschitz continuity of the nonlinear operator, which is reasonable and widely used in case of the nonlinear system. We also extent the regularity result of the semilinear case [6] to the equation (NE). with the aid of the intermediate property and the contraction mapping principle. The main research direction is to find conditions on the nonlinear term such that the regularity result of (NE) is preserved under perturbation and show that the mapping $H \times L^2(0,T;V) \times L^2(0,T;V^*) \ni (g^0, g^1, k) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$ is continuous in view of the monotonicity of A.

2. Assumptions and main theorem

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. Thus, in terms of the intermediate theory we may assume that

$$(V, V^*)_{1/2,2} = H$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \leqslant |u| \leqslant ||u||, \quad u \in V.$$

We note that a nonlinear operator A is said to be hemicontinuous on V if

$$w - \lim_{t \to 0} A(x + ty) = Ax$$

for every $x,\ y \in V$ where "$w - \lim$" indicates the weak convergence on V.

Let $A : V \longrightarrow V^*$ be given a monotone operator and hemicontinuous from V to V^* such that

$$A(0) = 0, \quad (Au - Av, u - v) \geqslant \omega_1 ||u - v||^2 - \omega_2 |u - v|^2, \tag{A1}$$

$$||Au||_* \leqslant \omega_3 (||u|| + 1) \tag{A2}$$

for every $u, v \in V$ where ω_2 is a real number and ω_1, ω_3 are some positive constants.

Here, we note that if $0 \neq A(0)$ we need the following assumption

$$(Au, u) \geqslant \omega_1 ||u||^2 - \omega_2 |u|^2$$

for every $u \in V$. It is also known that A is maximal monotone and $R(A) = V^*$ where $R(A)$ denotes the range of A.

Let $\mathscr{L}$ and $\mathscr{B}$ be the Lebesgue σ-field on $[0, \infty)$ and the Borel σ-field on $[-h, 0]$ for some $h > 0$, respectively. Let μ be a Borel measure on $[-h, 0]$ and $g : [0, \infty) \times [-h, 0] \times V \times V \to H$ be a nonlinear mapping satisfying the following:

(i) For any $x,y\in V$ the mapping $g(\cdot,\cdot,x,y)$ is strongly $\mathscr{L}\times\mathscr{B}$-measurable;
(ii) There exist positive constants L_0,L_1,L_2 such that

$$|g(t,s,x,y)-g(t,s,\hat{x},\hat{y})|\leqslant L_1||x-\hat{x}||+L_2||y-\hat{y}||, \tag{G1}$$

$$|g(t,s,0,0)|\leqslant L_0 \tag{G2}$$

for all $(t,s)\in[0,\infty)\times[-h,0]$ and $x,\hat{x},y,\hat{y}\in V$.

REMARK 1. The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [8].

For $x\in L^2(-h,T;V)$, $T>0$ we set

$$G(t,x)=\int_{-h}^{0}g(t,s,x(t),x(t+s))\mu(ds). \tag{2.1}$$

Here as in [8] we consider the Borel measurable corrections of $x(\cdot)$.

The main theorems of this paper are as follows.

THEOREM 1. (Main) *Let the assumptions (A1), (A2), (G1) and (G2) be satisfied. Then, for every $k\in L^2(0,T;V^*)$ and $(g^0,g^1)\in H\times L^2(0,T;V)$ the equation (NE) has a unique solution*

$$x\in L^2(0,T;V)\cap C([0,T];H)\cap W^{1,2}(0,T;V^*)$$

and there exists a constant C_1 depending on T such that

$$||x||_{L^2\cap C\cap W^{1,2}}\leqslant C_1(1+|g^0|+||g^1||_{L^2(0,T;V)}+||k||_{L^2(0,T;V^*)}). \tag{2.2}$$

As a corollary to Theorem 1, we have the following result.

COROLLARY 1. *Let the assumptions (A1), (A2), (G1) and (G2) be satisfied. Let the operator B be a monotone set in $H\times H$. Then for every $k\in L^2(0,T;V^*)$ and $(g^0,g^1)\in H\times L^2(0,T;V)$, the Cauchy problem*

$$\begin{cases} x'(t)\in(A+B)x(t)+G(t,x(t))+k(t), & 0<t\leqslant T,\\ x(0)=g^0, \quad x(s)=g^1(s) & s\in[-h,0)\end{cases}$$

has a unique solution

$$x\in L^2(0,T;V)\cap C([0,T];H)$$

and there exists a constant C_2 depending on T such that

$$||x||_{L^2\cap C}\leqslant C_2(1+|g^0|+||g^1||_{L^2(0,T;V)}+||k||_{L^2(0,T;V^*)}).$$

THEOREM 2. *Let the assumptions (A1), (A2), (G1) and (G2) be satisfied and $(g^0,g^1,k)\in H\times L^2(0,T;V)\times L^2(0,T;V^*)$, Then the solution x of the equation (NE) belongs to $L^2(0,T;V)\cap C([0,T];H)$ and the mapping*

$$H\times L^2(0,T;V)\times L^2(0,T;V^*)\ni(g^0,g^1,k)\mapsto x\in L^2(0,T;V)\cap C([0,T];H)$$

is continuous.

The proofs will be given in section 3.

3. Proofs of the main theorems

Let us consider with the quasi-autonomous differential equation

$$\begin{cases} x'(t)+Ax(t)=k(t), & 0<t\leqslant T, \\ x(0)=g^0, \end{cases} \tag{3.1}$$

where A is given and satisfies the hypotheses mentioned in section 2. The following result is from Theorem 2.6 of Chapter III in [3].

LEMMA 1. *Let $g^0\in H$ and $k\in L^2(0,T;V^*)$. Then there exists a unique solution x of (3.1) belonging to*

$$C([0,T];H)\cap L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$$

and satisfying

$$|x(t)|^2+\int_0^t ||x(s)||^2 ds\leqslant C_3(|g^0|^2+\int_0^t ||k(s)||_*^2 ds), \tag{3.2}$$

$$\int_0^t \Big|\Big|\frac{dx(s)}{ds}\Big|\Big|_*^2 dt\leqslant C_3(|g^0|^2+\int_0^t ||k(s)||_*^2 ds), \tag{3.3}$$

where C_3 is a constant.

Acting on both sides of (3.1) by $x(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2+\omega_1||x(t)||^2\leqslant \omega_2|x(t)|^2+(k(t),x(t)).$$

As is seen Theorem 2.6 in [3], integrating from 0 to t we can determine the constant C_3 in this lemma.

The following Lemma is from Brézis [4, Lemma A.5]

LEMMA 2. *Let $m\in L^1(0,T;\mathbb{R})$ satisfying $m(t)\geqslant 0$ for all $t\in(0,T)$ and $a\geqslant 0$ be a constant. Let b be a continuous function on $[0,T]$ satisfying the following inequality:*

$$\frac{1}{2}b^2(t)\leqslant \frac{1}{2}a^2+\int_0^t m(s)b(s)ds, \quad t\in[0,T].$$

Then,

$$|b(t)|\leqslant a+\int_0^t m(s)ds, \quad t\in[0,T].$$

Proof. Let

$$\beta_\varepsilon(t)=\frac{1}{2}(a+\varepsilon)^2+\int_0^t m(s)b(s)ds, \quad \varepsilon>0.$$

Then

$$\frac{d\beta_\varepsilon(t)}{dt} = m(t)b(t), \quad \tau \in (0,T),$$

and

$$\frac{1}{2}b^2(t) \leqslant \beta_0(t) \leqslant \beta_\varepsilon(t), \quad t \in [0,T]. \tag{3.4}$$

Hence, we have

$$\frac{d\beta_\varepsilon(t)}{dt} \leqslant m(t)\sqrt{2}\sqrt{\beta_\varepsilon(t)}.$$

Since $t \to \beta_\varepsilon(t)$ is absolutely continuous and

$$\frac{d}{dt}\sqrt{\beta_\varepsilon(t)} = \frac{1}{2\sqrt{\beta_\varepsilon(t)}}\frac{d\beta_\varepsilon(t)}{dt}$$

for all $t \in (0,T)$, it holds

$$\frac{d}{dt}\sqrt{\beta_\varepsilon(t)} \leqslant \frac{1}{\sqrt{2}}m(t),$$

that is,

$$\sqrt{\beta_\varepsilon(t)} \leqslant \sqrt{\beta_\varepsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)ds, \quad t \in (0,T).$$

Therefore, combining this with (3.4), we conclude that

$$\begin{aligned} |b(t)| &\leqslant \sqrt{2}\sqrt{\beta_\varepsilon(t)} \leqslant \sqrt{2}\sqrt{\beta_\varepsilon(0)} + \int_0^t m(s)ds \\ &= a + \varepsilon + \int_0^t m(s)ds, \quad t \in [0,T] \end{aligned}$$

for arbitrary $\varepsilon > 0$. □

LEMMA 3. *Let $x \in L^2(-h,T;V)$, $T > 0$. Then the nonlinear term $G(\cdot,x)$ defined by (2.1) belongs to $L^2(0,T;H)$ and*

$$\begin{aligned} &||G(\cdot,x)||_{L^2(0,T;H)} \\ &\leqslant \mu([-h,0])\{L_0\sqrt{T} + (L_1+L_2)||x||_{L^2(0,T;V)} + L_2||g^1||_{L^2(-h,0;V)}\}. \end{aligned} \tag{3.5}$$

Moreover if $x_1,\ x_2 \in L^2(-h,T;V)$, then

$$\begin{aligned} &||G(\cdot,x_1) - G(\cdot,x_2)||_{L^2(0,T;H)} \\ &\leqslant \mu([-h,0])\{(L_1+L_2)||x_1-x_2||_{L^2(0,T;V)} + L_2||x_1-x_2||_{L^2(-h,0;V)}\}. \end{aligned} \tag{3.6}$$

Proof. From (G1) and (G2) it is easily seen that

$$\begin{aligned} ||G(\cdot,x)||_{L^2(0,T;H)} &\leqslant \mu([-h,0])\{L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2||x||_{L^2(-h,T;V)}\} \\ &\leqslant \mu([-h,0])\{L_0\sqrt{T} + (L_1+L_2)||x||_{L^2(0,T;V)} + L_2||x||_{L^2(-h,0;V)}\}. \end{aligned}$$

The proof of (3.6) is similar. □

LEMMA 4. *For* $(g_i^0, g_i^1, k_i) \in H \times L^2(-h,0;V) \times L^2(0,T;V^*)$ $(i=1,2)$, *let us consider the following equation:*

$$\begin{cases} y_i'(t) + Ay_i(t) = G(t,x_i) + k_i(t), & 0 < t \leqslant T, \\ y_i(0) = g_i^0, \quad y_i(s) = g_i^1(s) & s \in [-h,0). \end{cases} \tag{3.7}$$

Then for $c < \omega_1$, *we have*

$$\begin{aligned} & e^{-\omega_2 t}|y_1(t) - y_2(t)| \\ \leqslant\, & e^{-\omega_2 t}(|g_1^0 - g_2^0| + \sqrt{2c^{-1}}||k_1 - k_2||_{L^2(0,T;V^*)}) + \int_0^t e^{-\omega_2 s}|G(s,x_1) - G(s,x_2)|ds. \end{aligned} \tag{3.8}$$

Proof. Invoking Lemma 1 and Lemma 3, we obtain that the problem

$$\begin{cases} y'(t) + Ay(t) = G(t,x) + k(t), & 0 < t \leqslant T, \\ y(0) = g^0, \quad y(s) = g^1(s) & s \in [-h,0) \end{cases}$$

has a unique solution $y \in L^2(0,T;V) \cap C([0,T];H)$. Let y_1, y_2 be the solutions of (3.7) with x replaced by x_1, $x_2 \in L^2(0,T;V)$, respectively. From (3.7) it follows that

$$\begin{cases} y_1'(t) - y_2'(t) + Ay_1(t) - Ay_2(t) = G(t,x_1) - G(t,x_2) + k_1(t) - k_2(t), & t > 0, \\ y_1(0) - y_2(0) = g_1^0 - g_2^0, \quad y_1(s) - y_2(s) = g_1^1(s) - g_2^1(s), & s \in [-h,0). \end{cases} \tag{3.9}$$

Multiplying on both sides of $y_1(t) - y_2(t)$ and by the assumption (A1), we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|y_1(t) - y_2(t)|^2 + \omega_1||y_1(t) - y_2(t)||^2 \\ \leqslant\, & \omega_2|y_1(t) - y_2(t)|^2 + |G(t,x_1) - G(t,x_2)|\,|y_1(t) - y_2(t)| \\ & + ||k_1(t) - k_2(t)||_*\,||y_1(t) - y_2(t)||. \end{aligned} \tag{3.10}$$

Putting

$$H(t) = |G(t,x_1) - G(t,x_2)|\,|y_1(t) - y_2(t)|$$

and we can choose a constant $c > 0$ such that $\omega_1 - c > 0$ and

$$||k_1(t) - k_2(t)||_*\,||y_1(t) - y_2(t)|| \leqslant \frac{1}{c}||k_1(t) - k_2(t)||_*^2 + c||y_1(t) - y_2(t)||^2.$$

By integrating (3.10) over $(0,t)$, this yields that

$$\begin{aligned} & \frac{1}{2}|y_1(t) - y_2(t)|^2 + (\omega_1 - c)\int_0^t ||y_1(s) - y_2(s)||^2 ds \\ \leqslant\, & \frac{1}{2}|g_1^0 - g_2^0|^2 + \frac{1}{c}||k_1 - k_2||_{L^2(0,T;V)}^2 + \omega_2\int_0^t |y_1(s) - y_2(s)|^2 ds + \int_0^t H(s)ds. \end{aligned} \tag{3.11}$$

From (3.11) it follows that

$$\begin{aligned} &\frac{d}{dt}\{e^{-2\omega_2 t}\int_0^t |y_1(s)-y_2(s)|^2 ds\} \\ &= 2e^{-2\omega_2 t}\{\frac{1}{2}|y_1(t)-y_2(t)|^2 - \omega_2\int_0^t |y_1(s)-y_2(s)|^2 ds\} \\ &\leqslant 2e^{-2\omega_2 t}\{\ \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^t H(s)ds\}. \end{aligned} \tag{3.12}$$

Integrating (3.12) over $(0,t)$ we have

$$\begin{aligned} &e^{-2\omega_2 t}\int_0^t |y_1(s)-y_2(s)|^2 ds \\ &\leqslant \frac{e^{-2\omega_2 t}}{\omega_2}\{\ \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*V)}\}+2\int_0^t e^{-2\omega_2\tau}\int_0^\tau H(s)dsd\tau \\ &= \frac{e^{-2\omega_2 t}}{\omega_2}\{\ \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}\}+2\int_0^t\int_s^t e^{-2\omega_2\tau}d\tau H(s)ds \\ &= \frac{e^{-2\omega_2 t}}{\omega_2}\{\ \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}\}+2\int_0^t \frac{e^{-2\omega_2 s}-e^{-2\omega_2 t}}{2\omega_2}H(s)ds \\ &= \frac{e^{-2\omega_2 t}}{\omega_2}\{\ \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}\}+\frac{1}{\omega_2}\int_0^t (e^{-2\omega_2 s}-e^{-2\omega_2 t})H(s)ds, \end{aligned}$$

thus, we get

$$\begin{aligned} &\omega_2\int_0^t |y_1(s)-y_2(s)|^2 ds \\ &\leqslant \frac{1}{2}|g_1^0-g_2^0|^2+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^t (e^{2\omega_2(t-s)}-1)H(s)ds. \end{aligned} \tag{3.13}$$

Combining (3.11) with (3.13) it holds that

$$\begin{aligned} &\frac{1}{2}|y_1(t)-y_2(t)|^2+(\omega_1-c)\int_0^t ||y_1(s)-y_2(s)||^2 ds \\ &\leqslant |g_1^0-g_2^0|^2+\frac{2}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^t e^{2\omega_2(t-s)}H(s)ds \\ &= |g_1^0-g_2^0|^2+\frac{2}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^t e^{2\omega_2(t-s)}|G(s,x_1)-G(s,x_2)|\,|y_1(s)-y_2(s)|ds, \end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned} &\frac{1}{2}(e^{-\omega_2 t}|y_1(t)-y_2(t)|)^2+(\omega_1-c)e^{-2\omega_2 t}\int_0^t ||y_1(s)-y_2(s)||^2 ds \\ &\leqslant e^{-2\omega_2 t}(|g_1^0-g_2^0|^2+\frac{2}{c}||k_1-k_2||_{L^2(0,T;V^*)})^2 \\ &\quad+\int_0^t e^{-\omega_2 s}|G(s,x_1)-G(s,x_2)|e^{-\omega_2 s}|y_1(s)-y_2(s)|ds. \end{aligned}$$

By using Lemma 2, we obtain that

$$
\begin{aligned}
& e^{-\omega_2 t}|y_1(t)-y_2(t)| \\
& \leqslant e^{-\omega_2 t}(|g_1^0-g_2^0|+\sqrt{2c^{-1}}||k_1-k_2||_{L^2(0,T;V^*)})+\int_0^t e^{-\omega_2 s}|G(s,x_1)-G(s,x_2)|ds. \quad \square
\end{aligned}
$$

Proof of Theorem 1. Let $(g_i^0, g_i^1) \in H \times L^2(-h,0;V)$ $(i=1,2)$. Consider the following equation:

$$
\begin{cases} y_i'(t)+Ay_i(t)=G(t,x_i)+k(t), & 0<t\leqslant T, \\ y_i(0)=g^0, \quad y_i(s)=g^1(s) & s\in[-h,0). \end{cases}
$$

Then it follows that

$$
\begin{cases} y_1'(t)-y_2'(t)+Ay_1(t)-Ay_2(t)=G(t,x_1)-G(t,x_2), & t>0, \\ y_1(0)-y_2(0)=0, \quad y_1(s)-y_2(s)=0, & s\in[-h,0). \end{cases}
$$

Let us fix $T_0>0$ such that

$$
\frac{1}{4\omega_1\omega_2}(e^{2\omega_2 T_0}-1)\mu([-h,0])^2(L_1+L_2)^2<1. \tag{3.15}
$$

By using Lemma 4, we are going to show that $x\mapsto y$ is strictly contractive from $L^2(0,T_0;V)$ to itself if the condition (3.15) is satisfied. From (3.14) and (3.8) it follows that

$$
\begin{aligned}
& \frac{1}{2}|y_1(t)-y_2(t)|^2+\omega_1\int_0^t ||y_1(s)-y_2(s)||^2 ds \qquad (3.16)\\
& \leqslant \int_0^t e^{2\omega_2(t-s)}|G(s,x_1)-G(s,x_2)|\int_0^s e^{\omega_2(s-\tau)}|G(\tau,x_1)-G(\tau,x_2)|d\tau ds \\
& = e^{2\omega_2 t}\int_0^t e^{-\omega_2 s}|G(s,x_1)-G(s,x_2)|\int_0^s e^{-\omega_2\tau}|G(\tau,x_1)-G(\tau,x_2)|d\tau ds \\
& = e^{2\omega_2 t}\int_0^t \frac{1}{2}\frac{d}{ds}\{\int_0^s e^{-\omega_2\tau}|G(\tau,x_1)-G(\tau,x_2)|d\tau\}^2 ds \\
& = \frac{1}{2}e^{2\omega_2 t}\{\int_0^t e^{-\omega_2\tau}|G(\tau,x_1)-G(\tau,x_2)|d\tau\}^2 \\
& \leqslant \frac{1}{2}e^{2\omega_2 t}\int_0^t e^{-2\omega_2\tau}d\tau\int_0^t |G(\tau,x_1)-G(\tau,x_2)|^2 d\tau \\
& = \frac{1}{2}e^{2\omega_2 t}\frac{1-e^{-2\omega_2 t}}{2\omega_2}\int_0^t |G(\tau,x_1)-G(\tau,x_2)|^2 d\tau \\
& = \frac{1}{4\omega_2}(e^{2\omega_2 t}-1)\int_0^t |G(s,x_1)-G(s,x_2)|^2 ds.
\end{aligned}
$$

From (3.6) of Lemma 3 it follows that for any $t>0$

$$
||G(\cdot,x_1)-G(\cdot,x_2)||_{L^2(0,t;H)}\leqslant \mu([-h,0])(L_1+L_2)||x_1-x_2||_{L^2(0,t;V)},
$$

and hence in view of (3.16) we have proved

$$\frac{1}{2}|y_1(t)-y_2(t)|^2+\omega_1\int_0^t||y_1(s)-y_2(s)||^2ds \tag{3.17}$$
$$\leqslant \frac{1}{4\omega_2}(e^{2\omega_2 t}-1)\mu([-h,0])^2(L_1+L_2)^2\int_0^t||x_1(s)-x_2(s)||^2ds.$$

Starting from the initial value $x_0(t)=g^0,\ x_0(s)=g^1(s)$ for $s\in[-h,0)$ consider a sequence $\{x_n(\cdot)\}$ satisfying

$$\begin{cases} \frac{d}{dt}x_{n+1}(t)+Ax_{n+1}(t)=G(t,x_n)+k(t), \quad 0<t\leqslant T,\\ x_n(0)=g^0, \quad x_n(0)=g^1(s), \quad s\in[-h,0). \end{cases}$$

Then from (3.8) of Lemma 4, it follows that

$$\frac{1}{2}|x_{n+1}(t)-x_n(t)|^2+\omega_1\int_0^t||x_{n+1}(s)-x_n(s)||^2ds \tag{3.18}$$
$$\leqslant \frac{1}{4\omega_2}(e^{2\omega_2 t}-1)\mu([-h,0])^2(L_1+L_2)^2\int_0^t||x_n(s)-x_{n-1}(s)||^2ds.$$

So by virtue of the condition (3.15) the contraction principle gives that there exists $x(\cdot)\in L^2(0,T_0;V)$ such that

$$x_n(\cdot)\to x(\cdot) \quad \text{in} \quad L^2(0,T_0;V),$$

and hence, from (3.18) there exists $x(\cdot)\in C([0,T_0];H)$ such that

$$x_n(\cdot)\to x(\cdot) \quad \text{in} \quad C(0,T_0;H).$$

Next we establish the estimates of solution. Let y be the solution of

$$\begin{cases} y'(t)+Ay(t)=k(t), \quad 0<t\leqslant T_0,\\ y(0)=g^0. \end{cases}$$

Then, since

$$(x'(t)-y'(t))+Ax(t)-Ay(t)=G(t,x),$$

by multiplying by $x(t)-y(t)$ and (A1), we obtain

$$\frac{1}{2}|x'(t)-y'(t)|^2+\omega_1||x(t)-y(t)||^2 \tag{3.19}$$
$$\leqslant \omega_2|x(t)-y(t)|^2+|G(t,x)|\,|x(t)-y(t)|.$$

By integrating on (3.19) over $(0,t)$ we have

$$\frac{1}{2}|x(t)-y(t)|^2+\omega_1\int_0^t||x(s)-y(s)||^2ds \tag{3.20}$$
$$\leqslant \omega_2\int_0^t|x(s)-y(s)|^2ds+\int_0^t|G(s,x)|\,|x(s)-y(s)|ds.$$

By the procedure similar to (3.16) we have

$$\begin{aligned}
&\frac{1}{2}|x(t)-y(t)|+\omega_1\int_0^t ||x(s)-y(s)||^2 ds\\
&\leqslant \int_0^t e^{2\omega_2(t-s)}|G(s,x)|\int_0^s e^{\omega_2(s-\tau)}|G(\tau,x)|d\tau ds\\
&=\frac{1}{4\omega_2}(e^{2\omega_2 t}-1)\int_0^t |G(s,x))|^2 ds\\
&\leqslant \frac{1}{4\omega_2}(e^{2\omega_2 t}-1)\int_0^t |G(s,x)|^2 ds.
\end{aligned}$$

Put

$$N=\frac{1}{4\omega_1\omega_2}(e^{2\omega_2 T_0}-1).$$

Then it holds

$$||x-y||_{L^2(0,T_0;V)}\leqslant N^{1/2}\mu([-h,0])\{L_0\sqrt{T}_0+(L_1+L_2)||x||_{L^2(0,T_0;V)}+L_2||g^1||_{L^2(-h,0;V)}\}.$$

and hence, from (3.2) of Lemma 1, we have that

$$\begin{aligned}
||x||_{L^2(0,T_0;V)} &\leqslant \frac{1}{1-N^{1/2}\mu([-h,0])(L_1+L_2)}||y||_{L^2(0,T_0;V)}\\
&\qquad+\frac{N^{1/2}\mu([-h,0])\{L_0\sqrt{T}_0+L_2||g^1||_{L^2(-h,0;V)}\}}{1-N^{1/2}\mu([-h,0])(L_1+L_2)}\\
&\leqslant \frac{\sqrt{C_3}}{1-N^{1/2}\mu([-h,0])(L_1+L_2)}(||g^0||+||k||_{L^2(0,T_0;V^*)})\\
&\qquad+\frac{N^{1/2}\mu([-h,0])\{L_0\sqrt{T}_0+L_2||g^1||_{L^2(-h,0;V)}\}}{1-N^{1/2}\mu([-h,0])(L_1+L_2)}\\
&\leqslant C_1(1+|g^0|+||g^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_0;V^*)})
\end{aligned} \tag{3.21}$$

for some positive constant C_1. Noting that

$$\begin{aligned}
L^2(0,T_0;V)\cap W^{1,2}(0,T_0;V^*)&\subset C([0,T_0];(V,V^*)_{1/2,2})\\
&=C([0,T_0];H).
\end{aligned} \tag{3.22}$$

It follows from (3.21), (3.22) that

$$|x(T_0)|\leqslant C_1(1+|g^0|+||g^1||_{L^2(0,T_0;V)}+||k||_{L^2(0,T_0;V^*)}).$$

Thus, since the condition (3.15) is independent of initial values, we can solve the equation in $[T_0,2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (3.21) holds for the solution under the condition (3.15). By repeating this process, the solution of (NE) can be extended the interval $[0,nT_0]$ for natural number n, i.e., for the initial $x(nT_0)$ in the interval $[nT_0,(n+1)T_0]$, as analogous estimate (3.21) holds for the solution in $[0,(n+1)T_0]$. Hence, the proof of Theorem 1 is complete. □

Proof of Theorem 2. If $(g^0,g^1)\in H\times L^2(0,T;V)$ and $k\in L^2(0,T;H)$ then x belongs to $L^2(0,T;V)\cap C([0,T];H)$ from Theorem 2.1. Let $(g_i^0,g_i^1,k_i)\in H\times L^2(0,T;V)\times L^2(0,T;H)$ and x_i be the solution of (NE) with (g_i^0,g_i^1,k_i) in place of (g^0,g^1,k) for $i=1,\ 2$. Multiplying on (NE) by $x_1(t)-x_2(t)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_1(t)-x_2(t)|^2+\omega_1||x_1(t)-x_2(t)||^2\\&\leqslant \omega_2|x_1(t)-x_2(t)|^2+|G(t,x_1)-G(t,x_2)|\,|x_1(t)-x_2(t)|\\&\quad+||k_1(t)-k_2(t)||_*\,||x_1(t)-x_2(t)||.\end{aligned}\tag{3.23}$$

Put

$$H(t)=|G(t,x_1)-G(t,x_2)|\,|x_1(t)-x_2(t)|.$$

Then by similar to (3.11), for $c<\omega_1$, we have

$$\begin{aligned}&\frac{1}{2}|x_1(t)-x_2(t)|^2+(\omega_1-c)\int_0^t||x_1(s)-x_2(s)||^2ds\\&\leqslant\frac{1}{2}|g_1^0-g_2^0|^2+\omega_2\int_0^t|x_1(s)-x_2(s)|^2ds+\frac{1}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^tH(s)ds.\end{aligned}$$

Thus, by the similar way to (3.14) and (3.8) we have

$$\begin{aligned}&\frac{1}{2}|x_1(t)-x_2(t)|^2+(\omega_1-c)\int_0^t||x_1(s)-x_2(s)||^2ds\\&\leqslant|g_1^0-g_2^0|^2+\frac{2}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}+\int_0^te^{2\omega_2(t-s)}H_1(s)ds.\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}e^{-\omega_2t}|x_1(t)-x_2(t)|&\leqslant e^{-\omega_2t}(|g_1^0-g_2^0|^2+\sqrt{2c^{-1}}||k_1-k_2||_{L^2(0,T;V^*)})\\&\quad+\int_0^te^{-\omega_2s}|G(s,x_1)-G(s,x_2)|ds.\end{aligned}\tag{3.25}$$

From (3.24) and (3.25) it follows that

$$\begin{aligned}&\frac{1}{2}|x_1(t)-x_2(t)|^2+(\omega_1-c)\int_0^t||x_1(s)-x_2(s)||^2ds\\&\leqslant|g_1^0-g_2^0|^2+\frac{2}{c}||k_1-k_2||^2_{L^2(0,T;V^*)}\\&\quad+(|g_1^0-g_2^0|^2+\sqrt{2c^{-1}}||k_1-k_2||_{L^2(0,T;V^*)})\int_0^te^{2\omega_2(t-s)}|G(s,x_1)-G(s,x_2)|ds\\&\quad+\int_0^te^{2\omega_2(t-s)}|G(s,x_1)-G(s,x_2)|\int_0^se^{\omega_2(s-\tau)}|G(\tau,x_1)-G(\tau,x_2)|d\tau ds.\end{aligned}\tag{3.26}$$

The third term of the right hand side of (3.22) is estimated as

$$\frac{(e^{2\omega_2t}-1)}{4\omega_2}\int_0^t|G(s,x_1)-G(s,x_2)|^2ds.\tag{3.27}$$

The second term of the right hand side of (3.22) is estimated as

$$\frac{1}{2}(|g_1^0-g_2^0|^2+\sqrt{2c^{-1}}||k_1-k_2||_{L^2(0,T;V^*)})^2+\frac{(e^{4\omega_2 t}-1)}{8\omega_2}\int_0^t|G(s,x_1)-G(s,x_2)|^2ds. \tag{3.28}$$

We note that from (3.6) of Lemma 3 for $0<t<T$

$$\begin{aligned} &||G(\cdot,x_1)-G(\cdot,x_2)||_{L^2(0,t;H)} \\ &\leqslant \mu([-h,0])\{(L_1+L_2)||x_1-x_2||_{L^2(0,t;V)}+L_2||g_1^1-g_2^2||_{L^2(-h,0;V)}\}. \end{aligned} \tag{3.29}$$

Let $T_1<T$ be such that

$$\omega_1-c-\min\Big\{\frac{e^{2\omega_2 T_1}-1)}{4\omega_2},\frac{(e^{4\omega_2 T_1}-1)}{8\omega_2}\Big\}[\mu([-h,0])(L_1+L_2)]^2>0.$$

Therefore, from (3.26)–(3.29) it follows that there exists a constant $C>0$ such that

$$\begin{aligned} &|x_1(T_1)-x_2(T_1)|^2+\int_0^{T_1}||x_1(s)-x_2(s)||^2ds \\ &\leqslant C(|g_1^0-g_2^0|^2+||g_1^1-g_2^2||_{L^2(-h,0;V)}+\int_0^{T_1}|k_1(s)-k_2(s)|^2ds) \end{aligned} \tag{3.30}$$

Suppose $(g_n^0,g_n^1,k_n)\to(g^0,g^1,k)$ in $\in H\times L^2(0,T;V)\times L^2(0,T;H)$, and let x_n and x be the solutions (NE) with (g_n^0,g_n^1,k_n) and (g^0,g^1,k), respectively. Then, by virtue of (3.30), we see that $x_n\to x$ in $L^2(0,T_1,V)\cap C([0,T_1];H)$. This implies that $x_n(T_1)\to x(T_1)$ in H. Therefore the same argument shows that $x_n\to x$ in

$$L^2(T_1,\min\{2T_1,T\};V)\cap C([T_1,\min\{2T_1,T\}];H).$$

Repeating this process, we conclude that $x_n\to x$ in $L^2(0,T;V)\cap C([0,T];H)$.

If $k\in L^2(0,T,V^*)$ we can choose a constant $c_0>0$ such that

$$\omega_1-\frac{c_0}{2}-\omega_2^{-1}\mu([-h,0])^2(L_1+L_2)^2(e^{2\omega_2 T_1}-1)>0.$$

and in (3.23)

$$||k_1(t)-k_2(t)||_*\,||x_1(t)-x_2(t)||\leqslant\frac{1}{2c_0}||k_1(t)-k_2(t)||_*+\frac{c_0}{2}||x_1(s)-x_2(s)||^2.$$

Thus, by the similar way to the proof of the case where $k\in L^2(0,T,H)$ we can obtain the results. □

REMARK 2. Let us assume that the embedding $V\subset H$ is compact. Let x_k be the solution of the equation (NE) corresponding to $k\in L^2(0,T;H)$. Hence if k is bounded in $L^2(0,T;V^*)$, then so is x_k in $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$ from Theorem 1. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset L^2(0,T;H))$ is compact in view of Theorem 2 of Aubin [2]. Hence, the mapping $k\mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.

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