Mathematical
Inequalities
& Applications
Volume 16, Number 4 (2013), 1185–1195 doi:10.7153/mia-16-92

APPROXIMATE PEXIDERIZED CAUCHY'S ADDITIVE TYPE MAPPINGS

YOUNG WHAN LEE

(*Communicated by Yeol Je Cho*)

Abstract. We prove the stability of the Pexiderized Cauchy's additive functional equation with a general form;

$$f(x+y) = g(x) + h(y) + \lambda(x,y)$$

where $\lambda(x,y)$ is a logarithm of a pseudo exponential function. From this result, we obtain the stability with the following form;

$$\frac{1}{1+\phi(x,y)} \leqslant \frac{f(x+y)}{e(x,y)g(x)h(y)} \leqslant 1+\phi(x,y),$$

where $e(x,y)$ is a pseudo exponential function. It is a generalized result for the stability of the Pexiderized Cauchy's functional equation.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [8]). Among those there was the question concerning the stability of homomorphisms: *Let* G_1 *be a group and let* G_2 *be a metric group with a metric* $d(\cdot,\cdot)$. *Given* $\varepsilon > 0$, *does there exist a* $\delta > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality* $d(h(xy), h(x)h(y)) \leqslant \delta$ *for all* $x,\ y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with* $d(h(x), H(x)) \leqslant \varepsilon$ *for all* $x \in G_1$*?* In the next year, D. H. Hyers [5] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Futhermore, the result of Hyers has been generalized by Th. M. Rassias [7].

The superstability of the functional equation $f(x+y) = f(x)f(y)$ was studied by J. Baker, J. Lawrence and F. Zorzitto [2]. They proved that if f is a functional on a real vector space W satisfying $|f(x+y) - (x)f(y)| \leqslant \delta$ for some fixed $\delta > 0$ and all $x,y \in W$, then either f is bounded or else $f(xy) = f(x)f(y)$ for all $x,y \in W$. This result was generalized with a simplified proof by J. Baker [1] as following: *Let* $\delta > 0$, S *be a semigroup and* $f : S \to C$ *satisfy* $|f(xy) - f(x)f(y)| \leqslant \delta$ *for all* $x,y \in S$. *Put* $\beta := (1+\sqrt{1+4\delta})/2$. *Then either* $f(x) \leqslant \beta$ *for all* $x \in S$ *or else* $f(xy) = f(x)f(y)$ *for all* $x,y \in S$. Since then, the stability and superstability problems of various functional equations have been investigated by many authors (see [3, 6]).

Mathematics subject classification (2010): 39B72, 38B22, 39B82.
Keywords and phrases: Functional equation, stability, superstability, gamma and beta functional equation, Cauchy functional equation, exponential functional equation.
The author of this work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant No. 2012-0008382).

In this paper we prove the stability of a Pexiderized Cauchy's additive functional equation with a general form;

$$f(x+y) = g(x) + h(y) + \lambda(x,y). \tag{1}$$

It is a generalized result for the stability of the Pexiderized Cauchy's functional equation. Also we obtain the stability with a functional inequality;

$$\frac{1}{1+\varphi(x,y)} \leqslant \frac{f(x+y)}{\exp(\lambda(x,y))g(x)h(y)} \leqslant 1+\varphi(x,y). \tag{2}$$

2. Definitions and solutions

Throughout this paper, we denote by D an additive subset ($x+y \in D$ for all $x,y \in D$) of $R^+ \cup \{0\}$ containing all nonnegative integers $Z^+ \cup \{0\}$.

DEFINITION 1. A function $e : D \times D \to R$ is *pseudo exponential* if $e(x,y)$ satisfies as follows;

(a) $e(x,y) = e(y,x) \quad (x,y \in D)$,
(b) $e(x,y) \geqslant 1 \quad (x,y \in D)$,
(c) $\frac{e(x,y)e(z,x+y)}{e(x,y+z)e(y,z)} = 1 \quad (x,y \in D)$,
(d) $e(x,n) \to \infty \quad$ (as $n \to \infty$ for $n \in Z^+$ and fixed $x \in D$),
(e) $e(0,x) = 1 \quad (x \in D)$.

DEFINITION 2. A function $\lambda : D \times D \to R$ is *a logarithm of a pseudo exponential function* if $\lambda(x,y)$ satisfies as follows;

(a) $\lambda(x,y) = \lambda(y,x) \quad (x,y \in D)$,
(b) $\lambda(x,y) \geqslant 0 \quad (x,y \in D)$,
(c) $\lambda(x,y) + \lambda(z,x+y) = \lambda(x,y+z) + \lambda(y,z) \quad (x,y \in D)$,
(d) $\lambda(x,n) \to \infty \quad$ (as $n \to \infty$ for $n \in Z^+$ and fixed $x \in D$),
(e) $\lambda(0,x) = 0 \quad (x \in D)$.

EXAMPLE 1. Let $\lambda(x,y) = xy$ for $x,y \in [0,\infty)$ then λ is a logarithm of a pseudo exponential function. Also if we let $\lambda(x,y) = a^{xy} (a > 1)$ and $e(x,y) = exp(\lambda(x,y))$, then $e(x,y)$ and $\lambda(x,y)$ satisfy Definition 1 and Definition 2, respectively.

DEFINITION 3. A function $f : D \to R$ is of *an approximate Cauchy's additive type* if there is a function $\phi : D \times D \to [0,\infty)$ and a logarithm of a pseudo exponential function $\lambda : D \times D \to R$ such that

$$|\, f(x+y) - f(x) - f(y) - \lambda(x,y)\,| \leqslant \phi(x,y)$$

for all $(x,y) \in D \times D$. In the case of $\phi = 0$, we call f a Cauchy's additive type function.

DEFINITION 4. A function $f : D \to R$ is of *an approximate Pexiderized Cauchy's additive type* if there is a function $\phi : D \times D \to [0,\infty)$, a logarithm of a pseudo exponential function $\lambda : D \times D \to R$ and some functions $g,h : D \to R$ such that

$$| f(x+y) - g(x) - h(y) - \lambda(x,y) | \leqslant \phi(x,y)$$

for all $(x,y) \in D \times D$. In the case of $\phi = 0$, we call f a Pexiderized Cauchy's additive type function.

EXAMPLE 2. If $f,g,h : [0,\infty) \to R$ are functions satisfying the equation (1) and $\lambda(x,y) = \ln a^{xy}$ $(a > 1)$, then λ is a logarithm of a pseudo exponential function, and $f(x) = \ln a^{\frac{x^2}{2}+1}, g(x) = \ln a^{\frac{x^2}{2}}, h(x) = \ln a^{\frac{x^2}{2}+1}$ are solutions of it. And also g is of a Cauchy's additive type function.

Now we consider the gamma-beta functional equation. If $f,g,h : (0,\infty) \to R$ are functions satisfying the equation (1) and $\beta(x,y)$ is the beta function then β^{-1} satisfies the conditions $(a) \sim (d)$ except (e) of Definition 1 (see, Corollary 4 in [6]) and $f(x) = \ln 6a^{x+1}\Gamma(x), g(x) = \ln 3a^{x}\Gamma(x), h(x) = \ln 2a^{x+1}\Gamma(x), \lambda(x,y) = \ln \beta^{-1}$ are solutions of the equation (1).

3. Stability of Cauchy's additive type functional equation

Throughout this section, we denote by $\phi : D \times D \to [0,\infty)$ a function such that

$$\Phi(x,y) := \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i y)}{2^{i+1}} < \infty$$

for all $(x,y) \in D \times D$ and by $\lambda : D \times D \to R$ a logarithm of a pseudo exponential function. The following theorems are the Hyers-Ulam stability of the Cauchy's additive type functional equations

$$f(x+y) = g(x) + h(y) + \lambda(x,y).$$

THEOREM 1. *Assume that a mapping $f : D \to R$ satisfies the functional inequality*

$$|f(x+y) - f(x) - f(y) - \lambda(x,y)| \leqslant \phi(x,y) \tag{3}$$

for all $x,y \in D$. Then there exists a unique mapping $g : D \to R$ such that

$$g(x+y) = g(x) + g(y) + \lambda(x,y)$$

for all $x,y \in D$ and

$$|f(x) - g(x)| \leqslant \Phi(x,x)$$

for all $x \in D$. In particular, g is defined by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

where

$$P_n(x) = \frac{f(2^n x)}{2^n} - \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \lambda(2^i x, 2^i x)$$

for all $x, y \in D$.

Proof. Let $e(x,y) = \exp(\lambda(x,y))$ for all $x, y \in D$. Then $e(x,y)$ is a pseudo exponential function and the inequality (3.1) is equivalent to the following inequality

$$|f(x+y) - f(x) - f(y) - \ln e(x,y)| \leqslant \phi(x,y) \tag{4}$$

for all $x, y \in D$. If we replace y by x and dividing 2 in (4), we get

$$\left| \frac{f(2x)}{2} - \ln e(x,x)^{\frac{1}{2}} - f(x) \right| \leqslant \frac{\phi(x,x)}{2} \tag{5}$$

for all $x \in D$. We use induction on n to prove

$$\left| \frac{f(2^n x)}{2^n} - \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - f(x) \right| \leqslant \sum_{i=0}^{n-1} \frac{\phi(2^i x, 2^i x)}{2^{i+1}} \tag{6}$$

for all $x \in D$. On account of (5), the inequality holds for $n = 1$. Suppose that inequality (6) holds true for some integer $n > 1$. Then (5) and (6) imply

$$\begin{aligned}
&\left| \frac{f(2^{n+1} x)}{2^{n+1}} - \ln \prod_{i=0}^{n} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - f(x) \right| \\
&\leqslant \left| \frac{f(2^n \cdot 2x)}{2 \cdot 2^n} - \frac{1}{2} \ln \prod_{i=0}^{n-1} e(2^i \cdot 2x, 2^i \cdot 2x)^{\frac{1}{2^{i+1}}} - \frac{f(2x)}{2} \right| + \left| \frac{f(2x)}{2} - \ln e(x,x)^{\frac{1}{2}} - f(x) \right| \\
&\leqslant \sum_{i=0}^{n} \frac{\phi(2^i x, 2^i x)}{2^{i+1}}
\end{aligned}$$

for any $x \in D$, which ends the proof of (6). For any $x \in D$ and for every positive integer n we define that

$$P_n(x) = \frac{f(2^n x)}{2^n} - \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}$$

for all $x, y \in D$. Let $m, n > 0$ be integers with $n > m$. Then it follows from (6) that for all $x \in D$

$$\begin{aligned}
|P_n(x) - P_m(x)| &= \frac{1}{2^m} \left| \frac{f(2^{n-m}(2^m x))}{2^{n-m}} - \ln \prod_{i=0}^{n-1} e(2^i x, 2^i x)^{\frac{1}{2^{i-m+1}}} - f(2^m x) \right| \\
&= \frac{1}{2^m} \left| \frac{f(2^{n-m}(2^m x))}{2^{n-m}} - \ln \prod_{i=0}^{n-m-1} e(2^i 2^m x, 2^i 2^m x)^{\frac{1}{2^{i+1}}} - f(2^m x) \right| \\
&\leqslant \sum_{i=m}^{n-1} \frac{\phi(2^i x, 2^i x)}{2^{i+1}} \to 0
\end{aligned}$$

as $m \to \infty$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function $g : D \to (0,\infty)$ by

$$g(x) := \lim_{n\to\infty} P_n(x)$$

for all $x \in D$. Now we prove that

$$g(x+y) = g(x) + g(y) + \ln e(x,y)$$

for all $x, y \in D$. For this, we consider the following property of the pseudo exponential function.

$$e(x+y,x+y) = \frac{e(x,x+y)e(y,y+2x)}{e(x,y)} = \frac{e(x,x)e(y,y)e(2x,2y)}{e(x,y)^2}$$

for all $x, y \in D$. By this property, we have the equation

$$\begin{aligned}
&\prod_{i=0}^{n-1}\left[\frac{e(2^i(x+y),2^i(x+y))}{e(2^ix,2^ix)e(2^iy,2^iy)}\right]^{\frac{1}{2^{i+1}}} = \prod_{i=0}^{n-1}\left[\frac{e(2^{i+1}x,2^{i+1}y)}{e(2^ix,2^iy)^2}\right]^{\frac{1}{2^{i+1}}}\\
&= \left[\frac{e(2x,2y)}{e(x,y)^2}\right]^{\frac{1}{2}} \cdot \left[\frac{e(2^2x,2^2y)}{e(2x,2y)^2}\right]^{\frac{1}{2^2}} \cdots \left[\frac{e(2^nx,2^ny)}{e(2^{n-1}x,2^{n-1}y)^2}\right]^{\frac{1}{2^n}}\\
&= \frac{e(2^nx,2^ny)^{\frac{1}{2^n}}}{e(x,y)}
\end{aligned} \tag{7}$$

for all $x,\ y \in D$. From this equation (7) we get

$$\begin{aligned}
&|g(x+y) - g(x) - g(y) - \ln e(x,y)|\\
&= \lim_{n\to\infty}\left|\frac{f(2^nx+2^ny)}{2^n} - \frac{f(2^nx)}{2^n} - \frac{f(2^ny)}{2^n} - \frac{1}{2^n}\ln e(2^nx,2^ny)\right|\\
&= \lim_{n\to\infty}\frac{\phi(2^nx,2^nx)}{2^n} = 0
\end{aligned}$$

for all $x, y \in D$ and thus

$$g(x+y) = g(x) + g(y) + \ln e(x,y)$$

for all $x, y \in D$. From the inequality (5), we have

$$|f(x) - g(x)| \leqslant \Phi(x,x)$$

for all $x \in D$. Now suppose that h satisfies the equation

$$h(x+y) = h(x) + h(y) + \ln e(x,y)$$

for all $x, y \in D$. and

$$|f(x) - h(x)| \leqslant \Phi(x,x)$$

for all $x \in D$. Then for all $x \in D$

$$|g(x)-h(x)| \leqslant \lim_{n\to\infty} \frac{1}{2^n}\left|f(2^n x)-h(2^n x)\right| + \left|\frac{h(2^n x)}{2^n} - \ln\prod_{i=0}^{n-1} e(2^i x, 2^i y)^{\frac{1}{2^{i+1}}} - h(x)\right|$$
$$\leqslant \lim_{n\to\infty} \frac{\Phi(2^n x, 2^n x)}{2^n} + 0 = \lim_{n\to\infty}\sum_{k=n}^{\infty} \frac{\phi(2^k x, 2^k x)}{2^k} = 0$$

as $n \to \infty$ and for all $x \in D$, and thus g is unique. □

COROLLARY 1. *Assume that a mapping $f : R^+ \to R^+$ satisfies the functional inequality*

$$|f(x+y)-f(x)-f(y)-xy| \leqslant \delta$$

for all $x,y \in R^+$. Then there exists a unique mapping $g : R^+ \to R^+$ such that

$$g(x+y) = g(x)+g(y)+xy$$

for all $x,y \in A$ and

$$|f(x)-g(x)| \leqslant \delta$$

for all $x \in R^+$.

Proof. From Theorem 1 with $\lambda(x,y) = xy$, we complete the proof. □

THEOREM 2. *Assume that mappings $f,g,h : D \to R$ satisfy the functional inequality*

$$|f(x+y)-g(x)-h(y)-\lambda(x,y)| < \phi(x,y) \tag{8}$$

for all $x,y \in D$. Then there exists a unique mapping $T : D \to R$ such that

$$T(x+y) = T(x)+T(y)+\lambda(x,y)$$

for all $x,y \in D$ and

$$|f(x)-T(x)| \leqslant \Psi(x) := \Phi(x,x)+\Phi(0,x)+\Phi(x,0)+|g(0)|+|h(0)|,$$
$$|g(x)-T(x)| \leqslant \phi(x,0)+|h(0)|+\Psi(x),$$
$$|h(x)-T(x)| \leqslant \phi(0,x)+|g(0)|+\Psi(x)$$

for all $x \in D$.

Proof. By the functional inequality in (8), we get

$$|f(x+y)-f(x)-f(y)-\lambda(x,y)|$$
$$\leqslant |f(x+y)-g(x)-h(y)-\lambda(x,y)| + |g(0)+h(y)+\lambda(0,y)-f(y)|$$
$$+ |g(x)+h(0)+\lambda(x,0)-f(x)| + |g(0)|+|h(0)|$$
$$\leqslant \phi(x,y)+\phi(0,y)+\phi(x,0)+|g(0)|+|h(0)|$$

for all $x \in D$. Note that

$$\sum_{i=0}^{\infty} \frac{\phi(0,2^i)}{2^{i+1}} = \Phi(0,y)$$

and

$$\sum_{i=0}^{\infty} \frac{|g(0)|}{2^{i+1}} = |g(0)|.$$

By Theorem 1, there exists a unique mapping $T : D \to R$ such that

$$T(x+y) = T(x) + T(y) + \lambda(x,y)$$

for all $x, y \in D$ and

$$|f(x) - T(x)| \leqslant \Psi(x) := \Phi(x,x) + \Phi(0,x) + \Phi(x,0) + |g(0)| + |h(0)|$$

for all $x \in D$. Then we have

$$\begin{aligned} |g(x) - T(x)| &\leqslant |g(x) - f(x)| + |f(x) - T(x)| \\ &\leqslant |g(x) + h(0) + \lambda(x,0) - f(x)| + |h(0)| + \Psi(x) \\ &\leqslant \phi(x,0) + |h(0)| + \Psi(x) \end{aligned}$$

and similarly

$$|h(x) - T(x)| \leqslant \phi(0,x) + |g(0)| + \Psi(x)$$

for all $x \in D$. $\square$

R. Ger [4] suggested a new type of stability for the exponential equation

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leqslant \delta.$$

Comparing this, we obtain the stability with a functional inequality;

$$\frac{1}{1+\phi(x,y)} \leqslant \frac{f(x+y)}{e(x,y)g(x)h(y)} \leqslant 1 + \phi(x,y).$$

Let $\varepsilon : D \times D \to [0,\infty)$ be a function such that

$$\varepsilon(x,y) := \sum_{i=0}^{\infty} \frac{\ln(1+\phi(2^i x, 2^i y))}{2^{i+1}}$$

for all $(x,y) \in D \times D$. Then

$$\varepsilon(x,y) \leqslant (1 + \Phi(x,y)) < \infty$$

for all $(x,y) \in D \times D$.

THEOREM 3. *If functions* $f,g,h : D \to (0,\infty)$ *satisfy the inequality*

$$\frac{1}{1+\phi(x,y)} \leqslant \frac{f(x+y)}{e(x,y)g(x)h(y)} \leqslant 1+\phi(x,y) \tag{9}$$

for all $(x,y) \in D \times D$ *and some pseudo exponential function* $e(x,y)$*, then there exists a unique function* $T : D \to (0,\infty)$ *such that*

$$T(x+y) = e(x,y)T(x)T(y)$$

for all $x,\ y \in D$ *and also*

$$\exp(-\Phi_1(x)) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(\Phi_1(x)),$$
$$\exp(-\Phi_2(x)) \leqslant \frac{T(x)}{g(x)} \leqslant \exp(\Phi_2(x)),$$

and

$$\exp(-\Phi_3(x)) \leqslant \frac{T(x)}{h(x)} \leqslant \exp(\Phi_3(x))$$

for all $x \in D$*, where*

$$\Phi_1(x) = \Phi(x,x) + \Phi(x,0) + \Phi(0,x) + 3 + |\ln g(0)| + |\ln h(0)|,$$
$$\Phi_2(x) = \Phi_1(x) + 1 + \phi(x,0) + |\ln h(0)|,$$

and

$$\Phi_3(x) = \Phi_1(x) + 1 + \phi(0,x) + |\ln g(0)|$$

for all $x \in D$*. In particular, if* $f = g = h$*, then*

$$\exp(-1 - \Phi(x,x)) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(1 + \Phi(x,x))$$

for all $x \in D$*.*

Proof. If we define functions $F,G,H : D \to R$ by

$$F(x) = \ln f(x),\ G(x) = \ln g(x),\ H(x) = \ln h(x)$$

for all $x \in D$, then the inequality (9) may be transformed into

$$|F(x+y) - G(x) - H(y) - \ln e(x,y)| \leqslant \ln(1 + \phi(x,y)). \tag{10}$$

By Theorem 2 with $\lambda(x,y) = \ln e(x,y)$, there exists a unique mapping $W : D \to R$ such that

$$W(x+y) = W(x) + W(y) + \ln e(x,y)$$

for all $x,y \in D$ and

$$|F(x)-W(x)| \leqslant \Phi(x,x)+\Phi(0,x)+\Phi(x,0)+3+|\ln g(0)|+|\ln h(0)| \\ = \Phi_1(x) \tag{11}$$

for all $x \in D$. Now we define a function $T : D \to R$ by

$$T(x) := \exp(W(x))$$

for all $x \in D$. Then

$$T(x+y) = \exp(W(x)+W(y)+\ln e(x,y)) = e(x,y)T(x)T(y)$$

for all $x,y \in D$, By (11) we have

$$-\Phi_1(x) \leqslant \ln T(x) - \ln f(x) \leqslant \Phi_1(x)$$

and thus

$$\exp(-\Phi_1(x)) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(\Phi_1(x))$$

for all $x \in D$.

By (10), we have

$$|F(x+y)-G(y)-H(x)-\ln e(x,y)| \leqslant \ln(1+\phi(y,x))$$

for all $x,y \in D$. Then

$$\begin{aligned}|G(x)-W(x)| &\leqslant |G(x)-F(x)|+|F(x)-W(x)| \\ &\leqslant |G(x)+H(0)+\ln e(x,0)-F(x)|+|H(0)|+\Phi_1(x) \\ &\leqslant 1+\phi(x,0)+|H(0)|+\Phi_1(x) = \Phi_2(x)\end{aligned}$$

and similarly,

$$|H(x)-W(x)| \leqslant 1+\phi(0,x)+|G(0)|+\Phi_1(x) = \Phi_3(x)$$

for all $x,y \in D$. Since $W(x) = \ln T(x)$, we have

$$\exp(-\Phi_2(x)) \leqslant \frac{T(x)}{g(x)} \leqslant \exp(\Phi_2(x)),$$

and

$$\exp(-\Phi_3(x)) \leqslant \frac{T(x)}{h(x)} \leqslant \exp(\Phi_3(x))$$

for all $x \in D$. If $f = g = h$, then we have

$$\exp(-1-\Phi(x,x)) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(1+\Phi(x,x))$$

for all $x \in D$, applying to Theorem 1 instead of Theorem 2. □

COROLLARY 2. *Let $\delta > 0$. If a function $f : R \to (0,\infty)$ satisfies the inequality*

$$\frac{1}{1+\delta} \leqslant \frac{f(x+y)}{f(x)f(y)} \leqslant 1+\delta$$

for all $x,y \in R$, then there exists a function $T : R \to (0,\infty)$ such that

$$T(x+y) = T(x)T(y)$$

for all $x,y \in R$ and

$$\exp(-1-\delta) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(1+\delta)$$

for all $x \in R$.

Proof. Let $\phi(x,y) = \delta$ and $e(x,y) = 1$ for all $x,y \in R$. Then

$$\Phi(x,y) = \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} = \delta$$

for all $x,y \in R$. By Theorem 3, we complete the proof. □

COROLLARY 3. *Let $\delta > 0$ and $a > 1$ be given. Suppose that $f : [0,\infty) \to (0,\infty)$ be a function such that*

$$\frac{1}{1+\delta} \leqslant \frac{f(x+y)}{a^{xy} f(x)f(y)} \leqslant 1+\delta$$

for all $x,\ y \in [0,\infty)$. Then there exists a unique function $T : [0,\infty) \to (0,\infty)$ such that

$$T(x+y) = a^{xy} T(x)T(y)$$

for all $x,y \in R$ and

$$\exp(-1-\delta) \leqslant \frac{T(x)}{f(x)} \leqslant \exp(1+\delta)$$

for all $x \in R$.

REFERENCES

[1] J. BAKER, *The stability of the cosine equations*, Proc. Amer. Math. Soc. 80 (1980), 411–416.

[2] J. BAKER, J. LAWRENCE AND F. ZORZITTO, *The stability of the equation $f(x+y) = f(x) + f(y)$*, Proc. Amer. Math. Soc. 74 (1979), 242–246.

[3] G. L. FORTI, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. 50 (1995), 146–190.

[4] R. GER, *Superstability is not natural*, Rocznik Naukowo-Dydaktyczny WSP Krakkowie, Prace Mat. 159 (1993), 109–123.

[5] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. 27 (1941), 222–224.

[6] Y. W. LEE, *Superatability and stability of the pexiderized multiplicative functional equation*, J. Inequalities and Appl. (2010) Article ID 486325, 1–15.
[7] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297–300.
[8] S. M. ULAM, *Problems in Modern Mathematics*, Proc. Chap. VI. Wiley. NewYork, 1964.

(Received November 20, 2012)

Young Whan Lee
Department of Computer Hacking and Information Security
College of Natural Science Daejeon University
Daejeon 300-716, Republic of Korea
e-mail: ywlee@dju.ac.kr