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TRIPLED BEST PROXIMITY POINT THEOREM IN METRIC SPACES

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Abstract. The purpose of this article is to first introduce the notion of tripled best proximity point and cyclic contraction pair. We also establish the existence and convergence theorems of tripled best proximity points in metric spaces. Moreover, we apply our results to setting of uniformly convex Banach space. Finally, we obtain some results on the existence and convergence of tripled fixed point in metric spaces and give illustrative examples of our theorems.

1. Introduction and preliminaries

In the two last decades, the theory of fixed points has appeared as a crucial technique in the study of nonlinear functional analysis. In particular, the techniques and tools in fixed point theory have application in many branches of applied mathematics and also in many research fields such as physics, chemistry, biology, economics, computer sciences, and many branches of engineering. The most significant result in fixed point theory, known as the Banach Contraction Mapping Principle (BCMP) is given by Banach in [4]. BCMP states that every contraction (self-mapping) $T : X \to X$ on a complete metric space (X,d) has a unique fixed point, that is, $Tx = x$. Due to its wide application potential, this celebrated principle has been generalized in many ways over the years [2, 10, 11, 23, 33].

On the other hand, the study of the existence of fixed point for non-self mapping on various abstract spaces is also very interesting. More precisely, for a given nonempty closed subsets A and B of a complete metric space (X,d), a contraction non-self mapping $T : A \to B$ does not necessarily yields a fixed point, that is, $d(Tx,x) \neq 0$. In this case, it is quite natural to investigate an element $x \in X$ such that $d(x,Tx)$ is minimum, that is, the points x and Tx are close proximity to each other.

Let A and B be closed subsets of a metric space (X,d) and $T : A \to B$ be a non-self mapping. A point x in A for which $d(x,Tx) = d(A,B)$ is called a best proximity point of T. If $A \cap B \neq \emptyset$ then the best proximity point becomes a fixed point of T. In other words, since a best proximity point reduces to a fixed point if the underlying mapping is assumed to be self mappings, the best proximity point theorems are natural generalizations of the BCMP. In this direction, the first result was given by Fan [13] in 1969. In these pioneering work, the author introduced and established a classical best approximation theorem:

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THEOREM 1. ([13]) *If A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is continuous mapping, then there exists an element $x \in A$ such that $d(x,Tx) = d(Tx,A)$.*

Following this initial paper, a number of authors have derived extensions of Fan's Theorem and best approximation theorem in many directions such as Prolla [27], Sehgal and Singh [28, 29], Wlodarczyk and Plebaniak [36, 37, 38, 39], Vetrivel et al. [35], Eldred and Veramani [12], Mongkolkeha and Kumam [24, 25, 26] and Basha and Veeramani [5, 6, 7, 8] (see also [3, 16, 17, 18, 19, 20, 21, 22] and reference therein).

One interesting and crucial notion is the one of coupled fixed point, introduced by Guo and Lakshmikantham [15] in 1987. Bhaskar and Laksmikantham [14] introduced the notion of mixed monotone mapping and proved some coupled fixed point theorems for mappings satisfying the mixed monotone property. In [14], the authors observed that their theorems can be used to investigate a large class of problems and discuss the existence and uniqueness of solution for a periodic boundary value problem. Several improvements and generalizations of [14] have recently appeared in the literature (see [1, 30, 31] and references therein).

Very recently, Berinde and Borcut [9] introduced the notions of tripled fixed point. They proved existence and uniqueness results of tripled fixed point in a partially ordered complete metric space. On the other hand, the concept of coupled best proximity point and property UC* are first introduced by Sintunavarat and Kumam [32]. They also give existence and convergence theorems of coupled best proximity point for cyclic contraction pairs.

Motivated by the interesting works [9] and [32], we first introduce the notions of tripled best proximity point and later establish the existence and convergence theorems of tripled best proximity point in metric spaces. Moreover, we apply these results in uniformly convex Banach space. We also study some results on the existence and convergence of tripled fixed point in metric spaces and give illustrative examples of our theorems.

We recall some basic definitions and examples that are related to the main results of this article. Throughout this article we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. For nonempty subsets A and B of a metric space (X,d), we set

$$d(A,B) = \inf\{d(x,y) : x \in A,\ \ y \in B\} \tag{1}$$

stands for the distance between A and B.

A Banach spaces X is said to be:

(1) *strictly convex* if the following implication holds: for all $x,y \in X$,

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \Longrightarrow \left\|\frac{x+y}{2}\right\| < 1.$$

(2) *uniformly convex* if, for any ε with $0 < \varepsilon \leqslant 2$, there exists $\delta > 0$ such that the following implication holds: for all $x,y \in X$,

$$\|x\| \leqslant 1,\ \|y\| \leqslant 1 \text{ and } \|x-y\| \geqslant \varepsilon \Longrightarrow \left\|\frac{x+y}{2}\right\| < 1-\delta.$$

It is easily to see that a uniformly convex Banach space X is strictly convex but the converge is not true.

DEFINITION 1. ([34]) Let A and B be nonempty subsets of a metric space (X,d). We say that the ordered pair (A,B) satisfies the *property UC* if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $d(x_n,y_n) \to d(A,B)$ and $d(z_n,y_n) \to d(A,B)$, then $d(x_n,z_n) \to 0$.

EXAMPLE 1. Let A and B be nonempty subsets of a metric space (X,d). The following statements are examples of pairs of nonempty subsets (A,B) satisfying the property UC.

(1) A pair (A,B) of nonempty subsets A,B of a metric space (X,d) such that $d(A,B)=0$.

(2) A pair (A,B) of nonempty subsets A,B of a uniformly convex Banach space X such that A is convex.

(3) A pair (A,B) of nonempty subsets A,B of a strictly convex Banach space such that A is convex and relatively compact and the closure of B is weakly compact.

DEFINITION 2. ([32]) Let A and B be nonempty subsets of a metric space (X,d). We say that the ordered pair (A,B) satisfies the *property UC** if (A,B) has property UC and the following condition holds:

If $\{x_n\}$, $\{z_n\}$ are two sequences in A and $\{y_n\}$ is a sequence in B satisfying the following conditions:

(1) $d(z_n,y_n) \to d(A,B)$,

(2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m,y_n) \leqslant d(A,B)+\varepsilon$ for all $m > n \geqslant N$,

then exists $N_1 \in \mathbb{N}$ such that $d(x_m,z_n) \leqslant d(A,B)+\varepsilon$ for all $m > n \geqslant N_1$.

EXAMPLE 2. ([32]) Let A and B be nonempty subsets of a metric space (X,d). The following statements are examples of a pair of nonempty subsets (A,B) satisfying the property UC*.

(1) A pair (A,B) of nonempty subsets A,B of a metric space (X,d) such that $d(A,B)=0$.

(2) A pair (A,B) of nonempty closed subsets A,B of a uniformly convex Banach space X such that A is convex.

DEFINITION 3. Let A and B be nonempty subsets of a metric space (X,d) and $T : A \to B$ be a mapping. A point $x \in A$ is called a *best proximity point* of T if the following condition holds:

$$d(x,Tx) = d(A,B).$$

It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

DEFINITION 4. ([9]) Let A be a nonempty subset of a metric space (X,d) and $F:A^3\to A$ be a mapping. A point $(x,y,z)\in A^3$ is called a *tripled fixed point* of F if the following conditions hold:

$$x=F(x,y,z),\ \ y=F(y,x,y),\ \ z=F(z,y,x).$$

2. Tripled best proximity point theorems

In this section we study the existence and convergence of tripled best proximity points for cyclic contraction pairs in metric spaces.

DEFINITION 5. Let A, B be nonempty subsets of a metric space (X,d) and $F:A^3\to B$ be mapping. An ordered tripled $(x,y,z)\in A^3$ is called a *tripled best proximity point* of F if,

$$d(x,F(x,y,z))=d(y,F(y,x,y))=d(z,F(z,y,x))=d(A,B).$$

It is easy to see that, if $A=B$ in Definition 5, then a tripled best proximity point reduces to a tripled fixed point.

Next, we introduce the notions of a cyclic contractions for a pair of mappings.

DEFINITION 6. Let A, B be nonempty subsets of a metric space $(X,d$ and $F:A^3\to B$, $G:B^3\to A$ be two mappings. The ordered pair (F,G) is called a *cyclic contraction* if there exists a non-negative number $\alpha<1$ such that

$$d(F(x,y,z),G(u,v,w))\leqslant\frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B)$$

for all $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$.

Note that, if (F,G) is a cyclic contraction, then the pair (G,F) is also a cyclic contraction.

EXAMPLE 3. Let $X=\mathbb{R}$ with the usual metric $d(x,y)=|x-y|$ and let $A=[2,6]$ and $B=[-6,-2]$. It easy to see that $d(A,B)=4$. Define two mappings $F:A^3\to B$ and $G:B^3\to A$ by

$$F(x,y,z)=\frac{-x-y-z-6}{6},\quad G(u,v,w)=\frac{-u-v-w+6}{6}$$

for all $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, respectively. For any $(x,y,z)\in A^3$, $(u,v,w)\in B^3$ and fixed $\alpha=\frac{1}{2}$, we get

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) &= \left|\frac{-x-y-z-6}{6}-\frac{-u-v-w+6}{6}\right| \\ &\leqslant \frac{|x-u|+|y-v|+|z-w|}{6}+2 \\ &= \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]+(1-\alpha)d(A,B). \end{aligned}$$

This implies that the pair (F,G) is a cyclic contraction with $\alpha=\frac{1}{2}$.

EXAMPLE 4. Let $X = \mathbb{R}^3$ with the metric

$$d((x,y,z),(u,v,w)) = \max\{|x-u|,|y-v|,|z-w|\}$$

for $(x,y,z),(u,v,w) \in X$ and let

$$A = \{(x,0,0) \in X : 0 \leqslant x \leqslant 1\}, \quad B = \{(x,1,1) \in X : 0 \leqslant x \leqslant 1\}.$$

It easy to prove that $d(A,B) = 1$. Define two mappings $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F((x,0,0),(y,0,0),(z,0,0)) = \left(\frac{x+y+z}{3},1,1\right),$$
$$G((u,1,1),(v,1,1),(w,1,1)) = \left(\frac{u+v+w}{3},0,0\right),$$

respectively. Then we obtain

$$\begin{aligned} &d(F((x,0,0),(y,0,0),(z,0,0)),G((u,1,1),(v,1,1),(w,1,1))) \\ &= d\left(\left(\frac{x+y+z}{3},1,1\right),\left(\frac{u+v+w}{3},0,0\right)\right) \\ &= 1. \end{aligned}$$

Also, if $(x,0,0),(y,0,0),(z,0,0) \in A$ and $(u,1,1),(v,1,1),(w,1,1) \in B$, then we have

$$\begin{aligned} &\frac{\alpha}{3}[d((x,0,0),(u,1,1)) + d((y,0,0),(v,1,1)) + d((z,0,0),(w,1,1))] + (1-\alpha)d(A,B) \\ &= \frac{\alpha}{3}[\max\{|x-u|,1,1\} + \max\{|y-v|,1,1\} + \max\{|z-w|,1,1\}] + (1-\alpha)d(A,B) \\ &= \frac{\alpha}{3} \times 3 + (1-\alpha) \\ &= 1 \end{aligned}$$

for any non-negative real number $\alpha < 1$. Therefore, letting

$$(\mathbf{X},\mathbf{Y},\mathbf{Z}) = ((x,0,0),(y,0,0),(z,0,0)) \in A^3,$$
$$(\mathbf{U},\mathbf{V},\mathbf{W}) = ((u,1,1),(v,1,1),(w,1,1)) \in B^3,$$

we get

$$d(F(\mathbf{X},\mathbf{Y},\mathbf{Z}),G(\mathbf{U},\mathbf{V},\mathbf{W})) \leqslant \frac{\alpha}{3}[d(\mathbf{X},\mathbf{U}) + d(\mathbf{Y},\mathbf{V}) + d(\mathbf{Z},\mathbf{W})] + (1-\alpha)d(A,B)$$

for any non-negative real number $\alpha < 1$. This implies that the pair (F,G) is a cyclic contraction.

The following lemmas play an important role in our main results.

LEMMA 1. *Let A, B be nonempty subsets of a metric space (X,d) and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. If $(x_0,y_0,z_0) \in A^3$ and we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}) \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, then we have

$$\begin{aligned} d(x_{2n},x_{2n+1}) &\to d(A,B), \quad & d(x_{2n+1},x_{2n+2}) &\to d(A,B), \\ d(y_{2n},y_{2n+1}) &\to d(A,B), \quad & d(y_{2n+1},y_{2n+2}) &\to d(A,B), \\ d(z_{2n},z_{2n+1}) &\to d(A,B), \quad & d(z_{2n+1},z_{2n+2}) &\to d(A,B). \end{aligned}$$

Proof. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} & d(x_{2n},x_{2n+1}) \\ = {} & d(x_{2n},F(x_{2n},y_{2n},z_{2n})) \\ = {} & d(G(x_{2n-1},y_{2n-1},z_{2n-1}), \\ & \quad F(G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(z_{2n-1},y_{2n-1},x_{2n-1}))) \\ \leqslant {} & \frac{\alpha}{3}[d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))+d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1})) \\ & +d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))]+(1-\alpha)d(A,B) \\ = {} & \frac{\alpha}{3}\Big[d(F(x_{2n-2},y_{2n-2},z_{2n-2}), \\ & \quad G(F(x_{2n-2},y_{2n-2},z_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2}),F(z_{2n-2},y_{2n-2},x_{2n-2}))) \\ & +d(F(y_{2n-2},x_{2n-2},y_{2n-2}), \\ & \quad G(F(y_{2n-2},x_{2n-2},y_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2}))) \\ & +d(F(z_{2n-2},y_{2n-2},x_{2n-2}), \\ & \quad G(F(z_{2n-2},y_{2n-2},x_{2n-2}),F(y_{2n-2},x_{2n-2},y_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2})))\Big] \\ & +(1-\alpha)d(A,B) \\ \leqslant {} & \frac{\alpha}{3}\Big[\frac{\alpha}{3}[d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))+d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2})) \\ & +d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))]+(1-\alpha)d(A,B) \\ & +\frac{\alpha}{3}[d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))+d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2})) \\ & +d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))]+(1-\alpha)d(A,B) \\ & +\frac{\alpha}{3}[d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))+d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2})) \\ & +d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))]+(1-\alpha)d(A,B)\Big] \\ & +(1-\alpha)d(A,B) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha^2}{3}[d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))+d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))\\
&\quad +d(y_{2n-2},F(y_{2n-2},x_{2n-2},y_{2n-2}))]+(1-\alpha^2)d(A,B).
\end{aligned}
$$

By induction, we see that

$$
\begin{aligned}
d(x_{2n},x_{2n+1}) \leqslant \frac{\alpha^{2n}}{3}&[d(x_0,F(x_0,y_0,z_0))+d(y_0,F(y_0,x_0,y_0))+d(z_0,F(z_0,y_0,x_0))]\\
&+(1-\alpha^{2n})d(A,B)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain

$$
d(x_{2n},x_{2n+1}) \to d(A,B). \tag{2}
$$

For all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&d(x_{2n+1},x_{2n+2})\\
&= d(x_{2n+1},G(x_{2n+1},y_{2n+1},z_{2n+1}))\\
&= d(F(x_{2n},y_{2n},z_{2n}),G(F(x_{2n},y_{2n},z_{2n}),F(y_{2n},x_{2n},y_{2n}),F(z_{2n},y_{2n},x_{2n})))\\
&\leqslant \frac{\alpha}{3}\big[d(x_{2n},F(x_{2n},y_{2n},z_{2n}))+d(y_{2n},F(y_{2n},x_{2n},y_{2n}))\\
&\quad +d(z_{2n},F(z_{2n},y_{2n},x_{2n})\big]+(1-\alpha)d(A,B)\\
&= \frac{\alpha}{3}\big[d(G(x_{2n-1},y_{2n-1},z_{2n-1}),\\
&\qquad F(G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(z_{2n-1},y_{2n-1},x_{2n-1})))\\
&\quad +d(G(y_{2n-1},x_{2n-1},y_{2n-1}),\\
&\qquad F(G(y_{2n-1},x_{2n-1},y_{2n-1}),G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1})))\\
&\quad +d(G(z_{2n-1},y_{2n-1},x_{2n-1}),\\
&\qquad F(G(z_{2n-1},y_{2n-1},x_{2n-1}),G(y_{2n-1},x_{2n-1},y_{2n-1}),G(x_{2n-1},y_{2n-1},z_{2n-1})))\big]\\
&\quad +(1-\alpha)d(A,B)\\
&\leqslant \frac{\alpha}{3}\Big[\frac{\alpha}{3}[d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))+d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))\\
&\quad +d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))+d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))\\
&\quad +d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))]+(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha}{3}[d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))+d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))\\
&\quad +d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))]+(1-\alpha)d(A,B)\Big]\\
&\quad +(1-\alpha)d(A,B)\\
&= \frac{\alpha^2}{3}[d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))+d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))\\
&\quad +d(y_{2n-1},G(y_{2n-1},x_{2n-1},y_{2n-1}))]+(1-\alpha^2)d(A,B).
\end{aligned}
$$

By induction, we see that

$$d(x_{2n+1},x_{2n+2}) \leqslant \frac{\alpha^{2n}}{3}[d(x_1,G(x_1,y_1,z_1))+d(y_1,G(y_1,x_1,y_1))+d(z_1,G(z_1,y_1,x_1))] \\ +(1-\alpha^{2n})d(A,B)$$

for all $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, we obtain

$$d(x_{2n+1},x_{2n+2}) \to d(A,B). \tag{3}$$

By the similar argument, we also have

$$\begin{aligned} &d(y_{2n},y_{2n+1}) \to d(A,B), \quad d(y_{2n+1},y_{2n+2}) \to d(A,B), \\ &d(z_{2n+1},z_{2n+2}) \to d(A,B), \quad d(z_{2n},z_{2n+1}) \to d(A,B). \end{aligned}$$

This completes the proof. □

LEMMA 2. *Let A, B be nonempty subsets of a metric space (X,d) such that the pairs (A,B) and (B,A) have the property UC and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}) \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N}\cup\{0\}$. Then, for any $\varepsilon > 0$, there exists a positive integer N_0 such that, for all $m > n \geqslant N_0$,

$$\frac{1}{3}[d(x_{2m},x_{2n+1})+d(y_{2m},y_{2n+1})+d(z_{2m},z_{2n+1})] < d(A,B)+\varepsilon. \tag{4}$$

Proof. By Lemma 1, we have

$$\begin{aligned} &d(x_{2n},x_{2n+1}) \to d(A,B), \quad d(x_{2n+1},x_{2n+2}) \to d(A,B), \\ &d(y_{2n},y_{2n+1}) \to d(A,B), \quad d(y_{2n+1},y_{2n+2}) \to d(A,B), \\ &d(z_{2n},z_{2n+1}) \to d(A,B), \quad d(z_{2n+1},z_{2n+2}) \to d(A,B). \end{aligned}$$

Since (A,B) has the property UC, we get

$$d(x_{2n},x_{2n+2}) \to 0.$$

The similar argument shows that

$$d(y_{2n},y_{2n+2}) \to 0 \text{ and } d(z_{2n},z_{2n+2}) \to 0.$$

Since (B,A) has the property UC, we also have

$$d(x_{2n+1},x_{2n+3}) \to 0, \quad d(y_{2n+1},y_{2n+3}) \to 0, \quad d(z_{2n+1},z_{2n+3}) \to 0.$$

Suppose that (4) does not hold. Then there exists $\varepsilon' > 0$ such that, for all $k \in \mathbb{N}$, there exists $m_k > n_k \geqslant k$ satisfying

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \geqslant d(A,B) + \varepsilon'.$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Then

$$\frac{1}{3}[d(x_{2m_k-2}, x_{2n_k+1}) + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k-2}, z_{2n_k+1})] < d(A,B) + \varepsilon'.$$

Therefore, we get

$$\begin{aligned}
& d(A,B) + \varepsilon' \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k+1}) + d(z_{2m_k}, z_{2m_k-2}) + d(z_{2m_k-2}, z_{2n_k+1})] \\
<\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k-2}) + d(y_{2m_k}, y_{2m_k-2}) + d(z_{2m_k}, z_{2m_k-2})] + d(A,B) + \varepsilon'.
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \to d(A,B) + \varepsilon'.$$

By using the triangle inequality, we get

$$\begin{aligned}
& \frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k+2}) + d(y_{2m_k+2}, y_{2n_k+3}) + d(y_{2n_k+3}, y_{2n_k+1}) \\
& + d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+2}, z_{2n_k+3}) + d(z_{2n_k+3}, z_{2n_k+1})] \\
=\ & \frac{1}{3}[d(x_{2m_k}, x_{2m_k+2}) \\
& + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}), F(x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2})) + d(x_{2n_k+3}, x_{2n_k+1}) \\
& + d(y_{2m_k}, y_{2m_k+2}) \\
& + d(G(y_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}), F(y_{2n_k+2}, x_{2n_k+2}, y_{2n_k+2})) + d(y_{2n_k+3}, y_{2n_k+1}) \\
& + d(z_{2m_k}, z_{2m_k+2}) \\
& + d(G(z_{2m_k+1}, y_{2m_k+1}, x_{2m_k+1}), F(z_{2n_k+2}, y_{2n_k+2}, x_{2n_k+2})) + d(z_{2n_k+3}, z_{2n_k+1})] \\
\leqslant\ & \frac{1}{3}\Big[d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{3}(d(x_{2m_k+1}, x_{2n_k+2}) + d(y_{2m_k+1}, y_{2n_k+2}) \\
& + d(z_{2m_k+1}, z_{2n_k+2})) + (1-\alpha)d(A,B) + d(x_{2n_k+3}, x_{2n_k+1})
\end{aligned}$$

$$
\begin{aligned}
&+d(y_{2m_k},y_{2m_k+2})+\frac{\alpha}{3}(d(y_{2m_k+1},y_{2n_k+2})+d(x_{2m_k+1},x_{2n_k+2})\\
&+d(y_{2m_k+1},y_{2n_k+2}))+(1-\alpha)d(A,B)+d(y_{2n_k+3},y_{2n_k+1})\\
&+d(z_{2m_k},z_{2m_k+2})+\frac{\alpha}{3}(d(z_{2m_k+1},z_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})\\
&+d(x_{2m_k+1},x_{2n_k+2}))+(1-\alpha)d(A,B)+d(z_{2n_k+3},z_{2n_k+1})\Big]\\
=&\frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha}{3}(d(x_{2m_k+1},x_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(z_{2m_k+1},z_{2n_k+2}))\\
&+(1-\alpha)d(A,B)\\
=&\frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha}{3}[d(F(x_{2m_k},y_{2m_k},z_{2m_k}),G(x_{2n_k+1},y_{2n_k+1},z_{2n_k+1}))\\
&+d(F(y_{2m_k},x_{2m_k},y_{2m_k}),G(y_{2n_k+1},x_{2n_k+1},y_{2n_k+1}))\\
&+d(F(z_{2m_k},y_{2m_k},x_{2m_k}),G(z_{2n_k+1},y_{2n_k+1},x_{2n_k+1}))]\\
&+(1-\alpha)d(A,B)\\
\leqslant&\frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})\\
&+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&+\frac{\alpha^2}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]\\
&+(1-\alpha^2)d(A,B).
\end{aligned}
$$

Taking $k\to\infty$, we get

$$
d(A,B)+\varepsilon'\leqslant\alpha^2[d(A,B)+\varepsilon']+(1-\alpha^2)d(A,B)=d(A,B)+\alpha^2\varepsilon',
$$

which is a contradiction. Therefore, we can conclude that (4) holds. This completes the proof. □

LEMMA 3. *Let A, B be nonempty subsets of a metric space (X,d) such that the pairs (A,B), (B,A) satisfy the property UC^*. Let $F:A^3\to B$, $G:B^3\to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For $(x_0,y_0,z_0)\in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$
\begin{aligned}
x_{2n+1}&=F(x_{2n},y_{2n},z_{2n}), & x_{2n+2}&=G(x_{2n+1},y_{2n+1},z_{2n+1}),\\
y_{2n+1}&=F(y_{2n},x_{2n},y_{2n}), & y_{2n+2}&=G(y_{2n+1},x_{2n+1},y_{2n+1}),\\
z_{2n+1}&=F(z_{2n},y_{2n},x_{2n}), & z_{2n+2}&=G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}
$$

for all $n\in\mathbb{N}\cup\{0\}$. Then the sequences $\{x_{2n}\}$, $\{y_{2n}\}$, $\{z_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n+1}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences.

Proof. By Lemma 1, we have

$$d(x_{2n},x_{2n+1}) \to d(A,B), \quad d(x_{2n+1},x_{2n+2}) \to d(A,B).$$

Since the pair (A,B) satisfies the property UC, we get $d(x_{2n},x_{2n+2}) \to 0$. Similarly, we also have $d(x_{2n+1},x_{2n+3}) \to 0$ since the pair (B,A) satisfies the property UC.

Now, we show that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_{2m},x_{2n+1}) \leqslant d(A,B) + \varepsilon \tag{5}$$

for all $m > n \geqslant N$. Suppose that (5) does not hold. Then there exists $\varepsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $m_k > n_k \geqslant k$ such that

$$d(x_{2m_k},x_{2n_k+1}) > d(A,B) + \varepsilon. \tag{6}$$

Further, corresponding to n_k, we can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying above relation. Now, we have

$$\begin{aligned} d(A,B) + \varepsilon &< d(x_{2m_k},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k-2}) + d(x_{2m_k-2},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k-2}) + d(A,B) + \varepsilon. \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k},x_{2n_k+1}) \to d(A,B) + \varepsilon$. By Lemma 2, there exists $N \in \mathbb{N}$ such that

$$\frac{1}{3}[d(x_{2m_k},x_{2n_k+1}) + d(y_{2m_k},y_{2n_k+1}) + d(z_{2m_k},z_{2n_k+1})] < d(A,B) + \varepsilon \tag{7}$$

for all $m > n \geqslant \mathbb{N}$. By using the triangle inequality, we get

$$\begin{aligned} & d(A,B) + \varepsilon \\ &< d(x_{2m_k},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k+2}) + d(x_{2m_k+2},x_{2n_k+3}) + d(x_{2n_k+3},x_{2n_k+1}) \\ &= d(x_{2m_k},x_{2m_k+2}) \\ &\quad + d(G(x_{2m_k+1},y_{2m_k+1},z_{2m_k+1}),F(x_{2n_k+2},y_{2n_k+2},z_{2n_k+2})) + d(x_{2n_k+3},x_{2n_k+1}) \\ &\leqslant d(x_{2m_k},x_{2m_k+2}) + \frac{\alpha}{3}[d(x_{2m_k+1},x_{2n_k+2}) + d(y_{2m_k+1},y_{2n_k+2}) + d(z_{2m_k+1},z_{2n_k+2})] \\ &\quad + (1-\alpha)d(A,B) + d(x_{2n_k+3},x_{2n_k+1}) \\ &= \frac{\alpha}{3}[d(F(x_{2m_k},y_{2m_k},z_{2m_k}),G(x_{2n_k+1},y_{2n_k+1},z_{2n_k+1})) \\ &\quad + d(F(y_{2m_k},x_{2m_k},y_{2m_k}),G(y_{2n_k+1},x_{2n_k+1},y_{2n_k+1})) \\ &\quad + d(F(z_{2m_k},y_{2m_k},x_{2m_k}),G(z_{2n_k+1},y_{2n_k+1},x_{2n_k+1}))] \\ &\quad + (1-\alpha)d(A,B) + d(x_{2m_k},x_{2m_k+2}) + d(x_{2n_k+3},x_{2n_k+1}) \\ &\leqslant \frac{\alpha}{3}\left[\frac{\alpha}{3}[d(x_{2m_k},x_{2n_k+1}) + d(y_{2m_k},y_{2n_k+1}) + d(z_{2m_k},z_{2n_k+1})] + (1-\alpha)d(A,B)\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{3}[d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})]+(1-\alpha)d(A,B)\\
&+\frac{\alpha}{3}[d(z_{2m_k},z_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})]+(1-\alpha)d(A,B)\Big]\\
&+(1-\alpha)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
=\ &\frac{\alpha^2}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]\\
&+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
<\ &\alpha^2(d(A,B)+\varepsilon)+(1-\alpha^2)d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})\\
=\ &\alpha^2\varepsilon+d(A,B)+d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1}).
\end{aligned}
$$

Taking $k\to\infty$, we get

$$d(A,B)+\varepsilon\leqslant d(A,B)+\alpha^2\varepsilon,$$

which is a contradiction. Therefore, the condition (5) holds. Since (5) holds and $d(x_{2n},x_{2n+1})\to d(A,B)$, by using the property UC* of (A,B), we deduce that $\{x_{2n}\}$ is a Cauchy sequence. In a similar way, we can prove that $\{y_{2n}\}$, $\{z_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n+1}\}$ and $\{z_{2n+1}\}$ are Cauchy sequences. This completes the proof. □

Here, we state the main result of this article on the existence and convergence of tripled best proximity points for cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property UC*.

THEOREM 2. *Let A, B be nonempty closed subsets of a metric space (X,d) such that the pairs (A,B) and (B,A) have the property UC* and $F:A^3\to B$, $G:B^3\to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For any $(x_0,y_0,z_0)\in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$
\begin{aligned}
x_{2n+1}&=F(x_{2n},y_{2n},z_{2n}), & x_{2n+2}&=G(x_{2n+1},y_{2n+1},z_{2n+1}),\\
y_{2n+1}&=F(y_{2n},x_{2n},y_{2n}), & y_{2n+2}&=G(y_{2n+1},x_{2n+1},y_{2n+1}),\\
z_{2n+1}&=F(z_{2n},y_{2n},x_{2n}), & z_{2n+2}&=G(z_{2n+1},y_{2n+1},x_{2n+1})
\end{aligned}
$$

for all $n\in\mathbb{N}\cup\{0\}$. Then F has a tripled best proximity point $(p,q,r)\in A^3$ and G has a tripled best proximity point $(p',q',r')\in B^3$. Moreover, we have

$$x_{2n}\to p,\ y_{2n}\to q,\ z_{2n}\to r,\ x_{2n+1}\to p',\ y_{2n+1}\to q',\ z_{2n+1}\to r'.$$

Furthermore, if $q=r$ and $q'=r'$, then

$$d(p,p')+d(q,q')+d(r,r')=3d(A,B).$$

Proof. By Lemma 1, we get $d(x_{2n},x_{2n+1})\to d(A,B)$. Using Lemma 3, we see that $\{x_{2n}\}$, $\{y_{2n}\}$ and $\{z_{2n}\}$ are Cauchy sequences. Thus there exist $p,q,r\in A$ such that $x_{2n}\to p$, $y_{2n}\to q$ and $z_{2n}\to r$. Now, we obtain

$$d(A,B)\leqslant d(p,x_{2n-1})\leqslant d(p,x_{2n})+d(x_{2n},x_{2n-1}). \tag{8}$$

Letting $n \to \infty$ in (8), we have $d(p, x_{2n-1}) \to d(A,B)$. By the similar argument, we also have $d(q, y_{2n-1}) \to d(A,B)$ and $d(r, z_{2n-1}) \to d(A,B)$. It follows that

$$\begin{aligned} d(x_{2n}, F(p,q,r)) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(p,q,r)) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n-1}, p) + d(y_{2n-1}, q) + d(z_{2n-1}, r)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $n \to \infty$, we get $d(p, F(p,q,r)) = d(A,B)$. Similarly, we can prove that

$$d(q, F(q,p,q)) = d(A,B), \quad d(r, F(r,q,p)) = d(A,B).$$

Therefore, (p,q,r) is a tripled best proximity point of F.

By the similar way, we can prove that there exist $p', q', r' \in B$ such that $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$. Moreover, we have

$$d(p', G(p',q',r')) = d(A,B), \quad d(q', F(q',p',q')) = d(A,B)$$

and

$$d(r', F(r',q',p')) = d(A,B)$$

and so (p',q',r') is a tripled best proximity point of G.

Finally, we assume that $q = r$ and $q' = r'$ and then we show that

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

For all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}), F(x_{2n}, y_{2n}, z_{2n})) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n}) + d(z_{2n-1}, z_{2n})] + (1-\alpha)d(A,B). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(p,p') \leqslant \frac{\alpha}{3}[d(p,p') + d(q,q') + d(r,r')] + (1-\alpha)d(A,B). \tag{9}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(G(y_{2n-1}, x_{2n-1}, y_{2n-1}), F(y_{2n}, x_{2n}, y_{2n})) \\ &\leqslant \frac{\alpha}{3}[d(y_{2n-1}, y_{2n}) + d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n})] + (1-\alpha)d(A,B). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(q,q') &\leqslant \frac{\alpha}{3}[d(q,q') + d(p,p') + d(q,q')] + (1-\alpha)d(A,B) \\ &= \frac{\alpha}{3}[d(q,q') + d(p,p') + d(r,r')] + (1-\alpha)d(A,B). \end{aligned} \tag{10}$$

Similarly, we have,

$$d(r,r') \leqslant \frac{\alpha}{3}[d(p,p') + d(q,q') + d(r,r')] + (1-\alpha)d(A,B). \tag{11}$$

It follows from (9), (10) and (11) that

$$d(p,p') + d(q,q') + d(r,r') \leqslant \alpha[d(p,p') + d(q,q') + d(r,r')] + 3(1-\alpha)d(A,B)$$

which implies that

$$d(p,p') + d(q,q') + d(r,r') \leqslant 3d(A,B). \tag{12}$$

Since $d(A,B) \leqslant d(p,p')$, $d(A,B) \leqslant d(q,q')$ and $d(A,B) \leqslant d(r,r')$, we have

$$d(p,p') + d(q,q') + d(r,r') \geqslant 3d(A,B). \tag{13}$$

From (12) and (13), we get

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B). \tag{14}$$

This completes the proof. □

Note that every pair of nonempty closed subsets A,B of a uniformly convex Banach space X such that A is convex satisfies the property UC*. Therefore, we obtain the following corollary.

COROLLARY 1. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and $F : A^3 \to B, G : B^3 \to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad x_{2n+2} = G(x_{2n+1},y_{2n+1},z_{2n+1}), \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad y_{2n+2} = G(y_{2n+1},x_{2n+1},y_{2n+1}), \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad z_{2n+2} = G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a tripled best proximity point $(p,q,r) \in A^3$ and G has a tripled best proximity point $(p',q',r') \in B^3$. Moreover, we have

$$x_{2n} \to p, \; y_{2n} \to q, \; z_{2n} \to r, \; x_{2n+1} \to p', \; y_{2n+1} \to q', \; z_{2n+1} \to r'.$$

Furthermore, if $q = r$ and $q' = r'$, then

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

Next, we give an example to illustrate Corollary 1.

EXAMPLE 5. Consider a uniformly convex Banach space $X = \mathbb{R}$ with the usual norm and let $A = [1,3]$ and $B = [-3,-1]$. Thus $d(A,B) = 2$. Define two mappings $F : A^3 \to B$ and $G : B^3 \to A$ by

$$F(x,y,z) = \frac{-x-y-z-3}{6}, \quad G(u,v,w) = \frac{-u-v-w+3}{6}$$

for all $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$, respectively. For any $(x,y,z) \in A^3$ and $(u,v,w) \in B^3$ and fixed $\alpha = \frac{1}{2}$, we get

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left| \frac{-x-y-z-3}{6} - \frac{-u-v-w+3}{6} \right| \\ &\leqslant \frac{|x-u|+|y-v|+|z-w|}{6} + 1 \\ &= \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)] + (1-\alpha)d(A,B) \end{aligned}$$

This implies that (F,G) is a cyclic contraction with $\alpha = \frac{1}{2}$. Since A and B are closed convex, the pairs (A,B) and (B,A) satisfy the property UC*. Therefore, all the hypothesis of Corollary 1 hold. Therefore, F has a tripled best proximity point and G has a tripled best proximity point. We note that a point $(1,1,1) \in A^3$ is a unique tripled best proximity point of F and a point $(-1,-1,-1) \in B^3$ is a unique tripled best proximity point of G. Furthermore, we get

$$d(1,-1)+d(1,-1)+d(1,-1) = 6 = 3d(A,B).$$

Next, we give the tripled best proximity point result in compact subsets of metric spaces.

THEOREM 3. *Let A, B be nonempty compact subsets of a metric space (X,d) and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$ we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}), \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}), \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a tripled best proximity point $(p,q,r) \in A^3$ and G has a tripled best proximity point $(p',q',r') \in B^3$. Moreover, we have

$$x_{2n} \to p, \ y_{2n} \to q, \ z_{2n} \to r, \ x_{2n+1} \to p', \ y_{2n+1} \to q', \ z_{2n+1} \to r'.$$

Furthermore, if $q = r$ and $q' = r'$, then

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B).$$

Proof. Since $x_0,y_0,z_0 \in A$ and

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad & x_{2n+2} &= G(x_{2n+1},y_{2n+1},z_{2n+1}) \\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad & y_{2n+2} &= G(y_{2n+1},x_{2n+1},y_{2n+1}) \\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad & z_{2n+2} &= G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $x_{2n},y_{2n},z_{2n} \in A$ and $x_{2n+1},y_{2n+1},z_{2n+1} \in A$ for all $n \in \mathbb{N} \cup \{0\}$. Since A is compact, the sequences $\{x_{2n}\}$, $\{y_{2n}\}$ and $\{z_{2n}\}$ have the convergent subsequences $\{x_{2n_k}\}$, $\{y_{2n_k}\}$ and $\{z_{2n_k}\}$, respectively, such that

$$x_{2n_k} \to p \in A, \;\; y_{2n_k} \to q \in A, \;\; z_{2n_k} \to r \in A.$$

Now, we have

$$d(A,B) \leqslant d(p, x_{2n_k-1}) \leqslant d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}). \tag{15}$$

By Lemma 1, we have $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. Taking $k \to \infty$ in (15), we get

$$d(p, x_{2n_k-1}) \to d(A,B).$$

By the similar argument, we observe that

$$d(q, x_{2n_k-1}) \to d(A,B), \;\; d(r, x_{2n_k-1}) \to d(A,B).$$

Note that

$$\begin{aligned} d(A,B) &\leqslant d(x_{2n_k}, F(p,q,r)) = d(G(x_{2n_k-1}, y_{2n_k-1}, z_{2n_k-1}), F(p,q,r)) \\ &\leqslant \frac{\alpha}{3}[d(x_{2n_k-1}, p) + d(y_{2n_k-1}, q) + d(z_{2n_k-1}, r)] + (1-\alpha)d(A,B). \end{aligned}$$

Taking $k \to \infty$, we get $d(p, F(p,q,r)) = d(A,B)$. Similarly, we can prove that

$$d(q, F(q,p,q)) = d(A,B), \quad d(r, F(r,q,p)) = d(A,B).$$

Thus F has a tripled best proximity $(p,q,r) \in A^3$. In a similar way, since B is compact, we can also prove that G has a tripled best proximity point $(p',q',r') \in B^3$. To prove

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B),$$

we follows the step of the proof of Theorem 2. This completes the proof. □

3. Tripled fixed point theorems

In this section, we give a new tripled fixed point theorem for a cyclic contraction pair and give one example to illustrate the result.

THEOREM 4. *Let A, B be nonempty closed subsets of a metric space (X,d) and $F : A^3 \to B$, $G : B^3 \to A$ be two mappings such that the ordered pair (F,G) is a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequence $\{x_n\},\{y_n\},\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}, z_{2n}), \quad x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}), \\ y_{2n+1} &= F(y_{2n}, x_{2n}, y_{2n}), \quad y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1}), \\ z_{2n+1} &= F(z_{2n}, y_{2n}, x_{2n}), \quad z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $d(A,B) = 0$, then F has a tripled fixed point $(p,q,r) \in A^3$ and G has a tripled fixed point $(p',q',r') \in B^3$. Moreover, we have

$$x_{2n} \to p, \;\; y_{2n} \to q, \;\; z_{2n} \to r, \;\; x_{2n+1} \to p', \;\; y_{2n+1} \to q', \;\; z_{2n+1} \to r'.$$

Furthermore, if $q = r$ and $q' = r'$, then F and G have a common tripled fixed point in $(A \cap B)^3$.

Proof. Since $d(A,B)=0$, it follows that the pairs (A,B) and (B,A) satisfy the property UC*. Therefore, by Theorem 2, we claim that F has a tripled best proximity point $(p,q,r)\in A^3$, that is,

$$d(p,F(p,q,r)) = d(q,F(q,p,q)) = d(r,F(r,q,p)) = d(A,B) \tag{16}$$

and G has a tripled best proximity point $(p',q',r')\in B^3$, that is,

$$d(p',G(p',q',r')) = d(q',G(q',p',q')) = d(r',G(r',q',p')) = d(A,B). \tag{17}$$

From (16) and $d(A,B)=0$, we conclude that

$$p = F(p,q,r), \quad q = F(q,p,q), \quad r = F(r,q,p),$$

that is, (p,q,r) is a tripled fixed point of F. It follows from (17) and $d(A,B)=0$ that

$$p' = G(p',q',r'), \quad q' = G(q',p',q'), \quad r' = G(r',q',p'),$$

that is, (p',q',r') is a tripled fixed point of G.

Next, we assume that $q=r$ and $q'=r'$ and then we show that F and G have a unique common tripled fixed point in $(A\cap B)^3$. From Theorem 2, we get

$$d(p,p') + d(q,q') + d(r,r') = 3d(A,B). \tag{18}$$

Since $d(A,B)=0$, we get

$$d(p,p') + d(q,q') + d(r,r') = 0$$

which implies that $p=p'$, $q=q'$ and $r=r'$. Therefore, we conclude that $(p,q,r)\in (A\cap B)^3$ is common tripled fixed point of F and G. This completes the proof. □

Next, we give one example to illustrate Theorem 4.

EXAMPLE 6. Consider a space $X=\mathbb{R}$ with the usual metric and let $A=[-2,0]$ and $B=[0,2]$. Define two mappings $F: A^3\to B$ and $G: B^3\to A$ by

$$F(x,y,z) = -\frac{x+y+z}{6}, \quad G(x,y,z) = -\frac{u+v+w}{6}$$

for all $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, respectively. Then $d(A,B)=0$ and the ordered pair (F,G) is a cyclic contraction with $\alpha=\frac{1}{2}$. Indeed, for any $(x,y,z)\in A^3$ and $(u,v,w)\in B^3$, we have

$$\begin{aligned} d(F(x,y,z),G(u,v,w)) &= \left| -\frac{x+y+z}{6} + \frac{u+v+w}{6} \right| \\ &\leqslant \frac{1}{6}(|x-u| + |y-v| + |z-w|) \\ &\leqslant \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)] + (1-\alpha)d(A,B). \end{aligned}$$

Therefore, all the hypothesis of Theorem 4 hold. Therefore, F and G have a common tripled fixed point and this point is $(0,0,0)\in (A\cap B)^3$.

If we take $A = B$ in Theorem 4, then we get the following results.

COROLLARY 2. *Let A be a nonempty closed subset of a complete metric space (X,d) and $F : A^3 \to A$, $G : A^3 \to A$ be two mappings such that the ordered pair (F,G) be a cyclic contraction. For any $(x_0,y_0,z_0) \in A^3$, we define the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ in X by*

$$\begin{aligned} x_{2n+1} &= F(x_{2n},y_{2n},z_{2n}), \quad x_{2n+2} = G(x_{2n+1},y_{2n+1},z_{2n+1}),\\ y_{2n+1} &= F(y_{2n},x_{2n},y_{2n}), \quad y_{2n+2} = G(y_{2n+1},x_{2n+1},y_{2n+1}),\\ z_{2n+1} &= F(z_{2n},y_{2n},x_{2n}), \quad z_{2n+2} = G(z_{2n+1},y_{2n+1},x_{2n+1}) \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Then F has a tripled fixed point $(p,q,r) \in A^3$ and G has a tripled fixed point $(p',q',r') \in A^3$. Moreover, we have

$$x_{2n} \to p, \; y_{2n} \to q, \; z_{2n} \to r, \; x_{2n+1} \to p', \; y_{2n+1} \to q', \; z_{2n+1} \to r'.$$

Furthermore, if $q = r$ and $q' = r'$, then F and G have a common tripled fixed point in A^3.

If we take $F = G$ in Corollary 2, then we get the following results.

COROLLARY 3. *Let A be nonempty closed subsets of a complete metric space (X,d) and $F : A^3 \to A$ be a mapping satisfying*

$$d(F(x,y,z),F(u,v,w)) \leqslant \frac{\alpha}{3}[d(x,u)+d(y,v)+d(z,w)]$$

for all $(x,y,z),(u,v,w) \in A^3$. Then F has a tripled fixed point $(p,q,r) \in A^3$.

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