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STRONG CONVERGENCE THEOREMS FOR BREGMAN QUASI–ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND EQUILIBRIUM PROBLEM IN REFLEXIVE BANACH SPACES

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Abstract. The purpose of this article is to propose an iteration algorithm for Bergman quasi-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of reflexive Banach spaces. As applications, we apply our results to a system of equilibrium problems. The results presented in the paper improve and extend the corresponding results of Reich and Sabach [Nonlinear Anal. 73 (2010), 122–135], Suantai et al. [Comput. Math. Appl. 64 (2012), 489–499], Nilsrakoo and Saejung [Appl. Math. Comput. 217:14 (2011), 6577–6586], Qin et al. [Applied Math Letters, 22 (2009), 1051–1055], Wang et al. [J. Comput. Appl. Math., 235 (2011), 2364–2371], Su et al. [Nonlinear Anal. 73 (2010), 390–3906], Nartinez-Yanes et al. [Nonlinear Anal., 64 (2006), 2400–2411] and others.

1. Introduction

Let E be a real reflexive Banach space with the dual E^*, C be a nonempty closed convex subset of E. In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping $T : C \to C$, $\mathscr{R}$ to denote the set of all real numbers.

Recall that a mapping $T : C \to C$ is *nonexpansive*, if $||Tx-Ty|| \leqslant ||x-y||$, $\forall x,y \in C$. T is said to be *quasi-nonexpansive*, if $F(T) \neq \emptyset$ and $||Tx-p|| \leqslant ||x-p||, \forall x \in C$, $p \in F(T)$.

It turns out that the fixed point theory of nonexpansive mappings can be applied to solutions of diverse problems such as finding zeroes of monotone mappings and solutions to certain evolution equations and solving convex feasibility, variational inequality and equilibrium problems. There are, in fact, many papers that deal with methods for finding fixed points of nonexpasive and quasi-nonexpansive mappings in Hilbert, uniformly convex and uniformly smooth Banach spaces (for example [2, 7–14, 18, 20].

When we try to extend this theory to general Banach spaces we encounter some difficulties, because many of the useful examples of nonexpansive mappings in Hilbert

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space are no longer nonexpansive in Banach spaces, for example, the resolvent $R_A := (I+A)^{-1}$ of a *maximal monotone mapping* $A : H \to 2^H$ and the *metric projection* P_C. There are several ways to overcome these difficulties. One of them is to use the Bergman distance (see Section 2) instead of the norm, Bergman (quasi-)nonexpansive mappings instead of the (quasi-)nonexpansive mappings (see Section 2) and the *Bergman projection* instead of the metric projection (see section 2).

In 2010, Reich and Sabach [15] introduced the concept of *Bergman strongly nonexpansive mapping* and study the convergence of two iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in reflexive Banach spaces. In 2012, Suantai et al. [19] also consider the strong convergence for Bergman strongly nonexpansive mappings in reflexive Banach spaces.

The purpose of this paper is to introduce the concept of *Bergman quasi-asymptotically nonexpansive mappings* which contains *Bergman strongly nonexpansive mapping* as its special case and by using *hybrid Bergman projection* to propose an iteration algorithm for Bergman quasi-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of reflexive Banach spaces. As applications, we apply our results to a system of equilibrium problems in reflexive Banach spaces. The results presented in the paper improve and extend the corresponding results of Reich and Sabach [15], Suantai et al. [19], Chang et al. [7, 8], Nilsrakoo and Saejung [12], Qin et al. [13, 14], Wang et al. [20], Su et al [18], Kang et al. [10] Martinez-Yanes and Xu [11] and others.

2. Preliminaries

In this section, we begin with by recalling some preliminaries and lemmas which will be used to prove our main results.

Throughout this paper E is a real reflexive Banach space, E^* is the dual space of E, $f : E \to (-\infty, +\infty]$ is a *proper, convex and lower-semi-continuous function* and $f^* : E^* \to (-\infty, +\infty]$ is the *Fenchel conjugate of* f defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*$$

We denote by $domf$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$.

For any $x \in intdomf$ and $y \in E$, we denote by $f^o(x,y)$ the *right-hand derivative* of f at x in the direction y, that is,

$$f^o(x,y) := \lim_{t \to +0} \frac{f(x+ty) - f(x)}{t}.$$

The function f is called *Gâteaux differentiable at* x, if $\lim_{t\to 0+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case $f^o(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f at x. The function f is said to be *Fréchet differentiable at* x, if this limit is attained uniformly in $||y|| = 1$. Finally, f is said to be *uniformly Fréchet differentiable on a subset* C *of* E, if the above limit is attained uniformly for $x \in C$ and $||y|| = 1$.

PROPOSITION 2.1. [15] *If* $f : E \to \mathscr{R}$ *is uniformly Fréchet differentiable and bounded on bounded subsets of* E, *then* f *is uniformly continuous on bounded subsets of* E *and* ∇f *is uniformly continuous on bounded subsets of* E *from the strong topology of* E *to the strong topology of* E^*.

DEFINITION 2.2. A function $f : E \to (-\infty, +\infty]$ is said to be "Legendre", if the following conditions are satisfied:

(L1) The interior of the domain of f, int dom$f$ is nonempty, f is Gâteaux differentiable on int dom$f$ and dom ∇f = int dom$f$;

(L2) The interior of the domain of f^*, int domf^* is nonempty, f^* is Gâteaux differentiable on int domf^* and dom ∇f^* = int domf^*.

REMARK 2.3. If E is a reflexive Banach space, and f is a Legendre function, then we have

(a) f is Legendre if and only if f^* is Legendre;

(b) $(\partial f)^{-1} = \partial f^*$;

(c) $\nabla f = (\nabla f^*)^{-1}$; ran ∇f = dom $\nabla f^* = intdomf^*$ and ran ∇f^* = dom $\nabla f = intdomf$;

(d) The function f and f^* are strictly convex on the interior of their respective domains.

Examples of Legendre functions are given in [1]. One important and interesting Legendre function is $\frac{1}{p}||x||^p$, $p \in (1, +\infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E, i.e., $\nabla f = J_p$, $p \in (1, \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert space.

From now on we always assume that the convex function $f : E \to (0, +\infty]$ is Legendre.

DEFINITION 2.4. [6] Let $f : E \to (-\infty, +\infty]$ be a convex and *Gâteaux* differentiable function. The function $D_f : donf \times intdomf \to [0, +\infty)$ defined by

$$D_f(y,x)) := f(y) - f(x) - \langle \nabla f(x),\ y - x\rangle$$

is called the *Bergman distance with respect to* f.

It should be noted that the Bergman distance is not a distance in the usual sense of the term. Clearly, $D_f(x,x) = 0$, but $D_f(x,y) = 0$ may not imply $x = y$. In our case when f is Legendre this indeed holds. In general, D_f is not symmetric and does not satisfy the triangle inequality. But it has the following important properties:

(1) (*The three point identity*): for any $x \in domf$ and $y,z \in intdomf$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y)\ x - y\rangle;$$

(2) (*The four point identity*): for any $y,w \in domf$ and $x,z \in intdomf$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x)\ y - w\rangle;$$

DEFINITION 2.5. [3, 4] Let $f: E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called *totally convex at a point* $x \in intdomf$, if its *modulus of total convexity at* x, $\nu_f : intdomf \times [0,+\infty) \to [0,+\infty]$, defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in domf,\ ||y-x|| = t\},$$

is positive whenever $t > 0$.

The function f is called *totally convex*, if it is totally convex at every point $x \in intdomf$. The function f is called *totally convex on bounded sets*, if $\nu_f(K,t)$ is positive for any nonempty bounded subset K of E and for any $t > 0$, where the modulus of total convexity of the function f on the set K is the function $\nu_f : intdomf \times (0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(K,t) := \inf\{\nu_f(x,t) : x \in K \cap intdomf\}.$$

Recall that the function $f : E \to \mathscr{R}$ is called *uniformly convex at* $x \in E$ [18], if $\mu_f(x,t) > 0$, $\forall t > 0$, where

$$\mu_f(x,t) = \inf\{f(x) + f(y) - 2f(\frac{x+y}{2}) : y \in domf,\ ||y-x|| = t\}.$$

The next proposition will be very useful in the proof our main results.

PROPOSITION 2.6. [15] *(1) If $x \in intdomf$, then the following statements are equivalent:*

(i) The function f is totally convex at x;

(ii) For any sequence $\{y_n\} \subset domf$,

$$\lim_{n\to+\infty} D_f(y_n,x) = 0 \Longrightarrow \lim_{n\to\infty} ||y_n - x|| = 0.$$

(2) If $f : E \to \mathscr{R}$ is a Gâteaux differentiable and totally convex function. If $x_0 \in E$, and the sequence $\{D_f(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.

Recall that the function f is called sequentially consistent [1], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $intdomf$ and $domf$, respectively, and $\{x_n\}$ is bounded, then

$$\lim_{n\to+\infty} D_f(y_n,x_n) = 0 \Longrightarrow \lim_{n\to+\infty} ||y_n - x_n|| = 0.$$

PROPOSITION 2.7. [4] *The function f is totally convex on bounded sets if and only if it is sequentially consistent.*

Recall that the Bregman projection (cf. [4]) of $x \in intdomf$ onto a nonempty, closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x),x) = \inf\{D_f(y,x) : y \in C\}.$$

Similarly to the metric projection in Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

PROPOSITION 2.8. [5] *Suppose that* f *is Gâteaux differentiable and totally convex on* $int dom f$. *Let* $x \in int dom f$ *and let* $C \subset int dom f$ *be a nonempty, closed and convex set. Then the following statements are equivalent:*

(i) $z \in C$ *is the Bregman projection of* x *onto* C *with respect to* f, *i.e.,* $z = P_C^f(x)$;

(ii) The vector z *is the unique solution of the following variational inequality*

$$\langle \nabla f(x) - \nabla f(z),\, z - y \rangle \geqslant 0,\ \forall y \in C;$$

(iii) z *is the unique solution of the following inequality*

$$D_f(y,z) + D_f(z,x) \leqslant D_f(y,x),\ \forall y \in C:$$

PROPOSITION 2.9. [16] *Let* $f : E \to \mathscr{R}$ *be a Gâteaux differentiable and totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_n, x_0)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded too.*

DEFINITION 2.10. Let C be a convex subset of E and let $T : C \to C$ be a mapping.

(1) A point $p \in C$ is said to be an *asymptotically fixed point of* T, if there exists a sequence $\{x_n\} \subset C$ such that x_n converges to p weakly, and $||x_n - Tx_n|| \to 0$.

In the sequel, we denote by $\hat{F}(T)$ the set of asymptotically fixed point of T.

(2) T is said to be *Bergman relatively nonexpansive*, if $F(T) \neq \emptyset$, $F(T) = \hat{F}(T)$ and $D_f(p, Tx) \leqslant D_f(p,x), \quad \forall x \in C, p \in F(T)$.

(3) $T : C \to C$ is said to be *Bergman strongly nonexpansive*, if $\hat{F}(T) \neq \emptyset$ and

$$D_f(p, Tx) \leqslant D_f(p,x), \quad \forall x \in C,\ p \in \hat{F}(T).$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \hat{F}(T)$ and

$$\lim_{n \to +\infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0,$$

it follows that

$$\lim_{n \to +\infty} (D_f(Tx_n, x_n) = 0.$$

(4) $T : C \to C$ is said to be *Bergman quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leqslant D_f(p,x), \quad \forall x \in C,\ p \in F(T).$$

(5) $T : C \to C$ is said to be *Bergman quasi-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ such that

$$D_f(p, T^n x) \leqslant k_n D_f(p,x),\ \forall n \geqslant 1,\ x \in C,\ p \in F(T). \tag{2.1}$$

(6) A mapping $T : C \to C$ is said to be *closed* if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

REMARK 2.11. From the definitions, it is obvious that if $F(T) = \hat{F}(T) \neq \emptyset$, then a Bergman strongly nonexpansive is a Bergman relatively nonexpansive mapping; A Bergman relatively nonexpansive mapping is a Bergman quasi-nonexpansive mapping. A Bergman quasi-nonexpansive mapping is a Bergman quasi-asymptotically nonexpansive mapping, but the converse is not true.

EXAMPLE 2.12. [15] Let E be a real reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone mapping and $f : E \to (-\infty, +\infty]$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E such that $A^{-1}0 \neq \emptyset$, then the resolvent

$$Res_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x) \tag{2.2}$$

is closed and *Bergman relatively nonexpansive* from E onto $D(A)$, so is a closed Bergman quasi-asymptotically -nonexpansive mapping;

EXAMPLE 2.13. [15, 17] Let E be a reflexive Banach space, C be a nonempty, closed and convex subset of E. If the Legendre function $f : E \to (-\infty + \infty]$ is uniformly Fréchet differentiable and bounded on bounded subset of E, then the Bergman projection P_C^f is a closed Bergman relatively nonexpansive mapping from E onto C, so is a closed Bergman quasi-asymptotically -nonexpansive mapping.

EXAMPLE 2.14. [15, 17] Let E be a reflexive Banach space, C be a nonempty closed and convex subset of E, $f : E \to (-\infty, +\infty]$ be a Legendre function and $g : C \times C \to \mathscr{R}$ be a bifunction satisfying the conditions: (A1) $g(x,x) = 0 \; \forall x \in C$; (A2) $g(x,y) + g(y,x) \leqslant 0 \; \forall x,y \in C$; (A3) for each $x,y,z \in C$, $\lim_{t \downarrow 0} g(tz + (1-t)x, y) \leqslant g(x,y)$; (A4) for each given $x \in C$, the function $y \mapsto g(x,y)$ is convex and lower semicontinuous.

The "so-called" *equilibrium problem for* g is to find a $x^* \in C$ such that $g(x^*, y) \geqslant 0$, $\forall y \in C$. The set of its solutions is denoted by $EP(g)$, and *the resolvent operator of a bifunction* $g : C \times C \to \mathscr{R}$, $Res_g^f : E \to 2^C$ is defined by

$$Res_g^f(x) = \{z \in C : g(z,y) + \langle \nabla f(z) - \nabla f(x),\, y - z \rangle \geqslant 0,\; \forall y \in C\}. \tag{2.3}$$

It is known that (1) Res_g^f is single-valued; (2) $F(Res_g^f) = EP(g)$; (3) Res_g^f is a closed Bergman quasi-nonexpansive mapping, so is a closed Bergman quasi-asymptotically nonexpansive mapping.

LEMMA 2.15. *Let E be a real reflexive Banach space and C be a nonempty, closed and convex subset of E and $f : X \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable and total convex on bounded subsets of E. Let $T : C \to C$ be a closed and Bergman quasi-asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$. If $F(T) \neq \emptyset$, then it is a closed and convex subset of C.*

Proof. Letting $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ (as $n \to \infty$), we prove that $p \in F(T)$. In fact, since $\{p_n\} \subset F(T)$ and $p_n \to p$, we have $Tp_n = p_n \to p$ (as $n \to \infty$). By the closeness of T, we have $Tp = p$. This implies that $F(T)$ is closed.

Next we prove that $F(T)$ is convex. For any $p,q \in F(T)$, $t \in (0,1)$, putting $w = tp+(1-t)q$, we prove that $w \in F(T)$. Indeed, in view of the definition of Bergman distance $D_f(x,y)$ we have

$$\begin{aligned} D_f(w,T^n w) &= f(w) - f(T^n w) - \langle \nabla f(T^n w),\ w - T^n w\rangle \\ &= f(w) - f(T^n w) - t\langle \nabla f(T^n w),\ p - T^n w\rangle - (1-t)\langle \nabla f(T^n w),\ q - T^n w\rangle \\ &= f(w) + tD_f(p,T^n w) + (1-t)D_f(q,T^n w) - tf(p) - (1-t)f(q). \end{aligned} \tag{2.4}$$

Since T is Bergman totally quasi-asymptotically nonexpansive, we have

$$\begin{aligned} tD_f(p,T^n w) + (1-t)D_f(q,T^n w) &\leqslant t\{D_f(p,w) + \nu_n\zeta(D_f(p,w)) + \mu_n\} \\ &\quad + (1-t)\{D_f(q,w) + \nu_n\zeta(D_f(q,w)) + \mu_n\} \\ &= t\{f(p) - f(w) - \langle \nabla f(w),\ p - w\rangle + \nu_n\zeta(D_f(p,w)) + \mu_n\} \\ &\quad + (1-t)\{f(q) - f(w) - \langle \nabla f(w),\ q - w\rangle + \nu_n\zeta(D_f(q,w)) + \mu_n\} \\ &= tf(p) + (1-t)f(q) - f(w) \\ &\quad + t\nu_n\zeta(D_f(p,w)) + (1-t)\nu_n\zeta(D_f(q,w)) + \mu_n. \end{aligned} \tag{2.5}$$

Substituting (2.5) into (2.4) and simplifying it we have

$$D_f(w,T^n w) \leqslant t\nu_n\zeta(D_f(p,w)) + (1-t)\nu_n\zeta(D_f(q,w)) + \mu_n \to 0\ (as\ n \to \infty).$$

Hence, $D_f(w,T^n w) \to 0$. By Proposition 2.6(ii), we have $T^n w \to w$. This implies that $TT^n w = T^{n+1} w \to w$. Since T is closed, we have $w = Tw$, i.e., $w \in F(T)$.

This completes the proof of Lemma 2.15. □

DEFINITION 2.16. [19] Let $f : E \to \mathscr{R}$ be a proper convex and lower semi-continuous Legendre function, then for any $z \in E$, for any $\{x_n\} \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$, the following holds

$$D_f(z, \nabla f^*(\sum_{i=1}^N t_i \nabla f(x_i)) \leqslant \sum_{i=1}^N t_i D_f(z,x_i). \tag{2.6}$$

DEFINITION 2.17. (1) A countable family of mappings $\{T_i\} : C \to C$ is said to be *uniformly Bergman quasi-asymptotically nonexpansive*, if $\mathscr{F} := \bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that

$$D_f(p, T_i^n x) \leqslant\ k_n D_f(p,x), \forall n \geqslant 1,\ i \geqslant 1\ \forall x \in C,\ p \in \mathscr{F}. \tag{2.7}$$

(2) A mapping $T : C \to C$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that $||T^n x - T^n y|| \leqslant L||x - y||$, $\forall x,y \in C$, $\forall n \geqslant 1$.

3. Main Results

THEOREM 3.1. *Let E be a real reflexive Banach space, C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, and $T_i : C \to C$, $i = 1, 2, \cdots$ be a countable family of closed and uniformly Bergman quasi-asymptotically nonexpansive mappings satisfying the condition (2.7) and for each $i \geqslant 1$, T_i is uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in [0, 1] such that* $\lim_{n\to\infty} \alpha_n = 0$. *Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \; C_1 = C, \\ y_{n,m} = \nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(T_m^n x_n)], \; m \geqslant 1, \\ C_{n+1} = \{z \in C_n : \sup\limits_{m \geqslant 1} D_f(z, y_{n,m}) \leqslant \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\} \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \; \forall n \geqslant 1, \end{cases} \tag{3.1}$$

where $\xi_n = (k_n - 1)\sup_{p \in \mathscr{F}} D_f(p, x_n)$. $P^f_{C_{n+1}}$ is the Bergman projection of E onto C_{n+1}. If $\mathscr{F} := \bigcap_{i=1}^{\infty} F(T_i)$ is bounded, then $\{x_n\}$ converges strongly to $P^f_{\mathscr{F}}(x_1)$.

Proof. **(I)** First we prove that $\mathscr{F}$ and C_n, $n \geqslant 1$ both are closed and convex subsets in C.

In fact, it follows from Lemma 2.15 that $F(T_i)$, $i \geqslant 1$ is a closed and convex subset of C. Therefore $\mathscr{F}$ is closed and convex in C.

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geqslant 1$. In view of the definition of Bergman distance with respect to f we have that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{m \geqslant 1} D_f(z, y_{n,m}) \leqslant \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\} \\ &= \bigcap_{m \geqslant 1} \{z \in C : D_f(z, y_{n,m}) \leqslant \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n) + \xi_n\} \bigcap C_n \\ &= \bigcap_{m \geqslant 1} \{z \in C : \alpha_n \langle \nabla f(x_1), \, z - x_1 \rangle + (1-\alpha_n) \langle \nabla f(x_n), \, z - x_n \rangle \\ &\quad - \langle \nabla f(y_{n,m}), \, z - y_{n,m} \rangle \leqslant -\alpha_n f(x_1) - (1-\alpha_n) f(x_n) + f(y_{n,m})\} \bigcap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex. The conclusions are proved.

(II) Now we prove that $\mathscr{F} \subset C_n$, $\forall n \geqslant 1$.

In fact, it is obvious that $\mathscr{F} \subset C_1 = C$. Suppose that $\mathscr{F} \subset C_n$ for some $n \geqslant 1$. Since $T_i : C \to C$, $i = 1, 2, \cdots$ is a countable family of uniformly Bergman quasi-asymptotically nonexpansive mappings satisfying (2.7), hence for any $u \in \mathscr{F} \subset C_n$, it follows from (2.7) that

$$\begin{aligned} D_f(u,y_{n,m}) &= D_f(u,\nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(T_m^n x_n)]) \\ &\leqslant \alpha_n D_f(u,x_1) + (1-\alpha_n) D_f(u,T_m^n x_n) \\ &\leqslant \alpha_n D_f(u,x_1) + (1-\alpha_n) k_n D_f(u,x) \\ &\leqslant \alpha_n D_f(u,x_1) + (1-\alpha_n) D_f(u,x_n) + (k_n-1) D_f(u,x_n)) \\ &= \alpha_n D_f(u,x_1) + (1-\alpha_n) D_f(u,x_n) + \xi_n, \ \ \forall m \geqslant 1. \end{aligned} \tag{3.2}$$

Therefore we have

$$\sup_{m\geqslant 1} D_f(u,y_{n,m}) \leqslant \alpha_n D_f(u,x_1) + (1-\alpha_n) D_f(u,x_n) + \xi_n. \tag{3.3}$$

This shows that $u \in C_{n+1}$, and so $\mathscr{F} \subset C_{n+1}$. The conclusion is proved.

(III) Now we prove that $\{x_n\}$ converges strongly to some point $p^* \in C$.

In fact, since $x_n = P_{C_n}^f(x_1)$, from Proposition 2.8 (ii) we have

$$\langle x_n - y, \ \nabla f(x_1) - \nabla f(x_n)\rangle \geqslant 0, \ \forall y \in C_n.$$

Again since $\mathscr{F} \subset C_n, \ \forall n \geqslant 1$, we have

$$\langle x_n - u, \ \nabla f(x_1) - \nabla f(x_n)\rangle \geqslant 0, \ \forall u \in \mathscr{F}.$$

It follows from Proposition 2.8(iii) that for each $u \in \mathscr{F}$ and for each $n \geqslant 1$

$$D_f(x_n,x_1) = D_f(P_{C_n}^f(x_1),x_1) \leqslant D_f(u,x_1) - D_f(u,x_n) \leqslant D_f(u,x_1). \tag{3.4}$$

Therefore $\{\phi(x_n,x_1)\}$ is bounded. It follows from Proposition 2.6(2) that $\{x_n\}$ is also bounded.

Since $x_n = P_{C_n}^f(x_1)$ and $x_{n+1} = P_{C_{n+1}}^f(x_1) \in C_{n+1} \subset C_n$, we have $D_f(x_n,x_1) \leqslant D_f(x_{n+1},x_1), \ \forall n \geqslant 1$. This implies that $\{D_f(x_n,x_1)\}$ is a nondecreasing sequence. Hence the limit $\lim_{n\to\infty} D_f(x_n,x_1)$ exists. Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup p^*$ (some point in $C = C_1$). Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $p^* \in C_n$ for each $n \geqslant 1$. In view of $x_{n_i} = P_{C_{n_i}}^f(x_1)$, we have

$$D_f(x_{n_i},x_1) \leqslant D_f(p^*,x_1), \quad \forall n_i \geqslant 1.$$

Since f is a lower semi-continuous function on convex set C, it is weakly lower semi-continuous on C. Hence we have

$$\begin{aligned} \liminf_{n_i\to\infty} D_f(x_{n_i},x_1) &= \liminf_{n_i\to\infty} \{f(x_{n_i}) - f(x_1) - \langle \nabla f(x_1), \ x_{n_i} - x_1\rangle\} \\ &\geqslant f(p^*) - f(x_1) - \langle \nabla f(x_1), \ p^* - x_1\rangle = D_f(p^*,x_1), \end{aligned}$$

and so

$$D_f(p^*,x_1) \leqslant \liminf_{n_i\to\infty} D_f(x_{n_i},x_1) \leqslant \limsup_{n_i\to\infty} D_f(x_{n_i},x_1) \leqslant D_f(p^*,x_1).$$

This implies that $\lim_{n_i\to\infty} D_f(x_{n_i},x_1) = D_f(p^*,x_1)$, and so $f(x_{n_i}) \to f(p^*)$. Since f is uniformly continuous, we have that

$$\lim_{n_i\to\infty} x_{n_i} = p^*. \tag{3.5}$$

Since $\{D_f(x_n,x_1)\}$ is convergent, this together with $\lim_{n\to\infty} D_f(x_{n_i},x_1) = D_f(p^*,x_1)$ shows that $\lim_{n\to\infty} D_f(x_n,x_1) = D_f(p^*,x_1)$. If there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q$, then from Proposition 2.8(iii) we have that

$$\begin{aligned} D_f(p^*,q) &= \lim_{n_i,n_j\to\infty} D_f(x_{n_i},x_{n_j}) \\ &= \lim_{n_i,n_j\to\infty} D_f(x_{n_i},P^f_{C_{n_j}}(x_1)) \\ &\leqslant \lim_{n_i,n_j\to\infty} (D_f(x_{n_i},x_1) - D_f(P^f_{C_{n_j}}(x_1),x_1)) \\ &= \lim_{n_i,n_j\to\infty} (D_f(x_{n_i},x_1) - D_f(x_{n_j},x_1)) \\ &= D_f(p^*,x_1) - D_f(p^*,x_1) = 0, \end{aligned}$$

i.e., $p^* = q$ and so

$$\lim_{n\to\infty} x_n = p^*. \tag{3.6}$$

By the way, from (3.6) it is easy to see that

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \{(k_n - 1) \sup_{p\in\mathscr{F}} D_f(p,x_n)\} = 0. \tag{3.7}$$

(IV) Now we prove that $p^* \in \mathscr{F}$.

In fact, since $x_{n+1} \in C_{n+1}$, from (3.1), (3.6) and (3.7) we have that

$$\sup_{m\geqslant 1} D_f(x_{n+1},y_{n,m}) \leqslant \alpha_n D_f(x_{n+1},x_1) + (1-\alpha_n) D_f(x_{n+1},x_n) + \xi_n \to 0 \; (as\; n\to\infty). \tag{3.8}$$

Since $x_n \to p^*$, it follows from (3.8) and Proposition 2.7 that for each $m \geqslant 1$

$$\lim_{n\to\infty} y_{n,m} = p^*. \tag{3.9}$$

Since $\{x_n\}$ is bounded and $\{T_m\}_{m=1}^\infty$ is uniformly Bergman quasi-asymptotically nonexpansive, $\{T_m^n x_n\}_{m,n}^\infty$ is uniformly bounded. In view of $\alpha_n \to 0$, hence from (3.1) we have that for each $m \geqslant 1$

$$\lim_{n\to\infty} ||\nabla f(y_{n,m}) - \nabla f(T_m^n x_n)|| = \lim_{n\to\infty} \alpha_n ||\nabla f(x_1) - \nabla f(T_m^n x_n)|| = 0. \tag{3.10}$$

Since ∇f is uniformly continuous, it follows from (3.9) that for each $m \geqslant 1$, $\nabla f(y_{n,m}) \to \nabla f(p^*)$. Hence from (3.10) we have that for each $m \geqslant 1$, $\lim_{n\to\infty} \nabla f(T_m^n x_n) = \nabla f(p^*)$. Since ∇f is uniformly continuous, it yields that for each $m \geqslant 1$

$$\lim_{n\to\infty} T_m^n x_n = p^*. \tag{3.11}$$

On the other hand, by the assumptions that for each $m \geqslant 1$, T_m is uniformly L_m-Lipschitz continuous, thus we have

$$\begin{aligned}||T_m^{n+1}x_n - T_m^n x_n|| &\leqslant ||T_m^{n+1}x_n - T_m^{n+1}x_{n+1}|| + ||T_m^{n+1}x_{n+1} - x_{n+1}|| \\ &\quad + ||x_{n+1} - x_n|| + ||x_n - T_m^n x_n|| \\ &\leqslant (L_m+1)||x_{n+1} - x_n|| + ||T_m^{n+1}x_{n+1} - x_{n+1}|| + ||x_n - T_m^n x_n||.\end{aligned} \tag{3.12}$$

From (3.11) and $x_n \to p^*$ we have that $\lim_{n\to\infty} ||T_m^{n+1}x_n - T_m^n x_n|| = 0$ and $\lim_{n\to\infty} T_m^{n+1}x_n = p^*$, i.e, $\lim_{n\to\infty} T_m T_m^n x_n = p^*$. In view of the closeness of T_m, it yields that $T_m p^* = p^*$, i.e., for each $m \geqslant 1$, $p^* \in F(T_m)$. By the arbitrariness of $m \geqslant 1$, we have $p^* \in \mathscr{F}$.

(V) Finally we prove that $p^* = P^f_{\mathscr{F}}(x_1)$, and so $x_n \to P^f_{\mathscr{F}}(x_1)$.

Let $w = P^f_{\mathscr{F}}(x_1)$. Since $w \in \mathscr{F} \subset C_n$ and $x_n = P^f_{C_n}(x_1)$, we have $D_f(x_n, x_1) \leqslant D_f(w, x_1)$, $\forall n \geqslant 1$. This implies that

$$D_f(p^*, x_1) = \lim_{n\to\infty} D_f(x_n, x_1) \leqslant D_f(w, x_1). \tag{3.13}$$

Since $w = P^f_{\mathscr{F}}(x_1)$, this implies that $p^* = w$. Therefore, $x_n \to p^* = P^f_{\mathscr{F}}(x_1)$.

The conclusion of Theorem 3.1 is proved. □

THEOREM 3.2. *Let E, C be the same as in Theorem 3.1. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a countable families of closed and Bergman quasi-nonexpansive mappings such that $\mathscr{F} := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in E \text{ chosen arbitrarily}; \ C_1 = C, \\ y_{n,m} = \nabla f^*[\alpha_n \nabla f(x_1) + (1-\alpha_n)\nabla f(T_m^n x_n)], \\ C_{n+1} = \{z \in C_n : \sup_{m\geqslant 1} D_f(z, y_{n,m}) \leqslant \alpha_n D_f(z, x_1) + (1-\alpha_n) D_f(z, x_n)\} \\ x_{n+1} = P^f_{C_{n+1}}(x_1), \ \forall n \geqslant 1. \end{cases} \tag{3.14}$$

where $\{\alpha_n\}$ is a sequence in [0, 1]. If $\alpha_n \to 0$, then $\{x_n\}$ converges strongly to $P^f_{\mathscr{F}}(x_1)$.

Proof. Since $\{T_i\}_{i=1}^{\infty}$ is a countable family of closed Bergman quasi-nonexpansive mappings, by Remark 2.12, it is a countable family of closed and uniformly Bergman quasi-asymptotically nonexpansive mappings with sequence $\{k_n = 1\}$. Hence $\eta_n = (k_n - 1)\sup_{u\in\mathscr{F}} \phi(u, x_n) = 0$. Therefore the conditions appeared in Theorem 3.1: "$\mathscr{F}$ is a bounded subset in C" and "for each $i \geqslant 1$, T_i is uniformly L_i-Lipschitz" are no use here. Therefore all conditions in Theorem 3.1 are satisfied. The conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately. □

REMARK 3.3. Theorems 3.1 and 3.2 improve and extend the corresponding results of Reich and Sabach [15], Suantai et al. [19], Chang et al. [7, 8], Nilsrakoo and Saejung [12], Qin et al. [13, 14], Wang et al. [20], Su et al [18], Kang et al. [10] Martinez-Yanes and Xu [11] and others.

4. Applications

In this section we shall utilize the results presented in section 3 to study *a system of equilibrium problems* in reflexive Banach spaces.

THEOREM 4.1. *Let C, E and $\{\alpha_n\}$ be the same as in Theorem 3.2. Let $\{g_m : C\times C\to\mathscr{R}\}$, $m=1,2,\cdots$, be a countable family of bifunctions satisfying conditions $(A1)-(A4)$ as given in Example 2.14. Let $Res^f_{g_m}(x): E\to 2^C$, $m=1,2,\cdots$, be the family of mappings defined by*

$$Res^f_{g_m}(x)=\{z\in C: g_m(z,y)+\langle\nabla f(z)-\nabla f(x),\, y-z\rangle\geqslant 0,\ \forall y\in C\},\ x\in E.$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_1\in E \text{ chosen arbitrarily};\ \ C_1=C,\\ g_m(u_n,y)+\langle\nabla f(u_n)-\nabla f(x_n),\, y-u_n\rangle\geqslant 0,\ \forall y\in C,\ m\geqslant 1;\\ y_{n,m}=\nabla f^*[\alpha_n\nabla f(x_1)+(1-\alpha_n)\nabla f(u_{n,m})],\\ C_{n+1}=\{z\in C_n:\sup\limits_{m\geqslant 1} D_f(z,y_{n,m})\leqslant\alpha_n D_f(z,x_1)+(1-\alpha_n)D_f(z,x_n)\}\\ x_{n+1}=P^f_{C_{n+1}}(x_1),\ \forall n\geqslant 1, \end{cases} \tag{4.1}$$

If $\mathscr{F}:=\bigcap_{i=1}^\infty F(Res^f_{g_i})\neq\emptyset$, then $\{x_n\}$ converges strongly to $P^f_{\mathscr{F}}(x_1)$ which is a common solution of the system of equilibrium problems for $\{g_m\}_{m=1}^\infty$.

Proof. In Example 2.14 we have pointed out that $u_{n,m}=Res^f_{g_m}(x_n)$, $F(Res^f_{g_m})=EP(g_m)$ for all $m\geqslant 1$ and $\{Res^f_{g_m}\}_{m=1}^\infty$ is a countable family of closed Bergman quasi-nonexpansive mappings. Hence (4.1) can be rewritten as follows:

$$\begin{cases} x_1\in E \text{ chosen arbitrarily};\ \ C_1=C,\\ y_{n,m}=\nabla f^*[\alpha_n\nabla f(x_1)+(1-\alpha_n)\nabla f(Res^f_{g_m}(x_n))],\\ C_{n+1}=\{z\in C_n:\sup\limits_{m\geqslant 1} D_f(z,y_{n,m})\leqslant\alpha_n D_f(z,x_1)+(1-\alpha_n)D_f(z,x_n)\}\\ x_{n+1}=P^f_{C_{n+1}}(x_1),\ \forall n\geqslant 1. \end{cases} \tag{4.2}$$

Therefore the conclusion of Theorem 4.1 can be obtained from Theorem 3.2. □

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