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STRONG CONVERGENCE THEOREMS OF STRONGLY RELATIVELY NONEXPANSIVE SEMI–GROUP IN BANACH SPACES

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Abstract. In this paper, we prove some strong convergence approximation theorems of strongly relatively nonexpansive semi-group in the framework of Banach spaces. Using the concept of duality theorems, we obtain analogue results for strongly generalized nonexpansive semi-group and for semi-group of firmly generalized nonexpansive type. The results presented in this paper improve and extend some corresponding results announced by some authors.

1. Introduction

Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote by R^+ and R the set of all nonnegative real numbers and the set of all real numbers, respectively. Also, we denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Recall that if E is smooth then J is single-valued and norm-to weak* continuous. We shall denote by J the single-value duality mapping.

A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} \leqslant 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leqslant 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geqslant \varepsilon$. E is said to be smooth (and the norm of E is called *Gâteaux* differentiable) if the limit

$$\lim_{n\to\infty} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. The norm of E is said to be uniformly *Gâteaux* differentiable (resp. *Fréchet* differentiable) if for each $y \in U$ ($x \in U$) the above limit is attained uniformly for any $x \in U$ (resp. uniformly for any $y \in U$). E is said to be uniformly smooth (and the norm of E is called uniformly *Fréchet* differentiable) if the above limit exists uniformly in $x, y \in U$.

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REMARK 1. The following basic properties of Banach space E can be founded in Cioranescu [4] and Takahashi [28].

(i) If E is an uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E;

(ii) If E is a reflective and strictly convex Banach space, then J^{-1} is norm-weak*-continuous;

(iii) If E is a smooth, reflective and strictly convex Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued, one-to one, and surjective;

(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;

(v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ [see, 4, 28] for more details;

(vi) If E is reflective, then E is smooth if and only if E^* is strictly convex;

(vii) If E has an uniformly *Gâteaux* differentiable norm, then J is uniformly norm-weak*-continuous on each bounded subset of E;

(viii) If E has a *Fréchet* differentiable norm, then J is norm-to-norm continuous on each bounded subset of E.

Next, we assume that E is a smooth, reflective and strictly convex Banach space. Consider the functional defined as in [1,2] by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x,y \in E. \tag{1.2}$$

It is obvious from the definition of φ that

$$(\|x\| - \|y\|)^2 \leqslant \varphi(x,y) \leqslant (\|x\| + \|y\|)^2, \ \forall x,y \in E. \qquad (1.3),$$

and

$$\varphi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz) \leqslant \lambda\varphi(x,y) + (1-\lambda)\varphi(x,z), \ \forall x,y \in E. \tag{1.4}$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = arginf_{y\in C}\varphi(y,x), \ \forall x \in E. \tag{1.5}$$

That is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem $\varphi(\overline{x},x) = \inf_{y\in C}\varphi(y,x)$.

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\varphi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [1, 2, 4, 13, 28]). In Hilbert space H, $\Pi_C = P_C$, where P_C is the projection from H onto C.

Suppose that C is a subset of a smooth Banach space E. A mapping $T : C \to E$ is relatively nonexpansive if the following properties are satisfied:

(i) $F(T) \neq \emptyset$, where $F(T)$ denotes the fixed points set of T;

(ii) $\varphi(p,Tx) \leqslant \varphi(p,x)$ for all $p \in F(T)$ and $x \in C$;

(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0, it follows that $p \in F(T)$.

If T satisfies (i) and (ii), then T is called relatively quasi-nonexpansive [20]. It is clear that in a Hilbert space H, (1.2) reduces to $\varphi(x,y) = \|x-y\|^2$, $\forall x,y \in H$. Hence, if $T : C \to H$ is nonexpansive mapping of a nonempty, closed and convex subset C of H, then it is relatively nonexpansive. If $T : C \to H$ is relatively quasi-nonexpansive, then it is quasi-nonexpansive, that is, $\|p - Tx\| \leqslant \|p - x\|$ for all $p \in F(T)$ and $x \in C$.

A relatively nonexpansive mapping $T : C \to E$ is strongly relatively nonexpansive [5, 24] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p,x_n) - \varphi(p,Tx_n) \to 0$,(as $n \to \infty$), for some $p \in F(T)$, it follows that $\varphi(Tx_n,x_n) \to 0$. Note that the notion of strongly nonexpansive mapping with respect to the norm was first introduced and studied in [3].

One parameter family $\Gamma := \{T(t) : t \geqslant 0\}$ of mappings from C into C is said to be nonexpansive semi-group, if the following conditions are satisfied:

(a) $T(0)x = x$, for all $x \in C$;

(b) $T(s+t) = T(s)T(t)$, $\forall s,t \geqslant 0$;

(c) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(d) for any $x,y \in C$, $\|T(t)x - T(t)y\| \leqslant \|x-y\|$, $\forall t \geqslant 0$.

We use $\mathscr{F}$ to denote the common fixed point set of the nonexpansive semi-group Γ, i.e., $\mathscr{F} = \bigcap_{t\geqslant 0} F(T(t))$.

DEFINITION 1. Let E be a real Banach space, C be a nonempty closed convex subset of E. $\Gamma := \{T(t) : t \geqslant 0\}$ be one parameter family of mappings from C into C. Γ is said to be relatively nonexpansive semi-group, if $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t)) \neq \emptyset$ and the following conditions are satisfied:

(i) $T(0)x = x$, for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$, $\forall s,t \geqslant 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\varphi(P,T(t)x) \leqslant \varphi(p,x)$ for all $p \in \mathscr{F}$, $t \geqslant 0$ and $x \in C$;

(v) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_n\}$ in C converges weakly to p and $\{x_n - T(t)x_n\}$ converges strongly to 0, it follows that $p \in \mathscr{F}$.

DEFINITION 2. Let E be a real Banach space, C be a nonempty closed convex subset of E. $\Gamma := \{T(t) : t \geqslant 0\}$ be one parameter family of mappings from C into C. Γ is said to be strongly relatively nonexpansive semi-group, if $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t)) \neq \emptyset$, the conditions of definition 1 $(i)-(v)$ and the following conditions are satisfied:

(vi) if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p,x_n) - \varphi(p,T(t)x_n)$ $\to 0$ for some $p \in \mathscr{F}$ it follows that $\varphi(T(t)x_n,x_n) \to 0$.

DEFINITION 3. A strongly relatively nonexpansive semi-group Γ is said to be uniformly Lipschitzian, if there exists a bounded measurable function $L : [0,\infty) \to [0,\infty)$ such that

$$\|T(t)x - T(t)y\| \leqslant L(t)\|x-y\|, \ \forall x,y \in C, \ t \geqslant 0. \tag{1.6}$$

In the sequel, we denote it by

$$L = sup_{t\geqslant 0} L(t) < \infty.$$

A family of mappings is the class of firmly nonexpansive mappings, where a mapping $T : C \to E$ is called firmly nonexpansive type [14] if

$$\varphi(Tx,Ty)+\varphi(Ty,Tx)+\varphi(Tx,x)+\varphi(Ty,y) \leqslant \varphi(Tx,y)+\varphi(Ty,x), \text{ for all } x,y \in C.$$

It is easily known that if T is firmly nonexpansive type with $I-T$ is demi-closed at zero, then it is strongly relatively nonexpansive. Furthermore, there is a mapping which is strongly relatively nonexpansive but is not firmly nonexpansive mapping [21].

DEFINITION 4. Let E be a real Banach space, C be a nonempty closed convex subset of E. $\Gamma := \{T(t) : t \geqslant 0\}$ be one parameter family of mappings from C into C. Γ is said to be firmly nonexpansive semi-group, if $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t)) \neq \emptyset$, the conditions of definition 1 $(i)-(v)$ and the following conditions are satisfied:

(vii) $\varphi(T(t)x,T(t)y)+\varphi(T(t)y,T(t)x)+\varphi(T(t)x,x)+\varphi(T(t)y,y) \leqslant \varphi(T(t)x,y)+\varphi(T(t)y,x)$, for all $x,y \in C$, $t \geqslant 0$.

Iterative approximation of fixed points for nonexpansive mapping, asymptotically nonexpansive mappings, relatively nonexpansive mappings, nonexpansive semi-group, asymptotically nonexpansive semi-group in Hilbert or Banach space have been studied extensively by many authors [6, 7, 8, 17, 18, 21, 22, 23, 25, 26].

Recently, Nilsrakoo et al. [22] introduced the following Halpern-Mann iterations for finding a fixed point of a relative nonexpansive mapping in Banach space:

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)(\beta_n Jx_n + (1-\beta_n)JTx_n)), \tag{1.5}$$

under suitable conditions, they obtained the strongly convergence theorems.

Then, for finding the set of fixed points strongly relatively nonexpansive mapping, Nilsrakoo [21] introduced the following iterative scheme:

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JTx_n), \tag{1.6}$$

under some control conditions, they proved that the sequence $\{x_n\}$ converges strongly to $q = \Pi_F u$.

Inspired and motivated by the recent work of Nilsrakoo et al. [21, 22], Chang et al. [10], etc., we propose an iteration scheme and prove some strong convergence theorems of strongly relatively nonexpansive semi-group in Banach spaces. Using the concept of duality theorems, we obtain analogue results for strongly generalized nonexpansive semi-group and for semi-group of firmly generalized nonexpansive type. The results presented in this paper improve and extend some recent corresponding results of Nilsrakoo [21], Su et al. [25], Matsushita et al. [17, 18], Suzuki [26], Chang [7], Nilsrakoo et al. [22], Kohsaka et al. [15] Saejung [27] and others.

2. Preliminaries

Throughout this paper, Let E be a real Banach space with the dual E^* and C be a nonempty closed convex subset of E. We denote the strong convergence, weak convergence of a sequence $\{x_n\}$ to a point $x \in E$ by $x_n \to x$, $x_n \rightharpoonup x$, respectively, and $F(T)$ is the fixed point set of a mapping T.

LEMMA 1. ([9]) *Let E be an uniformly convex and smooth Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n,y_n)\to 0$, then $x_n-y_n\to 0$.*

REMARK 2. For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in an uniformly convex and uniformly smooth Banach space E, we have

$$\varphi(x_n,y_n)\to 0 \Leftrightarrow x_n-y_n\to 0 \Leftrightarrow Jx_n-Jy_n\to 0.$$

LEMMA 2. ([1]) *Let E be a smooth, strictly convex and reflective Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*
(a) $\varphi(x,\Pi_C y)+\varphi(\Pi_C y,y)\leqslant\varphi(x,y)$, $\forall\, x\in C,\ y\in E$;
(b) If $x\in E$ and $z\in C$, then $z=\Pi_C x$ iff $\langle z-y,Jx-Jz\rangle\geqslant 0$, $\forall y\in C$;
(c) For $x,y\in E,\varphi(x,y)=0$ if and only if $x=y$;

REMARK 3. The generalized projection mapping Π_C above is relatively quasi-nonpansive and $F(\Pi_C)=C$.

Let E be a relatively, strictly convex and smooth Banach space. The duality mapping J^* from E^* onto $E^{**}=E$ coincides with the inverse of the duality mapping J from E onto E^*, that is, $J^*=J^{-1}$. We make use of the following mapping $V:E\times E^*\to R$ studied by Alber [1]

$$V(x,x^*)=\|x\|^2-2\langle x,x^*\rangle+\|x^*\|^2,\ \forall x\in E,\ \forall x^*\in E^*.$$

Obviously, $V(x,x^*)=\varphi(x,J^{-1}(x^*))$ for all $x\in E$ and $x^*\in E^*$.

LEMMA 3. ([1]) *Let E be a smooth and strictly convex and reflective Banach space, Then*

$$V(x,x^*)+2\langle J^{-1}(x^*)-x,y^*\rangle\leqslant V(x,x^*+y^*),\ for\ all\ x\in E\ and\ x^*,y^*\in E^*.$$

LEMMA 4. ([30]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}=(1-\alpha_n)a_n+\alpha_n\delta_n$$

for all $n\in N$, where the sequences $\{\alpha_n\}$ in (0,1) and $\{\delta_n\}$ in R satisfy the following conditions:
(1) $\lim_{n\to\infty}\alpha_n=0,\sum_{n=1}^{\infty}\delta_n=\infty$;
(2) $\limsup_{n\to\infty}\delta_n\leqslant 0$;
Then $\lim_{n\to\infty}a_n=0$.

LEMMA 5. ([19]) *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i}<a_{n_i+1}$ for all $i\in N$. Then there exists a sequence $\{m_k\}\subset N$ such that $m_k\to\infty$ and the following properties are satisfied for all (sufficiently large) number $k\in N$:*

$$a_{m_k}\leqslant a_{m_k+1}\ and\ a_k\leqslant a_{m_k+1}.$$

In fact, $m_k=max\{j\leqslant k:a_j\leqslant a_{j+1}\}$.

LEMMA 6. ([20]) *Let C be a nonempty, closed and convex subset of a strictly convex and smooth Banach space E and let $T : C \to E$ be a relatively quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

LEMMA 7. *Let C be a nonempty, closed and convex subset of an uniformly convex and smooth Banach space E and let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be a relatively nonexpansive semi-group, $x \in C$ and $\widehat{x} = \Pi_{F(T(t))}x$. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences such that $\varphi(T(t)x_n, x_n) \to 0$ and $\varphi(T(t)x_n, y_n) \to 0$. Then $\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - J\widehat{x}\rangle \leqslant 0$.*

Proof. From the uniformly convexity of E and Lemma 1,

$$T(t)x_n - x_n \to 0 \textit{ and } y_n - x_n \to 0.$$

By the definition of relatively nonexpansive semigroup Γ, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y \in F(T(t))$ and

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - \widehat{x}\rangle = \limsup_{n\to\infty}\langle x_n - \widehat{x}, Jx - \widehat{x}\rangle = \lim_{i\to\infty}\langle x_{n_i} - \widehat{x}, Jx - J\widehat{x}\rangle.$$

It follows from Lemma 2 that

$$\limsup_{n\to\infty}\langle y_n - \widehat{x}, Jx - \widehat{x}\rangle = \langle y - \widehat{x}, Jx - \widehat{x}\rangle \leqslant 0. \quad \square$$

LEMMA 8. ([15]) *Let C be a nonempty, closed and convex subset of an uniformly convex and uniformly smooth Banach space E and let $T : C \to E$ be a relatively nonexpansive mapping. Let U be the mapping defined by*

$$U = J^{-1}(\lambda J + (1-\lambda)JT),$$

where $\lambda \in (0,1)$, then $U : C \to E$ is strongly relatively nonexpansive and $F(U) = F(T)$.

3. Strong relatively nonexpansive semigroup

THEOREM 1. *Let C be a nonempty, closed and convex subset of an uniformly smooth and uniformly convex Banach space E. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be an uniformly Lipschitizan and strong relatively nonexpansive semi-group with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. For given $u \in C$ and arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows*

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JT(t_n)x_n), \tag{3.1}$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty}\alpha_n = 0$;

(2) $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to a $q = \Pi_{\mathscr{F}}u$, where $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t))$.

Proof. We first show that $\{x_n\}$ is bounded. Let $y_n = J^{-1}(\alpha_n Ju + (1-\alpha_n)JT(t_n)x_n)$, then $x_{n+1} = \Pi_C y_n$. Since $F((T(t))$, $t \geqslant 0$, is a closed and convex subset of C, therefore, $\mathscr{F}$ is closed and convex in C.

Since Π_C is relatively quasi-nonexpansive, $\varphi(q, x_{n+1}) = \varphi(q, \Pi_C y_n) \leqslant \varphi(q, y_n)$. Thus we have

$$\begin{aligned}\varphi(q,x_{n+1}) &\leqslant \varphi(q,y_n)\\ &= \varphi(q, J^{-1}(\alpha_n Ju + (1-\alpha_n)JT(t_n)x_n))\\ &\leqslant \alpha_n\varphi(q,u) + (1-\alpha_n)\varphi(q,T(t_n)x_n)\\ &\leqslant \max\{\varphi(q,u), \varphi(q,T(t_n)x_n)\}\\ &\leqslant \max\{\varphi(q,u), \varphi(q,x_1)\}.\end{aligned} \tag{3.2}$$

This implies that $\{x_n\}$ is bounded, and so is the sequence $\{T(t_n)x_n\}$.

Since,

$$\begin{aligned}\varphi(T(t_n)x_n, y_n) &\leqslant \alpha_n\varphi(T(t_n)x_n, u) + (1-\alpha_n)\varphi(T(t_n)x_n, T(t_n)x_n)\\ &= \alpha_n\varphi(T(t_n)x_n, u) \to 0,\ (as\ n \to \infty).\end{aligned} \tag{3.3}$$

From Remark 3, Lemma 3 and (1.4), we have

$$\begin{aligned}\varphi(q,x_{n+1}) &\leqslant \varphi(q,y_n) = V(q, Jy_n)\\ &\leqslant V(q, Jy_n - \alpha_n(Ju - Jq)) - 2\langle y_n - q, -\alpha_n(Ju - Jq)\rangle\\ &= V(q, \alpha_n Jq + (1-\alpha_n)JT(t_n)x_n) + 2\alpha_n\langle y_n - q, Ju - Jq\rangle\\ &= \varphi(q, J^{-1}(\alpha_n Jq + (1-\alpha_n)JT(t_n)x_n)) + 2\alpha_n\langle y_n - q, Ju - Jq\rangle\\ &\leqslant \alpha_n\varphi(q,q) + (1-\alpha_n)\varphi(q,T(t_n)x_n) + 2\alpha_n\langle y_n - q, Ju - Jq\rangle\\ &\leqslant (1-\alpha_n)\varphi(q,x_n) + 2\alpha_n\langle y_n - q, Ju - Jq\rangle.\end{aligned} \tag{3.4}$$

The rest of proof will be divided into two parts:

CASE 1. Suppose that there exists $n_0 \in N$ such that $\{\varphi(q,x_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(q,x_n)\}$ is then convergent. Then

$$\varphi(q,x_n) - \varphi(q,x_{n+1}) \to 0,\ (n \to \infty). \tag{3.5}$$

Notice that

$$\varphi(q,x_{n+1}) \leqslant \alpha_n\varphi(q,u) + (1-\alpha_n)\varphi(q,T(t_n)x_n). \tag{3.6}$$

As $n \to \infty$, it follows from (3.5) and (3.6) that

$$\begin{aligned}\varphi(q,x_n) - \varphi(q,T(t_n)x_n) &= \varphi(q,x_n) - \varphi(q,x_{n+1}) + \varphi(q,x_{n+1}) - \varphi(q,T(t_n)x_n)\\ &\leqslant \varphi(q,x_n) - \varphi(q,x_{n+1}) + \alpha_n[\varphi(q,u) - \varphi(q,T(t_n)x_n)] \to 0.\end{aligned}$$

Since T is strongly relatively nonexpansive, we have

$$\varphi(T(t_n)x_n, x_n) \to 0. \tag{3.7}$$

By Lemmas 1 and (3.7), we have

$$\lim_{n\to\infty}\|T(t_n)x_n - x_n\| = 0. \tag{3.8}$$

Since $\Gamma := \{T(t) : t \geqslant 0\}$ is uniformly Lipschitian, for any $k \in N$, by the definition 3, we have

$$\begin{aligned}\|T((k+1)t_n)x_n - T(kt_n)x_n\| &= \|T(kt_n)T(t_n)x_n - T(kt_n)x_n\| \\ &\leqslant L(kt_n)\|T(t_n)x_n - x_n\| \\ &\leqslant L\|T(t_n)x_n - x_n\|.\end{aligned} \tag{3.9}$$

From (3.8), for any $k \in N$, we have

$$\lim_{n\to\infty}\|T((k+1)t_n)x_n - T(kt_n)x_n\| = 0. \tag{3.10}$$

Since

$$\begin{aligned}\left\|T(t)x_n - T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n\right\| &= \left\|T\left(\left[\frac{t}{t_n}\right]t_n\right)T\left(t - \left[\frac{t}{t_n}\right]t_n\right)x_n - T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n\right\| \\ &\leqslant L\left\|T\left(t - \left[\frac{t}{t_n}\right]t_n\right)x_n - x_n\right\|,\end{aligned} \tag{3.11}$$

and $T(\cdot)$ is continuous, we have

$$\lim_{n\to\infty}\left\|T(t)x_n - T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n\right\| = 0. \tag{3.12}$$

In addition

$$\left\|x_n - T(t)x_n\right\| \leqslant \sum_{k=0}^{[\frac{t}{t_n}]-1}\left\|T((k+1)t_n)x_n - T(kt_n)x_n\right\| + \left\|T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n - T(t)x_n\right\|, \tag{3.13}$$

it follows from (3.9) and (3.12) that

$$\lim_{n\to\infty}\|T(t)x_n - x_n\| = 0. \tag{3.14}$$

Since

$$\|T(t)x_n - y_n\| = \|T(t)x_n - x_n\| + \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - y_n\|. \tag{3.15}$$

It follows from (3.3), (3.8) and (3.14), we have

$$\lim_{n\to\infty}\|T(t)x_n - y_n\| = 0. \tag{3.16}$$

It follows from Lemma 7, (3.3), (3.7), (3.14) and (3.16) that

$$\limsup_{n\to\infty}\langle y_n - q, Ju - Jq\rangle \leqslant 0. \tag{3.17}$$

Hence, By Lemmas 1 and 4, and (3.4) we have

$$\lim_{n\to\infty}\varphi(q, x_n) = 0, \tag{3.18}$$

This implies that $x_n \to q$, as $n \to \infty$.

CASE 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(q,x_{n_i}) < \varphi(q,x_{n_i+1}), \text{ for all } i \in N. \tag{3.19}$$

By Lemma 5, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$.

$$\varphi(q,x_{m_k}) < \varphi(q,x_{m_k+1}) \quad \text{and} \quad \varphi(q,x_k) < \varphi(q,x_{m_k+1}), \text{ for all } k \in N. \tag{3.20}$$

$$\begin{aligned}\varphi(q,x_{m_k}) - \varphi(q,T(t)x_{m_k}) &= \varphi(q,x_{m_k}) - \varphi(q,x_{m_k+1}) + \varphi(q,x_{m_k+1}) - \varphi(q,T(t)x_{m_k}) \\ &\leqslant \alpha_{m_k}(\varphi(q,u) - \varphi(q,T(t)x_{m_k})) \to 0.\end{aligned}$$

This implies that

$$\varphi(T(t)x_{m_k},x_{m_k}) \to 0, \ (as\ n \to \infty). \tag{3.21}$$

It follows from Lemma 7 and (3.3) that

$$\limsup_{k\to\infty} \langle y_{m_k} - q, Ju - Jq\rangle \leqslant 0. \tag{3.22}$$

By (3.4), we have

$$\varphi(q,x_{m_k+1}) \leqslant (1-\alpha_{m_k})\varphi(q,x_{m_k}) + 2\alpha_{m_k}\langle y_{m_k} - q, Ju - Jq\rangle. \tag{3.23}$$

Since $\varphi(q,x_{m_k}) \leqslant \varphi(q,x_{m_k+1})$, we have

$$\begin{aligned}\alpha_{m_k}\varphi(q,x_{m_k}) &\leqslant \varphi(q,x_{m_k}) - \varphi(q,x_{m_k+1}) + 2\alpha_{m_k}\langle y_{m_k} - q, Ju - Jq\rangle \\ &\leqslant 2\alpha_{m_k}\langle y_{m_k} - q, Ju - Jq\rangle.\end{aligned}$$

In particular, since $\alpha_{m_k} > 0$, we get

$$\varphi(q,x_{m_k}) \leqslant 2\langle y_{m_k} - q, Ju - Jq\rangle. \tag{3.24}$$

It follows from (3.6) that $\varphi(q,x_{m_k}) \to 0$. This together with (3.7) gives

$$\varphi(q,x_{m_k+1}) \to 0, \ (as\ n \to \infty).$$

But $\varphi(q,x_k) \leqslant \varphi(q,x_{m_k+1})$ for all $k \in N$, we conclude that $x_k \to q$. This implies that $x_n \to q$, $as\ n \to \infty$, and the proof is completed. □

In a Hilbert space, Theorem 1 is reduced to the following results.

COROLLARY 1. Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be a Lipschitizan and strong relatively nonexpansive semi-group with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. For given $u \in C$ and arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n,$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $q = P_{\mathscr{F}}u$, where $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t))$.

Proof. Since in Hilbert space, J is identity mapping, taking $\varphi(x,y) = \|x-y\|^2$ in Theorem 1, therefore all conditions in theorem 1 are satisfied. By the similar methods as given in the proof of Theorem 1, the conclusion of corollary 1 can be obtained from Theorem 1 immediately. $\square$

Applying Theorem 1 and Lemma 8, we have the following result.

COROLLARY 2. Let C be a nonempty, closed and convex subset of an uniformly smooth and uniformly convex Banach space E. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be an uniformly Lipschitizan and relatively nonexpansive semi-group with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. For given $u \in C$ and arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)(\lambda Jx_n + (1-\lambda)JT(t_n)x_n)),$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\lambda \in (0,1)$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $q = \Pi_{\mathscr{F}} u$, where $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t))$.

Firmly nonexpansive type mappings in an uniformly convex Banach space E whose norm is uniformly *Gâteaux* differentiable are strongly relative nonexpansive [14], we can obtain the following result from Theorem 1.

COROLLARY 3. Let C be a nonempty, closed and convex subset of an uniformly smooth and uniformly convex Banach space E whose norm is uniformly *Gâteaux* differentiable. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be an uniformly Lipschitzian and firmly nonexpansive semig-roup with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. For given $u \in C$ and arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows

$$x_{n+1} = \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)JT(t_n)x_n),$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $q = \Pi_{\mathscr{F}} u$, where $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t))$.

REMARK 4. In Theorem 1, we extend the mappings from strong relatively nonexpansive mappings and relatively nonexpansive mappings to strong relatively nonexpansive semigroup and relatively nonexpansive semigroup. So, our results presented in the article improve and extend the corresponding results of Nilsrakoo [21], Su et al. [25], Matsushita et al. [17, 18], Suzuki [26], Chang [7], Nilsrakoo et al. [22], Kohsaka et al. [15], Saejung [27] and others.

4. Strong generalized nonexpansive semigroup

Let C be a subset of a smooth Banach space E. In 2007, Ibaraki and Takahashi [12] introduced the following mapping. A mapping $T: C \to E$ is generalized nonexpansive if the following properties are satisfied:

(G1) $F(T) \neq \emptyset$;

(G2) $(Tx, p) \leqslant \varphi(x, p)$ for all $x \in C$ and $p \in F(T)$.

A generalized nonexpansive mapping $T: C \to E$ is strongly generalized nonexpansive [22] if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(x_n, p) - \varphi(Tx_n, p) \to 0$ for some $p \in F(T)$ it follows that $\varphi(x_n, Tx_n) \to 0$. A mapping $T: C \to E$ satisfies property $(G3)$ if whenever $\{x_n\}$ is a sequence in C such that $Jx_n \rightharpoonup^* Jp$ and $Jx_n - JTx_n \to 0$ it follows that $p \in F(T)$. Here $\rightharpoonup^*$ denotes the weak* convergence in the dual space. A mapping $R: E \to C$ is said to be a sunny generalized nonexpansive retraction if the following properties are satisfied:

(1) R is generalized nonexpansive;

(2) $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \geqslant 0$;

(3) $Rx = x$ for all $x \in C$.

A nonempty subset C of E is said to be a sunny generalized nonexpansive retract (resp. generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. generalized nonexpansive retraction) of E onto C (see [12] for more details). We know the following result.

LEMMA 9. ([16]) *Let C be a nonempty and closed subset of a reflective, strictly convex and smooth Banach space E. Then the following are equivalent:*

(i) C is a sunny generalized nonexpansive retract of E;

(ii) C is a generalized nonexpansive retract of E;

(iii) JC is closed and convex.

In this case, the sunny generalized nonexpansive retraction from E onto C is given by $J^{-1}\Pi_{JC}J$, where Π_{JC} is the generalized projection from E^* onto JC.

Let C be a nonempty subset of a smooth, strictly convex and reflective Banach space E and let $T: C \to E$ be a mapping. We define the duality $T^*: JC \to E$ of T by (see [11])

$$T^*x^* = JTJ^{-1}x^* \ for\ all\ x^* \in JC.$$

DEFINITION 5. Let E be a real Banach space, C be a nonempty closed convex subset of E. $\Gamma := \{T(t) : t \geqslant 0\}$ be one parameter family of mappings from C into C. Γ is said to be strongly generalized nonexpansive semi-group, if $\mathscr{F} := \bigcap_{t \geqslant 0} F(T(t)) \neq \emptyset$ and the following conditions are satisfied:

(i) $T(0)x = x$, for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$, $\forall s, t \geqslant 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\varphi(p, T(t)x) \leqslant \varphi(p, x)$ for all $p \in \mathscr{F}$ and $x \in C$;

(v) if whenever $\{x_n\}$ is a bounded sequence in C such that $\varphi(p, x_n) - \varphi(p, T(t)x_n) \to 0$ for some $p \in \mathscr{F}$, it follows that $\varphi(T(t)x_n, x_n) \to 0$.

A mapping $T: C \to E$ satisfies property (v) if whenever $\{x_n\}$ is a sequence in C such that $Jx_n \rightharpoonup^* Jp$ and $Jx_n - JT(t)x_n \to 0$ it follows that $p \in F(T(t))$.

LEMMA 10. *Let C be a nonempty subset of a reflective, strictly convex and smooth Banach space E. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be strongly generalized nonexpansive semigroup with property (v) and let $T(t)^* : JC \to E^*$ be the duality of $T(t)$. Then $T(t)^*$ is strongly relatively nonexpansive and $F(T(t)^*) = JF(T(t))$.*

Proof. In fact, we have that

$$\begin{aligned} x^* \in JF(T(t)) &\Leftrightarrow J^{-1}x^* \in F(T(t)) \\ &\Leftrightarrow TJ^{-1}x^* = J^{-1}x^* \\ &\Leftrightarrow JTJ^{-1}x^* = JJ^{-1}x^* \\ &\Leftrightarrow T^*x^* = x^* \\ &\Leftrightarrow x^* \in F(T(t)^*). \end{aligned}$$

This implies that $JF(T(t)) = F(T(t)^*)$. $\square$

THEOREM 2. *Let C be a nonempty, closed and sunny generalized nonexpansive retract of an uniformly smooth Banach space E whose dual space has a Fréchet differentiable norm. $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be a Lipschitizan and strong generalized nonexpansive semigroup with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. Let $\{x_n\}$ be a sequence in C defined by $u \in C, x_1 \in C$ and*

$$x_{n+1} = R_C(\alpha_n u + (1-\alpha_n)T(t_n)x_n),$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $R_{F(T(t)))}u$, where $R_{F(T(t))}$ is the unique sunny generalized nonexpansive retraction from E onto $F(T(t))$.

Proof. Suppose that $T(t)^* : JC \to E^*$ is the duality of $T(t)$. From Lemma 10, $T(t)^*$ is strongly generalized nonexpansive semigroup and $F(T^*) = JF((t))$. Let $x_n^* \equiv Jx_n$ and $u^* = Ju$. Using (3.22) and $R_C = J^{-1}\Pi_{JC}J$, we obtain

$$\begin{aligned} x_{n+1} &= \Pi_{JC}J(\alpha_n J^{-1}u^* + (1-\alpha_n)J^{-1}T(t_n)^* x_n^*). \\ &= \Pi_{JC}J^{*-1}(\alpha_n J^* u^* + (1-\alpha_n)J^* T(t_n)^* x_n^*) \end{aligned}$$

for all $n \in N$, $t \geqslant 0$. Applying Theorem 1 gives $x_n^* \to \Pi_{F(T^*)}u^*$. Since J^{-1} is norm-to-norm continuous,

$$x_n = J^{-1}x_n^* \to J^{-1}\Pi_{F(T^*)}u^* = J^{-1}\Pi_{JF(T)}(Ju) = R_{F(T(t))}u. \quad \square$$

DEFINITION 6. Let C be a nonempty subset of a smooth Banach space E. $\Gamma := \{T(t) : t \geqslant 0\}$ be one parameter family of mappings from C into C. Γ is said to be firmly generalized nonexpansive semi-group, if $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t)) \neq \emptyset$ and the following conditions are satisfied:

(i) $T(0)x = x$, for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$, $\forall s,t \geqslant 0$;

(iii) for each $x \in C$, the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\varphi(p,T(t)x) \leqslant \varphi(p,x)$ for all $p \in \mathscr{F}$ and $x \in C$;

(v) $\varphi(T(t)x,T(t)y) + \varphi(T(t)y,T(t)x) + \varphi(x,T(t)x) + \varphi(y,T(t)y) \leqslant \varphi(y,T(t)x) + \varphi(x,T(t)y),\ for\ all\ x,y \in C.$

It is not hard to show that the duality of a firmly generalized nonexpansive type semigroup is firmly nonexpansive type. As a consequence of corollary 3, we have the following result.

COROLLARY 4. Let C be a nonempty, closed and convex and sunny generalized nonexpansive retract of an uniformly smooth and uniformly convex Banach space E. Let $\Gamma := \{T(t) : t \geqslant 0\} : C \to C$ be a Lipschitizan and firmly generalized nonexpansive semigroup with a bounded measurable function $L : [0,\infty) \to [0,\infty)$. For given $u \in C$ and arbitrarily chosen $x_1 \in C$, $\{x_n\}$ is defined as follows

$$x_{n+1} = R_C(\alpha_n u + (1-\alpha_n)T(t_n)x_n),$$

where $\{t_n\}$ is a sequence of real numbers satisfying $t_n > 0$, $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying:

(1) $\lim_{n\to\infty} \alpha_n = 0$;

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}$ converges strongly to $q = \Pi_{\mathscr{F}} u$, where $\mathscr{F} := \bigcap_{t\geqslant 0} F(T(t))$.

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