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STABILITY OF CAUCHY ADDITIVE FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstract. In this article, we prove the generalized Hyers–Ulam stability of the following Cauchy additive functional equation

$$f\left(\frac{x-y}{n}+z\right)+f\left(\frac{y-z}{n}+x\right)+f\left(\frac{z-x}{n}+y\right)=f(x+y+z)$$

in fuzzy Banach spaces for any fixed nonzero integer n.

1. Introduction

In 1984, A. K. Katsaras [11] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [6, 13, 23]. In particular, T. Bag and S. K. Samanta [2], following S. C. Cheng and J. M. Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of J. Kramosil and Michalek type [12]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given [2, 16, 17] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the Cauchy additive functional equation in the fuzzy normed vector space setting.

DEFINITION 1. [2, 16, 17] Let X be a real vector spaces. A function $N : X\times\mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leqslant 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ for $c\neq 0$;
(N_4) $N(x+y,s+t)\geqslant\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;

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(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed vector space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [16].

DEFINITION 2. [2, 16, 17] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converges to x if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

DEFINITION 3. [2, 16, 17] Let (X,N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X (see [3]).

We recall the fixed point theorem from [14], which is needed in Section 3.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x,y,z \in X$.

THEOREM 1. [4, 14] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geqslant n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;

(4) $d(y,y^*) \leqslant \frac{1}{1-L} d(y,Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th. M. Rassias [10] may be the first to provide applications of fixed point theorems to the proof of stability theory of functional equations. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 14, 19, 20]).

The stability problem of functional equations originated from a question of S. M. Ulam [22] concerning the stability of group homomorphisms. D. H. Hyers [9] gave a first affirmative partial answer to the question of S. M. Ulam for additive mappings on

Banach spaces. Hyers's theorem was generalized by T. Aoki [1] for additive mappings and by Th. M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtain by P. Găvruta [7] by replacing the unbounded Cauchy difference by a general control function.

In 2009, M. S. Moslehian and A. Najati [18] introduced the Hyers–Ulam stability of the Cauchy additive functional inequality

$$\left\|f\left(\frac{x-y}{2}+z\right)+f\left(\frac{y-z}{2}+x\right)+f\left(\frac{z-x}{2}+y\right)\right\| \leqslant \|f(x+y+z)\| \tag{1}$$

and then have investigated the general solution and the Hyers–Ulam stability problem for the functional inequality.

In this paper, we consider a modified and generalized Cauchy additive functional equation

$$f\left(\frac{x-y}{n}+z\right)+f\left(\frac{y-z}{n}+x\right)+f\left(\frac{z-x}{n}+y\right)=f(x+y+z) \tag{2}$$

for any fixed nonzero integer n. First of all, it is easy to see that a function f satisfies the equation (2) if and only if f is additive. Thus the equation (2) may be called the Cauchy additive functional equation and the general solution of equation (2) may be called the Cauchy additive function. In Section 2, using direct method by iteration, we prove the generalized Hyers–Ulam stability of the Cauchy additive functional equation (2) in fuzzy Banach spaces. In Section 3, using fixed point alternative by contraction mappings, we prove the generalized Hyers–Ulam stability of (2) in fuzzy Banach spaces.

2. Stability of equation (2) by direct method

Throughout this paper, we assume that X is a vector space, (Y,N) is a fuzzy Banach space and (Z,N') is a fuzzy normed space.

For notational convenience, given a mapping $f: X \to Y$, we define the difference operator $Df: X^3 \to Y$ of the inequality (2) by

$$Df(x,y,z) := f\left(\frac{x-y}{n}+z\right)+f\left(\frac{y-z}{n}+x\right)+f\left(\frac{z-x}{n}+y\right)-f(x+y+z)$$

for all $x,y,z \in X$.

THEOREM 2. *Assume that there are a constant $s \in \mathbb{R}$ satisfying $0<|s|<2$ and a mapping $\varphi: X^3 \to Z$ for which a mapping $f: X \to Y$ with $f(0)=0$ satisfies the inequality*

$$N(Df(x,y,z),t) \geqslant N'(\varphi(x,y,z),t), \tag{3}$$
$$N'(\varphi(2x,2y,2z),t) \geqslant N'(s\varphi(x,y,z),t) \tag{4}$$

for all $x,y,z \in X$ and all $t>0$. Then we can find a unique Cauchy additive mapping $A: X \to Y$, defined as $A(x) := N\text{-}\lim_{n\to\infty} \frac{f(2^k x)}{2^k}$, $x \in X$, satisfying the equation

$DA(x,y,z)=0$ *and the inequality*

$$N(f(x)-A(x),t) \geqslant \min\Big\{N'\Big(\frac{4}{2-|s|}\varphi\Big(\frac{n(n-1)x}{n^2+3},\frac{n^2x}{n^2+3},\frac{-n(2n-1)x}{n^2+3}\Big),t\Big),$$
$$N'\Big(\frac{2}{2-|s|}\varphi\Big(\frac{2n^2x}{n^2+3},\frac{nx}{n^2+3},\frac{-n(2n+1)x}{n^2+3}\Big),t\Big)\Big\} \quad (5)$$

for all $x \in X$ *and all* $t>0$.

Proof. We observe from (4) that

$$N'(\varphi(2^ix,2^iy,2^iz),t) \geqslant N'(s^i\varphi(x,y,z),t) = N'\Big(\varphi(x,y,z),\frac{t}{|s|^i}\Big), \quad (6)$$
$$N'(\varphi(2^ix,2^iy,2^iz),|s|^it) \geqslant N'(\varphi(x,y,z),t), \quad t>0 \quad (7)$$

for all $x,y,z \in X$ and all $i \in \mathbb{N}$. Replacing z by $-x-y$ in (3), we obtain

$$N\Big(f\Big(\frac{(1-n)x-(n+1)y}{n}\Big)+f\Big(\frac{(n+1)x+2y}{n}\Big)+f\Big(\frac{-2x+(n-1)y}{n}\Big),t\Big)$$
$$\geqslant N'(\varphi(x,y,-x-y),t),$$

that is,

$$N(f(-(n-1)x-(n+1)y)+f((n+1)x+2y)+f(-2x+(n-1)y),t)$$
$$\geqslant N'(\varphi(nx,ny,-nx-ny),t) \quad (8)$$

for all $x,y \in X$ and all $t>0$. Putting $x := \frac{(n-1)x+(n+1)y}{n^2+3}$ and $y := \frac{nx-(n-1)y}{n^2+3}$ in (8), one has

$$N(f(-x)+f(x+y)+f(-y),t)$$
$$\geqslant N'\Big(\varphi\Big(\frac{n(n-1)x+n(n+1)y}{n^2+3},\frac{n^2x-n(n-1)y}{n^2+3},\frac{-n(2n-1)x-2ny}{n^2+3}\Big),t\Big) \quad (9)$$

for all $x,y \in X$ and all $t>0$. Putting $y:=0$ in (9), we get

$$N(f(-x)+f(x),t) \geqslant N'\Big(\varphi\Big(\frac{n(n-1)x}{n^2+3},\frac{n^2x}{n^2+3},\frac{-n(2n-1)x}{n^2+3}\Big),t\Big) \quad (10)$$

for all $x \in X$ and all $t>0$. Replacing y by x in (9), we have

$$N(2f(-x)+f(2x),t) \geqslant N'\Big(\varphi(\tfrac{2n^2x}{n^2+3},\tfrac{nx}{n^2+3},\tfrac{-n(2n+1)x}{n^2+3}),t\Big),$$
$$\text{and} \quad N\Big(f(-x)+\tfrac{f(2x)}{2},\tfrac{t}{2}\Big) \geqslant N'\Big(\varphi(\tfrac{2n^2x}{n^2+3},\tfrac{nx}{n^2+3},\tfrac{-n(2n+1)x}{n^2+3}),t\Big) \quad (11)$$

for all $x \in X$ and all $t>0$. It follows from (10) and (11) that

$$N\Big(f(x)-\frac{f(2x)}{2},\frac{t}{2}\Big)$$
$$\geqslant \min\Big\{N\Big(f(x)+f(-x),\frac{t}{4}\Big),N\Big(f(-x)+\frac{f(2x)}{2},\frac{t}{4}\Big)\Big\}$$

$$\geqslant \min\Big\{N'\Big(\varphi\Big(\frac{n(n-1)x}{n^2+3},\frac{n^2x}{n^2+3},\frac{-n(2n-1)x}{n^2+3}\Big),\frac{t}{4}\Big),$$
$$N'\Big(\varphi\Big(\frac{2n^2x}{n^2+3},\frac{nx}{n^2+3},\frac{-n(2n+1)x}{n^2+3}\Big),\frac{t}{2}\Big)\Big\}$$
$$:\equiv N''(\psi(x),t) \tag{12}$$

for all $x\in X$ and all $t>0$. Therefore it follows from (7) and (12) that

$$N\Big(\frac{f(2^ix)}{2^i}-\frac{f(2^{i+1}x)}{2^{i+1}},\frac{|s|^it}{2^{i+1}}\Big)\geqslant N''(\psi(2^ix),|s|^it)\geqslant N''(\psi(x),t),\quad t>0$$

for all $x\in X$ and any integer $i\geqslant 0$. So

$$\begin{aligned}N\Big(f(x)-\frac{f(2^kx)}{2^k},\sum_{i=0}^{k-1}\frac{|s|^it}{2^{i+1}}\Big)&=N\left(\sum_{i=0}^{k-1}\Big(\frac{f(2^ix)}{2^i}-\frac{f(2^{i+1}x)}{2^{i+1}}\Big),\sum_{i=0}^{k-1}\frac{|s|^it}{2^{i+1}}\right)\\&\geqslant\min_{0\leqslant i\leqslant k-1}\Big\{N\Big(\frac{f(2^ix)}{2^i}-\frac{f(2^{i+1}x)}{2^{i+1}},\frac{|s|^it}{2^{i+1}}\Big)\Big\}\\&\geqslant N''(\psi(x),t),\quad t>0,\end{aligned}\tag{13}$$

which yields

$$\begin{aligned}N\Big(\frac{f(2^mx)}{2^m}-\frac{f(2^{m+k}x)}{2^{m+k}},\sum_{i=m}^{m+k-1}\frac{|s|^it}{2^{i+1}}\Big)&=N\Big(\sum_{i=m}^{m+k-1}\Big(\frac{f(2^ix)}{2^i}-\frac{f(2^{i+1}x)}{2^{i+1}}\Big),\sum_{i=m}^{m+k-1}\frac{|s|^it}{2^{i+1}}\Big)\\&\geqslant\min_{m\leqslant i\leqslant m+k-1}\Big\{N\Big(\frac{f(2^ix)}{2^i}-\frac{f(2^{i+1}x)}{2^{n+1}},\frac{|s|^it}{2^{i+1}}\Big)\Big\}\\&\geqslant N''(\psi(x),t),\quad t>0\end{aligned}$$

for all $x\in X$ and any integers $k>0$, $m\geqslant 0$. Hence one obtains

$$N\Big(\frac{f(2^mx)}{2^m}-\frac{f(2^{m+k}x)}{2^{m+k}},t\Big)\geqslant N''\left(\psi(x),\frac{t}{\sum_{i=m}^{m+k-1}\frac{|s|^i}{2^{i+1}}}\right)\tag{14}$$

for all $x\in X$ and any integers $k>0$, $m\geqslant 0$, $t>0$. Since $\sum_{i=m}^{m+k-1}\frac{|s|^i}{2^i}$ is convergent series, we see by taking the limit $m\to\infty$ in the last inequality that a sequence $\{\frac{f(2^kx)}{2^k}\}$ is Cauchy in the fuzzy Banach space (Y,N) and so it converges in Y. Therefore a mapping $A:X\to Y$ defined by

$$A(x):=N\text{-}\lim_{k\to\infty}\frac{f(2^kx)}{2^k}$$

is well defined for all $x\in X$. It means that $\lim_{k\to\infty}N(\frac{f(2^kx)}{2^k}-A(x),t)=1$, $t>0$ for all $x\in X$. In addition, we see from (13) that

$$N\Big(f(x)-\frac{f(2^kx)}{2^k},t\Big)\geqslant N''\left(\psi(x),\frac{t}{\sum_{i=0}^{k-1}\frac{|s|^i}{2^{i+1}}}\right)\tag{15}$$

and so

$$N(f(x)-A(x),t) \geqslant \min\left\{N\left(f(x)-\frac{f(2^kx)}{2^k},(1-\varepsilon)t\right),N\left(\frac{f(2^kx)}{2^k}-A(x),\varepsilon t\right)\right\}$$
$$\geqslant N''\left(\psi(x),\frac{(1-\varepsilon)t}{\sum_{i=0}^{k-1}\frac{|s|^i}{2^{i+1}}}\right)$$
$$\geqslant N''(\psi(x),(1-\varepsilon)(2-|s|)t),\quad 0<\varepsilon<1, \tag{16}$$

for sufficiently large k and for all $x \in X$ and all $t > 0$. Since ε is arbitrary and N' is left continuous, we obtain

$$N(f(x)-A(x),t) \geqslant N''(\psi(x),(2-|s|)t),\quad t>0$$

for all $x \in X$, which yields the approximation (5).

In addition, it is clear from (3), (6) and (N_5) that the following relation

$$N\left(\frac{Df(2^kx,2^ky,2^kz)}{2^k},t\right) \geqslant N'(\varphi(2^kx,2^ky,2^kz),2^kt)$$
$$\geqslant N'\left(\varphi(x,y,z),\frac{2^k}{|s|^k}t\right)$$
$$\to 1 \text{ as } k\to\infty$$

holds for all $x,y,z \in X$ and all $t > 0$. Therefore, we obtain by use of $\lim_{k\to\infty} N(\frac{f(2^kx)}{2^k} - A(x),t) = 1\ (t>0)$ that

$$N(DA(x,y,z),t)$$
$$\geqslant \min\left\{N\left(DA(x,y,z)-\frac{Df(2^kx,2^ky,2^kz)}{2^k},\frac{t}{2}\right),N\left(\frac{Df(2^kx,2^ky,2^kz)}{2^k},\frac{t}{2}\right)\right\}$$
$$\to 1 \text{ as } k\to\infty$$

which implies $DA(x,y,z)=0$ by (N_2). Thus we find that A is a Cauchy additive mapping satisfying the equation (2) and the inequality (5) near the approximate additive mapping $f : X \to Y$.

To prove the afore-mentioned uniqueness, we assume now that there is another Cauchy additive mapping $A' : X \to Y$ which satisfies the inequality (5). Then one establishes by the equality $A'(2^kx) = 2^kA'(x)$ and (5) that

$$N(A(x)-A'(x),t) = N\left(\frac{A(2^kx)}{2^k}-\frac{A'(2^kx)}{2^k},t\right)$$
$$\geqslant \min\left\{N\left(\frac{A(2^kx)}{2^k}-\frac{f(2^kx)}{2^k},\frac{t}{2}\right),N\left(\frac{f(2^kx)}{2^k}-\frac{A'(2^kx)}{2^k},\frac{t}{2}\right)\right\}$$
$$\geqslant N''(\psi(2^kx),\frac{(2-|s|)2^kt}{2})$$
$$\geqslant N''\left(\psi(x),\frac{(2-|s|)2^kt}{2|s|^k}\right),\quad t>0$$

which tends to 1 as $k \to \infty$ by (N_5). Therefore one obtains $A(x) = A'(x)$ for all $x \in X$, completing the proof of uniqueness. □

THEOREM 3. *Assume that there are a constant $s \in \mathbb{R}$ satisfying $|s| > 2$ and a mapping $\varphi : X^3 \to Z$ for which a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x,y,z),t) \geqslant N'(\varphi(x,y,z),t), \tag{17}$$

$$N'\Big(\varphi\Big(\frac{x}{2},\frac{y}{2},\frac{z}{2}\Big),t\Big) \geqslant N'\Big(\frac{1}{s}\varphi(x,y,z),t\Big) \tag{18}$$

for all $x,y,z \in X$ and all $t > 0$. Then we can find a unique Cauchy additive mapping $A : X \to Y$, defined as $A(x) := N\text{-}\lim_{n\to\infty} 2^k f\big(\frac{x}{2^k}\big)$, $x \in X$, satisfying the equation $DA(x,y,z) = 0$ and the inequality

$$\begin{aligned} N(f(x)-A(x),t) \geqslant \min\Big\{&N'\Big(\frac{4}{|s|-2}\varphi\Big(\frac{n(n-1)x}{n^2+3},\frac{n^2x}{n^2+3},\frac{-n(2n-1)x}{n^2+3}\Big),t\Big),\\ &N'\Big(\frac{2}{|s|-2}\varphi\Big(\frac{2n^2x}{n^2+3},\frac{nx}{n^2+3},\frac{-n(2n+1)x}{n^2+3}\Big),t\Big)\Big\} \end{aligned} \tag{19}$$

for all $x \in X$ and all $t > 0$.

Proof. It follows from (12) and (18) that

$$N\Big(f(x)-2f\Big(\frac{x}{2}\Big),\frac{t}{|s|}\Big) \geqslant N''(\psi(x),t), \quad t > 0$$

for all $x \in X$. Therefore it follows that

$$N\Big(f(x)-2^k f\Big(\frac{x}{2^k}\Big),\sum_{i=0}^{k-1}\frac{2^i}{|s|^{i+1}}t\Big) \geqslant N''(\psi(x),t), \quad t > 0$$

for all $x \in X$ and any integer $k > 0$. Thus we see from the last inequality that

$$\begin{aligned} N\Big(f(x)-2^k f\Big(\frac{x}{2^k}\Big),t\Big) &\geqslant N''\Big(\psi(x),\frac{t}{\sum_{i=0}^{k-1}\frac{2^i}{|s|^{i+1}}}\Big)\\ &\geqslant N''(\psi(x),(|s|-2)t), \quad t > 0. \end{aligned}$$

The remaining assertion goes through by the similar way to the corresponding part of Theorem 2. □

We obtain the following corollary concerning the stability for approximate mappings controlled by a sum of powers of norms.

COROLLARY 1. *Let X be a normed space and $(\mathbb{R}, N')$ be a fuzzy normed space. Assume that there exist real numbers $\theta \geqslant 0$ and p be non negative real numbers with $p \neq 1$. If a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$N(Df(x,y,z),t) \geqslant N'(\theta(\|x\|^p+\|y\|^p+\|z\|^p),t)$$

for all $x, y, z \in X$ *and all* $t > 0$. *Then we can find a unique Cauchy additive mapping* $A : X \to Y$ *satisfying the equation* $DA(x, y, z) = 0$ *and the inequality*

$$\begin{aligned}
&N(f(x) - A(x), t) \\
&\geqslant \min\Big\{N'\Big(\Big(\Big|\frac{n(n-1)}{n^2+3}\Big|^p + \Big|\frac{n^2}{n^2+3}\Big|^p + \Big|\frac{-n(2n-1)}{n^2+3}\Big|^p\Big)\frac{4\theta}{|2-2^p|}\|x\|^p, t\Big), \\
&\qquad N'\Big(\Big(\Big|\frac{2n^2}{n^2+3}\Big|^p + \Big|\frac{n}{n^2+3}\Big|^p + \Big|\frac{-n(2n+1)}{n^2+3}\Big|^p\Big)\frac{2\theta}{|2-2^p|}\|x\|^p, t\Big)\Big\}
\end{aligned}$$

for all $x \in X$ *and all* $t > 0$.

3. Stability of equation (2) by fixed point method

Now, in the next theorem we are going to consider a stability problem concerning the stability of the equation (2) by using a fixed point theorem of the alternative for contraction mappings on generalized complete metric spaces due to B. Margolis and J. B. Diaz [14].

THEOREM 4. *Assume that there exist constant* $s \in \mathbb{R}$ *and* $q > 0$ *satisfying* $0 < |s|^{\frac{1}{q}} < 2$ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$N(Df(x,y,z), t_1 + t_2 + t_3) \geqslant \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q), N'(\varphi(z), t_3^q)\} \tag{20}$$

for all $x, y, z \in X$ *and all* $t_i > 0$ $(i = 1, 2, 3)$ *and* $\varphi : X \to Z$ *is a mapping satisfying*

$$N'(\varphi(2x), t) \geqslant N'(s\varphi(x), t) \tag{21}$$

for all $x \in X$ *and all* $t > 0$. *Then there exists a unique Cauchy additive mapping* $A : X \to Y$ *satisfying the equation* $DA(x, y, z) = 0$ *and the inequality*

$$\begin{aligned}
&N(f(x) - A(x), t) \\
&\geqslant \min\Big\{\min\Big\{N'\Big(\frac{12^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{n(n-1)x}{n^2+3}\Big), t^q\Big), N'\Big(\frac{12^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{n^2x}{n^2+3}\Big), t^q\Big), \\
&\qquad N'\Big(\frac{12^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{-n(2n-1)x}{n^2+3}\Big), t^q\Big)\Big\}, \\
&\quad \min\Big\{N'\Big(\frac{6^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{2n^2x}{n^2+3}\Big), t^q\Big), N'\Big(\frac{6^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{nx}{n^2+3}\Big), t^q\Big), \\
&\qquad N'\Big(\frac{6^q}{(2-|s|^{\frac{1}{q}})^q}\varphi\Big(\frac{-n(2n+1)x}{n^2+3}\Big), t^q\Big)\Big\}\Big\}
\end{aligned} \tag{22}$$

for all $x \in X$ *and all* $t > 0$.

Proof. We consider the set of functions

$$\Omega := \{g : X \to Y | g(0) = 0\}$$

and define a generalized metric on Ω as follows

$$d_\Omega(g,h) := \inf\{K \in [0,\infty] | N(g(x)-h(x),Kt) \geqslant N'''(\psi(x),t^q), \forall x \in X, t > 0\},$$

where

$$\begin{aligned} &N'''(\psi(x),t^q)\\ &\equiv \min\Big\{ \min\Big\{N'\Big(\varphi(\tfrac{n(n-1)x}{n^2+3}),(\tfrac{t}{4})^q\Big),N'\Big(\varphi(\tfrac{n^2x}{n^2+3}),(\tfrac{t}{4})^{\frac{1}{q}}\Big),N'\Big(\varphi(\tfrac{-n(2n-1)x}{n^2+3}),(\tfrac{t}{4})^q\Big)\Big\},\\ &\qquad \min\Big\{N'\Big(\varphi(\tfrac{2n^2x}{n^2+3}),(\tfrac{t}{2})^q\Big),N'\Big(\varphi(\tfrac{nx}{n^2+3}),(\tfrac{t}{2})^q\Big),N'\Big(\varphi(\tfrac{-n(2n+1)x}{n^2+3}),(\tfrac{t}{2})^q\Big)\Big\}\Big\}. \end{aligned}$$

Then one can easily see that (Ω, d_Ω) is a complete generalized metric space [8, 15].

Now, we define an operator $J : \Omega \to \Omega$ as

$$Jg(x) = \frac{g(2x)}{2}$$

for all $g \in \Omega$, $x \in X$. We first prove that J is strictly contractive on Ω. For any $g,h \in \Omega$, let $\varepsilon \in [0,\infty)$ be any constant with $d_\Omega(g,h) \leqslant \varepsilon$. Then we deduce from the use of (21) and the definition of $d_\Omega(g,h)$ that

$$\begin{aligned} &N(g(x)-h(x),\varepsilon t) \geqslant N'''(\psi(x),t^q), \quad \forall x \in X,\ t > 0,\\ &\Rightarrow N\Big(\frac{g(2x)}{2} - \frac{h(2x)}{2}, \frac{|s|^{\frac{1}{q}}\varepsilon t}{2}\Big) \geqslant N'''(\psi(2x),|s|t^q) \geqslant N'''(\psi(x),t^q),\\ &\Rightarrow N\Big(Jg(x)-Jh(x),\frac{|s|^{\frac{1}{q}}\varepsilon t}{2}\Big) \geqslant N'''(\psi(x),t^q), \quad \forall x \in X,\ t > 0,\\ &\Rightarrow d_\Omega(Jg,Jh) \leqslant \frac{|s|^{\frac{1}{q}}\varepsilon}{2}. \end{aligned}$$

Since ε is arbitrary constant with $d_\Omega(g,h) \leqslant \varepsilon$, we see that for any $g,h \in \Omega$,

$$d_\Omega(Jg,Jh) \leqslant \frac{|s|^{\frac{1}{q}}}{2} d_\Omega(g,h),$$

which implies that J is strictly contractive with constant $\frac{|s|^{\frac{1}{q}}}{2} < 1$ on Ω.

We now want to show that $d(f,Jf) < \infty$. If we put $t_i := t$ $(i = 1,2,3)$ in (20), then we arrive at

$$N(Df(x,y,z),3t) \geqslant \min\{N'(\varphi(x),t^q),N'(\varphi(y),t^q),N'(\varphi(z),t^q)\} \tag{23}$$

for all $x,y,z \in X$ and all $t > 0$. Therefore it follows from (23) and the same process (8)–(12) that

$$N\Big(f(x) - \frac{f(2x)}{2}, \frac{3t}{2}\Big) \geqslant N'''(\psi(x),t^q),$$

which yields $d_\Omega(f,Jf) \leqslant \frac{3}{2}$ and so $d_\Omega(J^k f, J^{k+1} f) \leqslant d_\Omega(f,Jf) \leqslant \frac{3}{2}$ for all $k \in \mathbb{N}$.

Using the fixed point theorem of the alternative for contraction J on generalized complete metric space Ω due to Theorem 1, we see the followings (i),(ii) and (iii) as follows:

(i) There is a mapping $A: X \to Y$ with $A(0) = 0$ such that

$$d_\Omega(f,A) \leqslant \frac{1}{1-\frac{|s|^{\frac{1}{q}}}{2}} d_\Omega(f,Jf) \leqslant \frac{3}{2-|s|^{\frac{1}{q}}}$$

and A is a fixed point of the operator J, that is, $\frac{1}{2}A(2x) = JA(x) = A(x)$ for all $x \in X$. Thus we can get

$$N\Big(f(x)-A(x), \frac{3t}{2-|s|^{\frac{1}{q}}}\Big) \geqslant N'''(\psi(x), t^q),$$

$$N(f(x)-A(x),t) \geqslant N'''\Big(\psi(x), \frac{(2-|s|^{\frac{1}{q}})^q}{3^q} t^q\Big) = \Big(\frac{3^q}{(2-|s|^{\frac{1}{q}})^q}\psi(x), t^q\Big)$$

for all $t > 0$ and all $x \in X$, which yields the approximation (22);

(ii) We find that $d_\Omega(J^k f, A) \to 0$ as $k \to \infty$. Thus we obtain that

$$\begin{aligned} N\Big(\frac{f(2^k x)}{2^k} - A(x), t\Big) &= N(f(2^k x) - A(2^k x), 2^k t) \\ &\geqslant N'''\Big(\frac{3^q \psi(2^k x)}{(2-|s|^{\frac{1}{q}})^q}, 2^{kq} t^q\Big) \\ &\geqslant N'''\Big(\frac{3^q \psi(x)}{(2-|s|^{\frac{1}{q}})^q}, \Big(\frac{2^q}{|s|}\Big)^k t^q\Big) \\ &\to 1 \text{ as } k \to \infty, \end{aligned}$$

for all $t > 0$ and all $x \in X$, that is, the mapping $A: X \to Y$ given by

$$N\text{-}\lim_{k\to\infty} \frac{f(2^k x)}{2^k} = A(x) \tag{24}$$

is well-defined for all $x \in X$. In addition, it follows from the conditions (21), (23) and (N_4) that

$$\begin{aligned} &N\Big(\frac{Df(2^k x, 2^k y, 2^k z)}{2^k}, t\Big) \\ &\geqslant \min\Big\{N'\Big(\varphi(2^k x), \frac{2^{kq} t^q}{3^q}\Big), N'\Big(\varphi(2^k y), \frac{2^{kq} t^q}{3^q}\Big), N'\Big(\varphi(2^k z), \frac{2^{kq} t^q}{3^q}\Big)\Big\} \\ &\geqslant \min\Big\{N'\Big(|s|^k \varphi(x), \frac{2^{kq} t^q}{3^q}\Big), N'\Big(|s|^k \varphi(y), \frac{2^{kq} t^q}{3^q}\Big), N'\Big(|s|^k \varphi(z), \frac{2^{kq} t^q}{3^q}\Big)\Big\} \\ &\geqslant \min\Big\{N'\Big(\varphi(x), \Big(\frac{2^q}{|s|}\Big)^k \frac{t^q}{3^q}\Big) N'\Big(\varphi(y), \Big(\frac{2^q}{|s|}\Big)^k \frac{t^q}{3^q}\Big) N'\Big(\varphi(z), \Big(\frac{2^q}{|s|}\Big)^k \frac{t^q}{3^q}\Big)\Big\} \\ &\to 1 \text{ as } k \to \infty, \quad t > 0 \end{aligned} \tag{25}$$

for all $x,y,z \in X$. Therefore we obtain by use of (N_4), (24) and (25)

$$\begin{aligned}
&N(DA(x,y,z),t)\\
&\geqslant \min\Big\{N\Big(DA(x,y,z)-\frac{Df(2^kx,2^ky,2^kz)}{2^k},\frac{t}{2}\Big),N\Big(\frac{Df(2^kx,2^ky,2^kz)}{2^k},\frac{t}{2}\Big)\Big\}\\
&\to 1 \text{ as } k\to\infty, \quad t>0
\end{aligned}$$

which implies $DA(x,y,z)=0$ by (N_2), and so the mapping A is Cauchy additive satisfying the equation (2);

(iii) We know that the mapping A is a unique fixed point of the operator J in the set $\Delta=\{g\in\Omega | d_\Omega(f,g)<\infty\}$. Thus if we assume that there exists another Cauchy additive mapping $A':X\to Y$ satisfying the inequality (22), then

$$A'(x)=\frac{A'(2x)}{2}=JA'(x), \quad d_\Omega(f,A')\leqslant \frac{3}{(2-|s|^{\frac{1}{q}})}<\infty,$$

and so A' is a fixed point of the operator J and $A'\in\Delta=\{g\in\Omega|d_\Omega(f,g)<\infty\}$. By the uniqueness of the fixed point of J in Δ, we find that $A=A'$, which proves the uniqueness of A satisfying the inequality (22). This ends the proof of the theorem. □

COROLLARY 2. *Let X be a normed space and $(\mathbb{R}, N')$ be a fuzzy normed space. Let $\xi:[0,\infty)\to[0,\infty)$ be a nontrivial function satisfying*

$$\xi(2t)=\xi(2)\xi(t), \quad (t\geqslant 0), \qquad 0<\xi(2)^{\frac{1}{q}}<2$$

for some $q>0$. If a mapping $f:X\to Y$ with $f(0)=0$ satisfies the functional inequality

$$N(Df(x,y,z),t_1+t_2+t_3)\geqslant \min\{N'(\xi(\|x\|),t_1^q),N'(\xi(\|y\|),t_2^q),N'(\xi(\|z\|),t_3^q)\}$$

for all $x,y,z\in X$, $t_i>0$ $(i=1,2,3)$, then there exists a unique Cauchy additive mapping $A:X\to Y$ such that

$$\begin{aligned}
&N(f(x)-A(x),t)\\
&\geqslant \min\Big\{\min\Big\{N'\Big(\frac{12^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{n(n-1)}{n^2+3}\Big|\|x\|\Big),t^q\Big),\\
&\qquad N'\Big(\frac{12^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{n^2}{n^2+3}\Big|\|x\|\Big),t^q\Big),\\
&\qquad N'\Big(\frac{12^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{-n(2n-1)}{n^2+3}\Big|\|x\|\Big),t^q\Big)\Big\},\\
&\quad \min\Big\{N'\Big(\frac{6^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{2n^2}{n^2+3}\Big|\|x\|\Big),t^q\Big),\\
&\qquad N'\Big(\frac{6^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{n}{n^2+3}\Big|\|x\|\Big),t^q\Big),\\
&\qquad N'\Big(\frac{6^q}{(2-\xi(2)^{\frac{1}{q}})^q}\xi\Big(\Big|\frac{-n(2n+1)}{n^2+3}\Big|\|x\|\Big),t^q\Big)\Big\}\Big\}
\end{aligned}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $\varphi(x) = \xi(\|x\|)$, and applying Theorem 4 with $s := \xi(2)$, we obtain the desired result. □

THEOREM 5. *Assume that there exist constant* $s \in \mathbb{R}$ *and* $q > 0$ *satisfying* $|s|^{\frac{1}{q}} > 2$ *such that a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$N(Df(x,y,z), t_1 + t_2 + t_3) \geqslant \min\{N'(\varphi(x), t_1^q), N'(\varphi(y), t_2^q), N'(\varphi(z), t_3^q)\}$$

for all $x, y, z \in X$, $t_i > 0$ $(i = 1, 2, 3)$ *and* $\varphi : X \to Z$ *is a mapping satisfying*

$$N'\Big(\varphi(\frac{x}{2}), t\Big) \geqslant N'\Big(\frac{\varphi(x)}{s}, t\Big)$$

for all $x \in X$ *and all* $t > 0$. *Then there exists a unique Cauchy additive mapping* $A : X \to Y$ *satisfying the equation* $DA(x,y,z) = 0$ *and the inequality*

$$\begin{aligned}
&N(f(x) - A(x), t)\\
&\geqslant \min\Big\{ \min\Big\{N'\Big(\frac{12^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{n(n-1)x}{n^2+3}\Big), t^q\Big), N'\Big(\frac{12^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{n^2 x}{n^2+3}\Big), t^q\Big),\\
&\qquad N'\Big(\frac{12^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{-n(2n-1)x}{n^2+3}\Big), t^q\Big)\Big\},\\
&\qquad \min\Big\{N'\Big(\frac{6^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{2n^2 x}{n^2+3}\Big), t^q\Big), N'\Big(\frac{6^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{n}{n^2+3}\Big), t^q\Big),\\
&\qquad N'\Big(\frac{6^q}{(|s|^{\frac{1}{q}} - 2)^q}\varphi\Big(\frac{-n(2n+1)x}{n^2+3}\Big), t^q\Big)\Big\}\Big\}
\end{aligned}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof of this theorem is similar to that of Theorem 4. □

COROLLARY 3. *Let* X *be a normed space and* $(\mathbb{R}, N')$ *be a fuzzy normed space. Let* $\xi : [0, \infty) \to [0, \infty)$ *be a nontrivial function satisfying*

$$\xi\Big(\frac{t}{2}\Big) = \xi\Big(\frac{1}{2}\Big)\xi(t), \quad (t \geqslant 0), \qquad \xi\Big(\frac{1}{2}\Big)^{-\frac{1}{q}} > 2$$

for some $q > 0$. *If a mapping* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the functional inequality*

$$N(Df(x,y,z), t_1 + t_2 + t_3) \geqslant \min\{N'(\xi(\|x\|), t_1^q), N'(\xi(\|y\|), t_2^q), N'(\xi(\|z\|), t_3^q)\}$$

for all $x, y, z \in X$, $t_i > 0$ $(i = 1, 2, 3)$, *then there exists a unique Cauchy additive mapping* $A : X \to Y$ *such that*

$$\begin{aligned}
&N(f(x) - A(x), t)\\
&\geqslant \min\Big\{ \min\Big\{ N'\Big(\frac{12^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{n(n-1)}{n^2+3}\Big|\|x\|\Big), t^q\Big),\\
&\qquad N'\Big(\frac{12^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{n^2}{n^2+3}\Big|\|x\|\Big), t^q\Big),\\
&\qquad N'\Big(\frac{12^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{-n(2n-1)}{n^2+3}\Big|\|x\|\Big), t^q\Big)\Big\},\\
&\quad \min\Big\{ N'\Big(\frac{6^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{2n^2}{n^2+3}\Big|\|x\|\Big), t^q\Big),\\
&\qquad N'\Big(\frac{6^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{n}{n^2+3}\Big|\|x\|\Big), t^q\Big),\\
&\qquad N'\Big(\frac{6^q}{(\frac{1}{\xi(\frac{1}{2})}^{\frac{1}{q}} - 2)^q}\xi\Big(\Big|\frac{-n(2n+1)}{n^2+3}\Big|\|x\|\Big), t^q\Big)\Big\}\Big\}
\end{aligned}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Letting $\varphi(x) = \xi(\|x\|)$ and applying Theorem 5 with $s := \frac{1}{\xi(\frac{1}{2})}$, we lead to the approximation. □

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