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SUPERSTABILITY OF THE DIFFERENCE–FORM FUNCTIONAL EQUATIONS RELATED TO DISTANCE MEASURES

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Abstract. The present work extends the study on the stability of the functional equation $f(pr,qs)+f(ps,qr)=f(p,q)f(r,s)$, which arises in the characterization of symmetrically compositive sum-form distance measures, and as a products of some multiplicative functions.

In this paper, we obtain the superstability of the functional equations

$$f(pr,qs)-f(ps,qr)=f(p,q)g(r,s)$$
$$f(pr,qs)-f(ps,qr)=g(p,q)f(r,s)$$
$$f(pr,qs)-f(ps,qr)=g(p,q)g(r,s)$$
$$f(pr,qs)-f(ps,qr)=g(p,q)h(r,s),$$

for all $p,q,r,s\in G$, where G is an Abelian group. These functional equations arise in the characterization of the nonsymmetrically compositive difference-form related to distance measures, products of some multiplicative functions. In reduction, they can be represented as exponential functional equations.

1. Introduction

Let I denote the open unit interval $(0,1)$. Let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $(G,\cdot)$ and $(A,\cdot)$ be a semigroup and Abelian group, respectively. Let $\mathbb{R}_+=\{x\in\mathbb{R}\,|\,x>0\}$ be a set of positive real numbers.

Chung et al. [1] characterized symmetrically compositive sum-form distance measures with a measurable generating function. The following functional equation

$$f(pr,qs)+f(ps,qr)=f(p,q)f(r,s) \qquad (FE)$$

holding for all $p,q,r,s\in G$ was instrumental in the characterization of symmetrically compositive sum-form distance measures. They proved the following theorem giving the general solution of the equation (FE).

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Suppose $f: I^2 \to \mathbb{R}$ satisfies the functional equation (FE), that is,

$$f(pr,qs)+f(ps,qr)=f(p,q)f(r,s)$$

for all $p,q,r,s \in I$. Then,

$$f(p,q)=M_1(p)M_2(q)+M_1(q)M_2(p)$$

where $M_1, M_2 : \mathbb{R} \to \mathbb{C}$ are multiplicative functions. Further, either M_1 and M_2 are both real or M_2 is the complex conjugate of M_1. The converse is also true.

The stability of the functional equation (FE), as well as the four generalizations of (FE), namely

$$f(pr,qs)+f(ps,qr)=f(p,q)g(r,s) \qquad (FE_{fg})$$
$$f(pr,qs)+f(ps,qr)=g(p,q)f(r,s) \qquad (FE_{gf})$$
$$f(pr,qs)+f(ps,qr)=g(p,q)g(r,s) \qquad (FE_{gg})$$
$$f(pr,qs)+f(ps,qr)=g(p,q)h(r,s) \qquad (FE_{gh})$$

for all $p,q,r,s \in G$, were studied by the author and Sahoo in ([4], [5]). For other functional equations similar to (FE), the interested reader should refer to [2], [3], [6], [7], and [8].

Based on the above motivation, let us consider that the characterization of non-symmetrically compositive difference-form related to distance measures:

$$f(pr,qs)-f(ps,qr)=f(p,q)f(r,s) \qquad (DM)$$

for all $p,q,r,s \in G$, which can be represented exponential functional equation in reduction.

This paper aims to investigate the superstability of four generalized functional equations of (DM), namely, as well as that of the following type functional equations:

$$f(pr,qs)-f(ps,qr)=f(p,q)g(r,s) \qquad (M_{fffg})$$
$$f(pr,qs)-f(ps,qr)=g(p,q)f(r,s) \qquad (M_{ffgf})$$
$$f(pr,qs)-f(ps,qr)=g(p,q)g(r,s) \qquad (M_{ffgg})$$
$$f(pr,qs)-f(ps,qr)=g(p,q)h(r,s). \qquad (M_{ffgh})$$

2. Superstability of the equations

In this section, we investigate the superstability of four generalized functional equations (M_{fffg}), (M_{ffgf}), (M_{ffgg}), and (M_{ffgh}).

THEOREM 1. *Let $f,g : G^2 \to \mathbb{R}$ and $\phi : G \to \mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - f(p,q)g(r,s)| \leqslant \varepsilon \qquad \forall\, p,q,r,s \in G. \tag{1}$$

Then either g is bounded or f and g satisfy the equation (M_{fffg})

Proof. Let g be an unbounded solution of inequality (1). Then, there exists a sequence $\{(x_n,y_n)|n \in N\}$ in G^2 such that $0 \neq |g(x_n,y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$, $s = y_n$ in (1), we have

$$f(px_n,qy_n) - f(py_n,qx_n) - f(p,q)g(x_n,y_n)| \leqslant \varepsilon, \tag{2}$$

which is

$$\left|\frac{f(px_n,qy_n) - f(py_n,qx_n)}{g(x_n,y_n)} - f(p,q)\right| \leqslant \frac{\varepsilon}{|g(x_n,y_n)|}. \tag{3}$$

Passing to the limit as $n \to \infty$, we obtain that

$$f(p,q) = \lim_{n\to\infty} \frac{f(px_n,qy_n) - f(py_n,qx_n)}{g(x_n,y_n)} \tag{4}$$

Letting $p = px_n$, $q = qy_n$ in (1), we have

$$|f(prx_n,qsy_n) - f(psx_n,qry_n) - f(px_n,qy_n)g(r,s)| \leqslant \varepsilon,$$

which is

$$\left|\frac{f(prx_n,qsy_n) - f(psx_n,qry_n)}{g(x_n,y_n)} - \frac{f(px_n,qy_n)}{g(x_n,y_n)}g(r,s)\right| \leqslant \frac{\varepsilon}{|g(x_n,y_n)|}.$$

Passing to the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{f(prx_n,qsy_n) - f(psx_n,qry_n)}{g(x_n,y_n)} = \lim_{n\to\infty} \frac{f(px_n,qy_n)}{g(x_n,y_n)} g(r,s) \tag{5}$$

Letting $p = py_n$, $q = qx_n$ in (1), we have

$$|f(pry_n,qsx_n) - f(psy_n,qrx_n) - f(py_n,qx_n)g(r,s)| \leqslant \varepsilon,$$

which is

$$\left|\frac{f(pry_n,qsx_n) - f(psy_n,qrx_n)}{g(x_n,y_n)} - \frac{f(py_n,qx_n)}{g(x_n,y_n)}g(r,s)\right| \leqslant \frac{\varepsilon}{|g(x_n,y_n)|}.$$

Passing to the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{f(pry_n, qsx_n) - f(psy_n, qrx_n)}{g(x_n, y_n)} = \lim_{n\to\infty} \frac{f(py_n, qx_n)}{g(x_n, y_n)} g(r,s) \tag{6}$$

Thus, from (4), (5), and (6), we obtain that

$$\begin{aligned} & f(pr,qs) - f(ps,qr) \\ &= \lim_{n\to\infty} \left(\frac{f(prx_n, qsy_n) - f(pry_n, qsx_n)}{g(x_n, y_n)} - \frac{f(psx_n, qry_n) - f(psy_n, qrx_n)}{g(x_n, y_n)} \right) \\ &= \lim_{n\to\infty} \frac{f(px_n, qy_n) - f(qy_n, px_n)}{g(x_n, y_n)} g(r,s) \\ &= f(p,q)g(r,s). \quad \square \end{aligned}$$

THEOREM 2. *Let $f, g : G^2 \to \mathbb{R}$ and $\phi : G \to \mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - g(p,q)f(r,s)| \leqslant \phi(p,q) \qquad \forall\, p,q,r,s \in G. \tag{7}$$

Then either f is bounded or g satisfies (FE).

Proof. Let f be an unbounded solution of inequality (7). Then, there exists a sequence $\{(x_n, y_n) | n \in N\}$ in G^2 such that $0 \neq |f(x_n, y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$, $s = y_n$ in (7), we have

$$f(px_n, qy_n) - f(py_n, qx_n) - g(p,q)f(x_n, y_n)| \leqslant \phi(p,q),$$

which is

$$\left| \frac{f(px_n, qy_n) - f(py_n, qx_n)}{f(x_n, y_n)} - g(p,q) \right| \leqslant \frac{\phi(p,q)}{|f(x_n, y_n)|}.$$

Passing to the limit as $n \to \infty$ yields

$$g(p,q) = \lim_{n\to\infty} \frac{f(px_n, qy_n) - f(py_n, qx_n)}{f(x_n, y_n)} \tag{8}$$

Letting $r = rx_n$, $s = sy_n$ in (7), we have

$$|f(prx_n, qsy_n) - f(psy_n, qrx_n) - g(p,q)f(rx_n, sy_n)| \leqslant \phi(p,q).$$

Dividing $f(x_n, y_n)$ in the two-side, and passing to the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{f(prx_n, qsy_n) - f(psy_n, qrx_n)}{f(x_n, y_n)} = g(p,q) \lim_{n\to\infty} \frac{f(rx_n, sy_n)}{f(x_n, y_n)}. \tag{9}$$

Letting $r = ry_n$, $s = sx_n$ in (7), we have

$$|f(pry_n, qsx_n) - f(psx_n, qry_n) - g(p,q)f(ry_n, sx_n)| \leqslant \phi(p,q).$$

Dividing $f(x_n,y_n)$, and passing to the limit as $n\to\infty$ yields

$$\lim_{n\to\infty}\frac{f(pry_n,qsx_n)-f(psx_n,qry_n)}{f(x_n,y_n)}=g(p,q)\lim_{n\to\infty}\frac{f(ry_n,sx_n)}{f(x_n,y_n)}. \tag{10}$$

Thus, from (8), (9), and (10), we obtain

$$\begin{aligned}
&g(pr,qs)+g(ps,qr)\\
&=\lim_{n\to\infty}\left(\frac{f(prx_n,qsy_n)-f(pry_n,qsx_n)}{f(x_n,y_n)}+\frac{f(psx_n,qry_n)-f(psy_n,qrx_n)}{f(x_n,y_n)}\right)\\
&=g(p,q)\lim_{n\to\infty}\frac{f(rx_n,sy_n)-f(ry_n,sx_n)}{f(x_n,y_n)}\\
&=g(p,q)g(r,s). \qquad \square
\end{aligned}$$

On inequality (7) of Theorem 2, we can easily check that if f is bounded, then g is bounded. Indeed, let f be bounded, for every $p,q,r,s\in G$, we have

$$|g(p,g)f(r,s)|\leqslant\phi(p,q)+|f(pr,qs)-f(ps,qr)|.$$

Therefore, g is bounded, thus we obtain the following.

COROLLARY 1. *Let $f,g:G^2\to\mathbb{R}$ be functions satisfying*

$$|f(pr,qs)-f(ps,qr)-g(p,q)f(r,s)|\leqslant\phi(p,q)\qquad\forall\, p,q,r,s\in G. \tag{11}$$

Then either g is bounded or g satisfies (FE).

THEOREM 3. *Let $f,g:G^2\to\mathbb{R}$ and $\phi:G\to\mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs)-f(ps,qr)-g(p,q)f(r,s)|\leqslant\phi(r,s)\qquad\forall\, p,q,r,s\in G. \tag{12}$$

and $|f(p,q)\pm g(p,q)|\leqslant M$ for all $p,q\in G$ and some constant M.

Then, either g is bounded or f satisfies the equation (DM).

In particular, if the above bounded M condition is replaced by g that satisfies (FE), then f and g satisfy (M_{ffgf}).

Proof. Let g be an unbounded solution of inequality (12). Then, there exists a sequence $\{(x_n,y_n)|n\in N\}$ in G^2 such that $0\neq|g(x_n,y_n)|\to\infty$ as $n\to\infty$.

Letting $p=x_n$, $q=y_n$ in (12), we have

$$f(x_nr,y_ns)-f(x_ns,y_nr)-g(x_n,y_n)f(r,s)|\leqslant\phi(r,s).$$

Dividing $f(x_n,y_n)$, and passing to the limit as $n\to\infty$, we obtain that

$$f(r,s)=\lim_{n\to\infty}\frac{f(x_nr,y_ns)-f(x_ns,y_nr)}{g(x_n,y_n)}, \tag{13}$$

which means

$$f(r,s) = -f(s,r) \qquad \forall\, r,s \in G. \tag{14}$$

Letting $p = x_n p$, $q = y_n q$ in (12), we have

$$f(x_n pr, y_n qs) - f(x_n ps, y_n qr) - g(x_n p, y_n q) f(r,s)| \leqslant \phi(r,s).$$

Dividing $f(x_n, y_n)$, passing to the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \frac{f(x_n pr, y_n qs) - f(x_n ps, y_n qr)}{g(x_n, y_n)} = \lim_{n\to\infty} \frac{g(x_n p, y_n q)}{g(x_n, y_n)} f(r,s). \tag{15}$$

Letting $p = x_n q$, $q = y_n p$ in (12), dividing $g(x_n, y_n)$, and passing to the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{f(x_n qr, y_n ps) - f(x_n qs, y_n pr)}{g(x_n, y_n)} = f(r,s) \lim_{n\to\infty} \frac{g(x_n q, y_n p)}{g(x_n, y_n)}. \tag{16}$$

Thus, from (13), (14), (15), and (16), we obtain

$$\begin{aligned}
f(pr,qs) - f(ps,qr) &= f(pr,qs) + f(qr,ps) \\
&= \lim_{n\to\infty} \left(\frac{f(x_n pr, y_n qs) - f(x_n ps, y_n qr)}{g(x_n, y_n)} + \frac{f(x_n qr, y_n ps) - f(x_n qs, y_n pr)}{g(x_n, y_n)} \right) \\
&= f(r,s) \left(\lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p)}{g(x_n, y_n)} \right) \\
&= f(r,s) \left(f(p,q) + \lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p) - f(x_n p, y_n q) + f(x_n q, y_n p)}{g(x_n, y_n)} \right) \\
&= f(p,q) f(r,s),
\end{aligned} \tag{17}$$

where we have from the bounded M condition

$$\begin{aligned}
&\lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p) - f(x_n p, y_n q) + f(x_n q, y_n p)}{g(x_n, y_n)} \\
&\qquad \leqslant \lim_{n\to\infty} \frac{2M}{g(x_n, y_n)} = 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

In particular, if g satisfies (FE), then, from (17) immediately, f and g satisfy that $f(pr,qs) - f(ps,qr) = g(p,q) f(r,s)$. □

COROLLARY 2. *Let $\varepsilon \geqslant 0$ and $f, g : G^2 \to \mathbb{R}$ be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - g(p,q) f(r,s)| \leqslant \varepsilon \qquad \forall\, p,q,r,s \in G. \tag{18}$$

Then,

(i) either f is bounded or g satisfies (FE).

(ii) either g is bounded, or g satisfies (FE) and also f and g satisfy (M_{ffgf}).

Proof. From Theorem 2, we obtain (i) immediately. For (ii), from Corollary 1 and equation (17) of Theorem 3, we have that f and g satisfy the equation (M_{ffgf}). □

THEOREM 4. *Let $f,g : G^2 \to \mathbb{R}$ and $\phi : G \to \mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - g(p,q)g(r,s)| \leqslant \phi(p,q) \qquad \forall\, p,q,r,s \in G, \tag{19}$$

and $|f(p,q) - g(p,q)| \leqslant M$ for all $p,q \in G$ and some constant M.
Then, either g is bounded or g satisfies (FE).

Proof. For g to be an unbounded solution of inequality (19), we can choose a sequence $\{(x_n,y_n) | n \in N\}$ in G^2 such that $0 \neq |g(x_n,y_n)| \to \infty$ as $n \to \infty$.

Letting $r = x_n$, $s = y_n$ in (19), we have

$$|f(px_n,qy_n) - f(py_n,qx_n) - g(p,q)g(x_n,y_n)| \leqslant \phi(p,q).$$

Dividing $g(x_n,y_n)$, and passing to the limit as $n \to \infty$, we obtain that

$$g(p,q) = \lim_{n\to\infty} \frac{f(px_n,qy_n) - f(py_n,qx_n)}{g(x_n,y_n)}. \tag{20}$$

Replacing $r = rx_n$, $s = sy_n$ in (19), we have

$$|f(prx_n,qsy_n) - f(psy_n,qrx_n) - g(p,q)g(rx_n,sy_n)| \leqslant \phi(p,q).$$

Dividing $g(x_n,y_n)$, and passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \frac{f(prx_n,qsy_n) - f(psy_n,qrx_n)}{g(x_n,y_n)} = g(p,q) \lim_{n\to\infty} \frac{g(rx_n,sy_n)}{g(x_n,y_n)}. \tag{21}$$

Replacing $r = ry_n$, $s = sx_n$ in (19), we have

$$|f(pry_n,qsx_n) - f(psx_n,qry_n) - g(p,q)g(ry_n,sx_n)| \leqslant \phi(p,q).$$

Dividing $g(x_n,y_n)$, and passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \frac{f(pry_n,qsx_n) - f(psx_n,qry_n)}{g(x_n,y_n)} = g(p,q) \lim_{n\to\infty} \frac{g(ry_n,sx_n)}{g(x_n,y_n)}. \tag{22}$$

Thus from (20), (21), and (22), we obtain

$$\begin{aligned}
&g(pr,qs) + g(ps,qr) \\
&\quad = \lim_{n\to\infty} \frac{f(prx_n,qsy_n) - f(pry_n,qsx_n)}{g(x_n,y_n)} + \lim_{n\to\infty} \frac{f(psx_n,qry_n) - f(psy_n,qrx_n)}{g(x_n,y_n)} \\
&\quad = \lim_{n\to\infty} \left(\frac{f(prx_n,qsy_n) - f(psy_n,qrx_n)}{g(x_n,y_n)} - \frac{f(pry_n,qsx_n) - f(psx_n,qry_n)}{g(x_n,y_n)} \right) \\
&\quad = g(p,q) \lim_{n\to\infty} \frac{g(rx_n,sy_n) - g(ry_n,sx_n)}{g(x_n,y_n)} \\
&\quad = g(p,q)\left(g(r,s) + \lim_{n\to\infty} \frac{g(rx_n,sy_n) - g(ry_n,sx_n) - f(rx_n,sy_n) + f(ry_n,sx_n)}{g(x_n,y_n)} \right) \\
&\quad = g(p,q)g(r,s),
\end{aligned}$$

where we have from the bounded M condition

$$\lim_{n\to\infty}\frac{g(rx_n,sy_n)-g(ry_n,sx_n)-f(rx_n,sy_n)+f(ry_n,sx_n)}{g(x_n,y_n)}$$
$$\leqslant \lim_{n\to\infty}\frac{2M}{g(x_n,y_n)}=0 \quad \text{as} \quad n\to\infty. \quad \square$$

THEOREM 5. *Let $f,g:G^2\to\mathbb{R}$ and $\phi:G\to\mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs)-f(ps,qr)-g(p,q)g(r,s)|\leqslant\phi(r,s) \qquad \forall\, p,q,r,s\in G, \tag{23}$$

and $|f(p,q)\pm g(p,q)|\leqslant M$ for all $p,q\in G$ and some constant M.
Then either g is bounded or g satisfies (DM).

Proof. The proof runs along the same proceedings as Theorem 4. Start by letting $p=x_n$, $q=y_n$ in (23). $\square$

COROLLARY 3. *Let $f,g:G^2\to\mathbb{R}$ and $\phi:G\to\mathbb{R}_+$ be nonzero functions satisfying*

$$|f(pr,qs)-f(ps,qr)-g(p,q)g(r,s)|\leqslant\varepsilon \qquad \forall\, p,q,r,s\in G.$$

If $|f(p,q)-g(p,q)|\leqslant M$ for all $p,q\in G$ and some constant M, then g is bounded.

Proof. Suppose that g is unbounded, then, from Theorem 4 and the proof's process (which is similar to the equation (17) in Theorem 3) of Theorem 5, g satisfies the equation (*FE*) and (*DM*), respectively. It implies that f is zero, which encounters the contradiction. Hence g is bounded. $\square$

THEOREM 6. *Let $f,g,h:G^2\to\mathbb{R}$ and $\phi:G\to\mathbb{R}_+$ be functions satisfying*

$$|f(pr,qs)-f(ps,qr)-g(p,q)h(r,s)|\leqslant\phi(r,s) \qquad \forall\, p,q,r,s\in G, \tag{24}$$

and $|f(p,q)\pm g(p,q)|\leqslant M$ for all $p,q\in G$ and some constant M.
Then either g is bounded or h satisfies (DM).

Proof. Let g be an unbounded. Then we can choose a sequence $\{(x_n,y_n)|n\in N\}$ in G^2 such that $0\neq|g(x_0,y_0)|\to\infty$ as $n\to\infty$.

Letting $p=x_n$, $q=y_n$ in (24), we have

$$|f(x_nr,y_ns)-f(x_ns,y_nr)-g(x_n,y_n)h(r,s)|\leqslant\phi(r,s),$$

which is

$$\left|\frac{f(x_nr,y_ns)-f(x_ns,y_nr)}{g(x_n,y_n)}-h(r,s)\right|\leqslant\frac{\phi(r,s)}{|g(x_n,y_n)|}. \tag{25}$$

Passing to the limit as $n\to\infty$, we obtain that

$$h(r,s)=\lim_{n\to\infty}\frac{f(x_nr,y_ns)-f(x_ns,y_nr)}{g(x_n,y_n)}=-h(s,r) \tag{26}$$

Letting $p = x_n p$, $q = y_n q$ in (24), we have

$$f(x_n pr, y_n qs) - f(x_n ps, y_n qr) - g(x_n p, y_n q)h(r,s)| \leqslant \phi(r,s).$$

Dividing $g(x_n, y_n)$, and passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \frac{f(x_n pr, y_n qs) - f(x_n ps, y_n qr)}{g(x_n, y_n)} = \lim_{n\to\infty} \frac{g(x_n p, y_n q)}{g(x_n, y_n)} h(r,s). \tag{27}$$

Letting $p = x_n q$, $q = y_n p$ in (24), we have

$$|f(x_n qr, y_n ps) - f(x_n qs, y_n pr) - g(x_n q, y_n p)h(r,s)| \leqslant \phi(r,s).$$

Dividing $g(x_n, y_n)$, and passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \frac{f(x_n qr, y_n ps) - f(x_n qs, y_n pr)}{g(x_n, y_n)} = \lim_{n\to\infty} \frac{g(x_n q, y_n p)}{g(x_n, y_n)} h(r,s). \tag{28}$$

From (26), (27), (28), and the bounded M condition, we obtain

$$\begin{aligned} h(pr,qs) - h(ps,qr) &= h(pr,qs) + h(qr,ps) \\ &= \lim_{n\to\infty} \frac{f(x_n pr, y_n qs) - f(x_n ps, y_n qr)}{g(x_n, y_n)} + \lim_{n\to\infty} \frac{f(x_n qr, y_n ps) - f(x_n qs, y_n pr)}{g(x_n, y_n)} \\ &= \lim_{n\to\infty} \frac{g(x_n p, y_n q) + g(x_n q, y_n p)}{g(x_n, y_n)} h(r,s) \\ &= \lim_{n\to\infty} \Big(\frac{g(x_n p, y_n q) - f(x_n p, y_n q) + g(x_n q, y_n p) + f(x_n q, y_n p)}{g(x_n, y_n)} + h(p,q) \Big) h(r,s) \\ &= h(p,q)h(r,s). \quad \square \end{aligned}$$

THEOREM 7. *Let* $f, g, h : G^2 \to \mathbb{R}$ *and* $\phi : G \to \mathbb{R}_+$ *be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - g(p,q)h(r,s)| \leqslant \phi(p,q) \qquad \forall\, p,q,r,s \in G, \tag{29}$$

and $|f(p,q) - g(p,q)| \leqslant M$ *for all* $p,q \in G$ *and some constant* M*.*
Then either h *is bounded or* g *satisfies* (*FE*)*.*

Proof. The proof runs along the same proceedings as Theorem 6. Start by letting $r = x_n$, $s = y_n$ in (29). $\square$

COROLLARY 4. *Let* $f, g, h : G^2 \to \mathbb{R}$ *and* $\phi : G \to \mathbb{R}_+$ *be functions satisfying*

$$|f(pr,qs) - f(ps,qr) - g(p,q)h(r,s)| \leqslant \varepsilon \qquad \forall\, p,q,r,s \in G.$$

(1) If $|f(p,q) - g(p,q)| \leqslant M$ *for all* $p,q \in G$ *and some constant* M*, then* h *is bounded or* g *satisfies* (*FE*)*.*

(2) If $|f(p,q) \pm g(p,q)| \leqslant M$ *for all* $p,q \in G$ *and some constant* M*, then* g *is bounded or* h *satisfies* (*DM*)*.*

3. Extension to the Banach space

Based on all the results presented in Section 2, the range of functions within the Abelian group can be extended to the Banach space. For simplicity, we will only present Theorem 2. Given that the other cases are similar, its illustration will be omitted.

THEOREM 8. *Let* $(E, \|\cdot\|)$ *be a semisimple commutative Banach space. Assume that* $f, g : G^2 \to E$ *satisfy the inequality:*

$$\|f(pr, qs) - f(ps, qr) - g(p,q)f(r,s)\| \leqslant \phi(p,q) \tag{30}$$

for all $p,q,r,s \in G$.

For an arbitrary linear multiplicative functional $x^* \in E^*$, *then, either* $x^* \circ f$ *(or* $x^* \circ g$ *) is bounded or* g *satisfies the equation (FE).*

Proof. Assume that (30) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned}\phi(p,q) &\geqslant \|f(pr,qs) - f(ps,qr) - g(p,q)f(r,s)\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(f(pr,qs) - f(ps,qr) - g(p,q)f(r,s)\right)\right| \\ &\geqslant \left| x^*\left(f(pr,qs)\right) + x^*\left(f(ps,qr)\right) - x^*\left(g(p,q)\right) x^*\left(f(r,s)\right)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield a solution of inequality (7) of Theorem 2. Since, by assumption, the superposition $x^* \circ f$ is unbounded, an appeal to Theorem 2 shows that the function $x^* \circ g$ solves (FE).

In other words, bearing the linear multiplicativity of x^* in mind, for all $p,q,r,s \in G$, the difference $\mathcal{D}g(p,q,r,s) : G^2 \to \mathbb{C}$ defined by

$$\mathcal{D}g(p,q,r,s) := g(pr,qs) - g(ps,qr) - g(p,q)g(r,s)$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$\mathcal{D}g(p,q,r,s) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\}$$

for all $p,q,r,s \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, that is,

$$\mathcal{D}g(p,q,r,s) = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed.

For the case $x^* \circ g$ is bounded, the proof of which runs along as like Corollary 1. □

REMARK 1. i) We obtain the same number of corollaries on the Banach space for all the theorems mentioned in Section 2.

The domains and operations of all the obtained results can be extended to the following cases:

ii) Abelian group : G^2 is replaced by the real field : R^2 and $I^2 = (0,1)^2$.

iii) the operator $\diamond : G^2 \to \mathbb{R}_+$ defined by $f(p,q) = p \diamond q$.

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