Mathematical
Inequalities
& Applications
Volume 16, Number 4 (2013), 1115–1122 doi:10.7153/mia-16-86

# A NOTE ON THE HERMITE NUMBERS AND POLYNOMIALS

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(*Communicated by Yeol Je Cho*)

*Abstract.* In this paper, we compute explicitly an integral involving the Hermite polynomials. From our computation, we derive the formula for a product of two Hermite polynomials. Finally, we give some interesting formulae for the product of two Hermite polynomials associated with Bernoulli polynomials like Carlitz did.

## 1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$ (see [1-12]).

In the special case $x=0$, $B_n(0)=B_n$ are called the $n$*-th Bernoulli numbers*. In [3], it is known that

$$\int_0^1 B_p(x)B_q(x)dx=(-1)^{p+1}\frac{B_{p+q}}{\binom{p+q}{q}}, \quad \text{where } p+q\geqslant 2. \tag{2}$$

Carlitz have shown that

$$B_m(x)B_n(x)=\sum_r\left[\binom{m}{2r}n+\binom{n}{2r}m\right]\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r}+(-1)^{m+1}\frac{B_{m+n}}{\binom{m+n}{n}}, \tag{3}$$

where $m,n\in\mathbb{Z}_+$ with $m+n\geqslant 2$ (see [3]).

Let $\mathbb{P}_n=\left\{\sum_i a_ix^i \mid a_i\in\mathbb{R}\right\}$ be the space of polynomials of degree less than or equal to $n$. The set of Bernoulli polynomials $\{B_0(x),B_1(x),\ldots,B_n(x)\}$ is a basis for $\mathbb{P}_n$ (see [5]). For $p(x)\in\mathbb{P}_n$, let $p(x)=\sum_{k=0}^n a_kB_k(x)$. Then we have

$$a_0=\int_0^1 p(x)dx, \quad a_k=\frac{1}{k!}\left(p^{(k-1)}(1)-p^{(k-1)}(0)\right), \quad (k=1,2,\ldots,n), \tag{4}$$

where $p^{(k)}(0)=\frac{d^kp(x)}{dx^k}|_{x=0}$ (see [5]).

*Mathematics subject classification* (2010): 11B68, 33C45.
*Keywords and phrases*: Hermite polynomials, Bernoulli polynomials.

As is well known, the Hermite polynomials are defined by the generating function to be

$$e^{2xt-t^2}=e^{H(x)t}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \tag{5}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$ (see [6]).

From (5), we have

$$H_n(x)=\left(\frac{\partial}{\partial t}\right)^n e^{2xt-t^2}\big|_{t=0}=(-1)^n e^{x^2}\left(\frac{d^n}{dx^n}e^{-x^2}\right). \tag{6}$$

In the special case, $x=0$, $H_n(0)=H_n$ are called the *$n$-th Hermite numbers*. It is known [6] that

$$H_n(x)=e^{\frac{x^2}{2}}\left(x-\frac{d}{dx}\right)^n e^{-\frac{x^2}{2}}, \tag{7}$$

and

$$H_{2n+1}=0,\quad H_{2n}=\frac{(-1)^n(2n)!}{n!},\quad (n\in\mathbb{Z}_+). \tag{8}$$

By (5), we easily see that

$$H_n(x)=(H+2x)^n=\sum_{l=0}^{n}\binom{n}{l}H_{n-l}2^l x^l, \tag{9}$$

and

$$\frac{d}{dx}H_n(x)=2nH_{n-1}(x),\quad H_n(-x)=(-1)^n H_n(x),\quad (\text{see [6]}). \tag{10}$$

The purpose of this paper is to give some formulae on the integral of the product of several Hermite polynomials.

Finally, we investigate some new and interesting identities for the product of two Hermite polynomials associated with Bernoulli polynomials like Carlitz did.

## 2. Some identities of Hermite polynomials

From (10), we note that

$$\begin{aligned}\int_0^1 H_n(x)dx&=\frac{1}{2(n+1)}\left(H_{n+1}(1)-H_{n+1}\right)\\&=\sum_{l=0}^{n}\frac{1}{l+1}\binom{n}{l}H_{n-l}2^l.\end{aligned} \tag{11}$$

Now, let us consider the following integral:

$$\int_0^1 x^n H_n(x)dx=\sum_{l=0}^{n}\binom{n}{l}H_{n-l}2^l\int_0^1 x^{n+l}dx=\sum_{l=0}^{n}\binom{n}{l}\frac{H_{n-l}2^l}{n+l+1}. \tag{12}$$

By (10), we get

$$\begin{aligned}\int_0^1 x^n H_n(x)dx &= \sum_{l=0}^{n}\binom{n}{l}2^l H_{n-l}(1)\int_0^1 x^n(x-1)^l dx\\ &= \sum_{l=0}^{n}\binom{n}{l}(-2)^l H_{n-l}(1)\int_0^1 x^n(1-x)^l dx\\ &= \sum_{l=0}^{n}\binom{n}{l}(-2)^l H_{n-l}(1)\frac{\Gamma(n+1)\Gamma(l+1)}{\Gamma(n+l+2)}\\ &= \sum_{l=0}^{n}\binom{n}{l}2^l(-1)^l H_{n-l}(1)\frac{n!l!}{(n+l+1)!}\\ &= \sum_{l=0}^{n}(-2)^l H_{n-l}(1)\frac{\binom{n}{l}}{(n+l+1)\binom{n+l}{l}}.\end{aligned} \tag{13}$$

Therefore, by (12) and (13), we obtain the following theorem.

THEOREM 2.1. *For $n\in\mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}\frac{H_{n-l}2^l}{n+l+1}=\sum_{l=0}^{n}(-2)^l H_{n-l}(1)\frac{\binom{n}{l}}{(n+l+1)\binom{n+l}{l}}.$$

Let $n\in\mathbb{N}$ with $n\geqslant 2$. Then we have

$$\begin{aligned}&\int_0^1 x^n H_n(x)dx=\frac{H_n(1)}{n+1}+(-1)\frac{2n}{n+1}\int_0^1 x^{n+1}H_{n-1}(x)dx\\ &=\frac{H_n(1)}{n+1}+(-1)\frac{2nH_{n-1}(1)}{(n+1)(n+2)}+(-1)^2\frac{2^2n(n-1)}{(n+1)(n+2)}\int_0^1 x^{n+2}H_{n-2}(x)dx\\ &=\cdots\\ &=\frac{H_n(1)}{n+1}+\sum_{l=2}^{n}\frac{2^{l-1}n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{l-1}H_{n-l+1}(1)\\ &\quad+\frac{(-1)^n2^n n!}{(n+1)(n+2)\cdots(2n)}\int_0^1 x^{2n}dx\\ &=\frac{H_n(1)}{n+1}+\sum_{l=2}^{n}\frac{2^{l-1}n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{l-1}H_{n-l+1}(1)\\ &\quad+(-1)^n\frac{2^n}{(2n+1)\binom{2n}{n}}.\end{aligned} \tag{14}$$

Therefore, by (13) and (14), we obtain the following theorem.

THEOREM 2.2. *For $n \in \mathbb{N}$ with $n \geqslant 2$, we have*

$$\frac{H_n(1)}{n+1}+\sum_{l=2}^{n}\frac{2^{l-1}n(n-1)\cdots(n-l+2)}{(n+1)(n+2)\cdots(n+l)}(-1)^{l-1}H_{n-l+1}(1)+\frac{(-2)^n}{(2n+1)\binom{2n}{n}}$$
$$=\sum_{l=0}^{n}(-2)^l H_{n-l}(1)\frac{\binom{n}{l}}{(n+l+1)\binom{n+l}{l}}.$$

For $n,k \in \mathbb{Z}_+$, let us compute the following integral for the product of Hermite polynomials and $x^k$:

$$\begin{aligned}
&\int_0^1 x^k H_n(x)dx \\
&=\sum_{l=0}^{n}\binom{n}{l}(-1)^l 2^l H_{n-l}(1)\int_0^1 x^k(1-x)^l dx \\
&=\sum_{l=0}^{n}\binom{n}{l}(-1)^l 2^l H_{n-l}(1)\frac{\Gamma(l+1)\Gamma(k+1)}{\Gamma(l+k+2)} \\
&=\sum_{l=0}^{n}\binom{n}{l}(-2)^l H_{n-l}(1)\frac{1}{(l+k+1)\binom{l+k}{l}}.
\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following proposition.

PROPOSITION 2.3. *For $n,k \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
\int_0^1 x^k H_n(x)dx &= \sum_{l=0}^{n}\binom{n}{l}(-1)^l 2^l H_{n-l}(1)\frac{\Gamma(l+1)\Gamma(k+1)}{\Gamma(l+k+2)} \\
&=\sum_{l=0}^{n}\binom{n}{l}(-2)^l H_{n-l}(1)\frac{1}{(l+k+1)\binom{l+k}{l}}.
\end{aligned}$$

For $k,n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_0^1 x^k H_n(x)dx = \frac{H_{n+1}(1)}{2(n+1)}-\frac{k}{2(n+1)}\int_0^1 x^{k-1}H_{n+1}(x)dx \\
&=\frac{H_{n+1}(1)}{2(n+1)}-\frac{k}{2(n+1)}\sum_{l=0}^{n+1}\binom{n+1}{l}H_{n+1-l}(1)(-2)^l\frac{\Gamma(k)\Gamma(l+1)}{\Gamma(k+l+1)} \\
&=\frac{H_{n+1}(1)}{2(n+1)}-\frac{1}{2(n+1)}\sum_{l=0}^{n+1}\binom{n+1}{l}H_{n+1-l}(1)(-2)^l\frac{1}{\binom{k+l}{l}}.
\end{aligned} \tag{16}$$

Therefore, by (15) and (16), we obtain the following theorem.

THEOREM 2.4. *For $n,k \in \mathbb{N}$, we have*

$$\frac{H_{n+1}(1)}{2(n+1)}=\sum_{l=0}^{n}\frac{\binom{n}{l}(-2)^l H_{n-l}(1)}{(k+l+1)\binom{k+l}{l}}+\frac{1}{2(n+1)}\sum_{l=0}^{n+1}\binom{n+1}{l}\frac{H_{n+1-l}(1)(-2)^l}{\binom{k+l}{l}}.$$

Let us assume that $p(x) = H_n(x) \in \mathbb{P}_n$. Then $H_n(x)$ is generated by $\{B_0(x), B_1(x), \dots, B_n(x)\}$ to be

$$H_n(x) = \sum_{k=0}^{n} a_k B_k(x), \tag{17}$$

where, by (4), for $k = 1, 2, \dots, n$ we get

$$\begin{aligned} a_k &= \frac{1}{k!}\left(H_n^{(k-1)}(1) - H_n^{(k-1)}(0)\right) \\ &= \frac{1}{k!} 2^{k-1} \frac{n!}{(n-k+1)!}\left(H_{n-k+1}(1) - H_{n-k+1}\right) \\ &= \frac{1}{n+1} 2^{k-1} \binom{n+1}{k}\left(H_{n-k+1}(1) - H_{n-k+1}\right), \end{aligned} \tag{18}$$

and

$$a_0 = \int_0^1 H_n(x)dx = \frac{H_{n+1}(1) - H_{n+1}}{2(n+1)}. \tag{19}$$

Therefore, by (17), (18) and (19), we obtain the following proposition.

PROPOSITION 2.5. *For $n \in \mathbb{Z}_+$, we have*

$$H_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} 2^{k-1} \binom{n+1}{k} \left(H_{n-k+1}(1) - H_{n-k+1}\right) B_k(x).$$

Let $m, n \in \mathbb{Z}_+$. Then we get

$$\begin{aligned} &H_n(x)H_m(x) \\ &= \left(\frac{1}{n+1}\sum_{k=0}^{n} 2^{k-1}\binom{n+1}{k}\left(H_{n-k+1}(1) - H_{n-k+1}\right)B_k(x)\right) \\ &\quad \times \left(\frac{1}{m+1}\sum_{l=0}^{m} 2^{l-1}\binom{m+1}{l}\left(H_{m-l+1}(1) - H_{m-l+1}\right)B_l(x)\right) \\ &= \frac{1}{(n+1)(m+1)}\sum_{p=0}^{n+m}\sum_{l=0}^{p}\binom{n+1}{l}\binom{m+1}{p-l}\left(H_{n-l+1}(1) - H_{n-l+1}\right) \\ &\quad \times \left(H_{m-p+l+1}(1) - H_{m-p+l+1}\right)B_{p-l}(x)B_l(x)2^{p-2} \\ &= \frac{1}{(n+1)(m+1)}\left(\frac{\left(H_{n+1}(1) - H_{n+1}\right)\left(H_{m+1}(1) - H_{m+1}\right)}{4}\right) + \frac{1}{n+1}\left(H_{n+1}(1) - H_{n+1}\right) \\ &\quad \times \left(H_m(1) - H_m\right)\left(\frac{x}{2} - \frac{1}{4}\right) + \frac{1}{m+1}\left(H_n(1) - H_n\right)\left(H_{m+1}(1) - H_{m+1}\right)\left(\frac{x}{2} - \frac{1}{4}\right) \\ &\quad + \frac{1}{(n+1)(m+1)}\sum_{p=2}^{n+m}\sum_{l=0}^{p} 2^{p-2}\binom{n+1}{l}\binom{m+1}{p-l}\left(H_{n-l+1}(1) - H_{n-l+1}\right) \\ &\quad \times \left(H_{m-p+l+1}(1) - H_{m-p+l+1}\right) \end{aligned}$$

$$\times\left\{\sum_r\left(\binom{p-l}{2r}l+\binom{l}{2r}(p-l)\right)\frac{B_{2r}B_{p-2r}(x)}{p-2r}+(-1)^{p-l+1}\frac{B_p}{\binom{p}{l}}\right\}. \tag{20}$$

Therefore, by (20), we obtain the following theorem.

THEOREM 2.6. *For $n,m\in\mathbb{Z}_+$, we have*

$$\begin{aligned}
&H_n(x)H_m(x)\\
=&\frac{1}{(n+1)(m+1)}\cdot\frac{(H_{n+1}(1)-H_{n+1})(H_{m+1}(1)-H_{m+1})}{4}\\
&+\frac{1}{n+1}(H_{n+1}(1)-H_{n+1})(H_m(1)-H_m)\left(\frac{x}{2}-\frac{1}{4}\right)\\
&+\frac{1}{m+1}(H_n(1)-H_n)(H_{m+1}(1)-H_{m+1})\left(\frac{x}{2}-\frac{1}{4}\right)\\
&+\frac{1}{(n+1)(m+1)}\sum_{p=2}^{n+m}\sum_{l=0}^{p}2^{p-2}\binom{n+1}{l}\binom{m+1}{p-l}(H_{n-l+1}(1)-H_{n-l+1})\\
&\times(H_{m-p+l+1}(1)-H_{m-p+l+1})\left\{\sum_r\left(\binom{p-l}{2r}l+\binom{l}{2r}(p-l)\right)\frac{B_{2r}B_{p-2r}(x)}{p-2r}\right.\\
&\left.+(-1)^{p-l+1}\frac{B_p}{\binom{p}{l}}\right\}.
\end{aligned}$$

It is known that $\mathbb{P}_n$ is an inner product space with the inner product

$$\langle p(x),q(x)\rangle=\int_{-\infty}^{\infty}e^{-x^2}p(x)q(x)dx,$$

where $p(x),q(x)\in\mathbb{P}_n$ (see [6]).

In [6], we note that

$$\int_{-\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)dx=2^n n!\sqrt{\pi}\delta_{n,m}, \tag{21}$$

where $\delta_{n,m}$ is the Kronecker symbol.

From (21), we can see that $\{H_0(x),H_1(x),\dots,H_n(x)\}$ is an orthogonal basis for the inner product space $\mathbb{P}_n$. Then $p(x)$ is generated by $\{H_0(x),H_1(x),\dots,H_n(x)\}$ to be

$$p(x)=\sum_{k=0}^{n}C_kH_k(x), \tag{22}$$

where

$$\begin{aligned}
C_k&=\frac{1}{2^k k!\sqrt{\pi}}\langle p(x),H_n(x)\rangle=\frac{1}{2^k k!\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-x^2}H_k(x)p(x)dx\\
&=\frac{(-1)^k}{2^k k!\sqrt{\pi}}\int_{-\infty}^{\infty}\left(\frac{d^k}{dx^k}e^{-x^2}\right)p(x)dx,\quad\text{(see [6])}.
\end{aligned} \tag{23}$$

It is known [6] that

$$H_n(x) = \begin{cases} \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^{\frac{n}{2}-k} n! 2^{2k} x^{2k}}{\left(\frac{n}{2}-k\right)!(2k)!}, & \text{if } n \equiv 0 \pmod 2, \\ \sum_{k=0}^{\frac{n-1}{2}} \frac{(-1)^{\frac{n-1}{2}-k} n! 2^{2k+1} x^{2k+1}}{\left(\frac{n-1}{2}-k\right)!(2k+1)!}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{24}$$

Let us assume that $n, m \in \mathbb{N}$ with $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$. Then, by (24), we get

$$\begin{aligned} H_m(x)H_{n-m}(x) &= \left( \sum_{k=0}^{\frac{m}{2}} \frac{(-1)^{\frac{m}{2}-k} m! 2^{2k} x^{2k}}{\left(\frac{m}{2}-k\right)!(2k)!} \right) \left( \sum_{l=0}^{\frac{n-m}{2}} \frac{(-1)^{\frac{n-m}{2}-l} (n-m)! 2^{2l} x^{2l}}{\left(\frac{n-m}{2}-l\right)!(2l)!} \right) \\ &= \sum_{p=0}^{\frac{n}{2}} \sum_{l=0}^{p} \left\{ \frac{(-1)^{\frac{n}{2}-p} m!(n-m)!}{\left(\frac{m}{2}-l\right)!(2l)!\left(\frac{n-m}{2}-p+l\right)!(2p-2l)!} \right\} x^{2p} 2^{2p}. \end{aligned} \tag{25}$$

Let us take $p(x) = H_m(x)H_{n-m}(x) \in \mathbb{P}_n$. Then, by (22), we get

$$H_m(x)H_{n-m}(x) = \sum_{k=0}^{n} C_k H_k(x), \quad (C_k \in \mathbb{R}), \tag{26}$$

where, by (23) and (25), we get

$$\begin{aligned} C_k &= \frac{1}{2^k k! \sqrt{\pi}} \langle p(x), H_k(x) \rangle = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k}{dx^k} e^{-x^2} \right) H_{n-m}(x) H_m(x) dx \\ &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \sum_{p=0}^{n} \sum_{l=0}^{p} \left\{ \frac{(-1)^{\frac{n}{2}-p} m!(n-m)!}{\left(\frac{m}{2}-l\right)!(2l)!\left(\frac{n-m}{2}-p+l\right)!(2p-2l)!} \right\} 2^{2p} \\ &\quad \times \int_{-\infty}^{\infty} \left( \frac{d^k}{dx^k} e^{-x^2} \right) x^{2p} dx. \end{aligned} \tag{27}$$

It is easy to show that

$$\int_{-\infty}^{\infty} \left( \frac{d^k}{dx^k} e^{-x^2} \right) x^{2p} dx = \frac{(-1)^k (2p)!}{2^{2p-k} \left(\frac{2p-k}{2}\right)!} \sqrt{\pi}, \tag{28}$$

for $k \leqslant 2p$ and $k \equiv 0 \pmod 2$, and 0 otherwise.

From (27) and(28), we have

$$C_k = \frac{1}{k!} \sum_{\frac{k}{2} \leqslant p \leqslant n} \sum_{0 \leqslant l \leqslant p} \frac{(2p)!(-1)^{\frac{n}{2}-p} m!(n-m)!}{\left(\frac{2p-k}{2}\right)! \left(\frac{m}{2}-l\right)!(2l)!\left(\frac{n-m}{2}-p+l\right)!(2p-2l)!}, \tag{29}$$

for $k \equiv 0 \pmod 2$, and 0 otherwise. Therefore, by (26) and (29), we obtain the following theorem.

THEOREM 2.7. *For $n,m \in \mathbb{N}$ with $n \equiv 0 \pmod 2$ and $m \equiv 0 \pmod 2$, we have*

$$H_m(x)H_{n-m}(x) = \sum_{0 \leqslant k \leqslant n,\ k \equiv 0 \pmod 2} \left\{ \frac{1}{k!} \sum_{\frac{k}{2} \leqslant p \leqslant n} \sum_{0 \leqslant l \leqslant p} \frac{(2p)!(-1)^{\frac{n}{2}-p} m!(n-m)!}{\left(\frac{2p-k}{2}\right)! \left(\frac{m}{2}-l\right)!(2l)! \left(\frac{n-m}{2}-p+l\right)!(2p-2l)!} \right\} \times H_k(x).$$

*Acknowledgements.*

This research was supported by Basic Science Research Program through the National Research Foundation of Korea NRF funded by the Ministry of Education, Science and Technology 2012R1A1A2003786.

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(Received October 24, 2012)

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