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EXISTENCE THEOREMS FOR SOME SYSTEMS OF QUASI–VARIATIONAL INEQUALITIES PROBLEMS ON UNIFORMLY PROX–REGULAR SETS

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Abstract. In this paper, some systems of quasi-variational inequality problems are considered on a class of nonconvex sets, as uniformly prox-regular sets. Some sufficient conditions for the existence solution of the considered problems are provided. Also, some interesting remarks are discussed. The results which are presented in this paper are more general, and may be viewed as an extension, improvement and refinement of the previously known results in the litterateurs.

1. Introduction

Let H be a real Hilbert space and $T : H \to H$ be a mapping. In the early 1960s, Stampacchia [24], introduced the problem of finding $x^* \in K$ such that

$$\langle Tx^*, x - x^* \rangle \geqslant 0, \quad \text{for all} \quad x \in K, \tag{1}$$

where K is a closed convex subset of H. The problem of type (1) is called the variational inequality problem. Since then, the variational inequality theory has become a powerful tool, which is a rich source of inspiration for study the problem in mathematics. Subsequently, it has been studied by many researchers, who generalized and extended such a mentioned problem and, moreover, it has been used to analyze many problems arising in mathematics, economics, engineering sciences and physics(see [3, 5, 7, 15, 21] and the references therein). Roughly speaking, the development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving. In the early 1970s, Bensoussan et al. [1] introduced and studied the concept of the quasi-variational inequality, which is the problem of finding $x^* \in C(x^*)$ such that

$$\langle Tx^*, x - x^* \rangle \geqslant 0, \quad \text{for all} \quad x^* \in C(x^*), \tag{2}$$

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where C is a set-valued mapping on H into itself. Since in many important problems the considered set also depends upon the solutions explicitly or implicitly, it is worth mentioning that the problem of type (2) is of interest to study; see [13] for more details. Obviously, the concept of the quasi-variational inequality contains the variational inequality, introduced by Stampacchia [24], as a special case. On the other direction, in 2001, R. U. Verma [25] introduced the concept of the system for variational inequality problem, by considering the problem of finding $x^*, y^* \in K$ such that

$$\begin{aligned} \langle \rho T y^* + x^* - y^*, x - x^* \rangle \geqslant 0, \quad \text{for all} \quad x \in K, \\ \langle \eta T x^* + y^* - x^*, x - y^* \rangle \geqslant 0, \quad \text{for all} \quad x \in K, \end{aligned} \tag{3}$$

where ρ, η are two fixed positive real numbers and K is a closed convex subset of H. Notice that, the concept of a system of variational inequality is very interesting, this is because a variety of equilibrium models, such as, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem, can be uniformly modelled as a system of variational inequalities.

It is worth to pointed out that the approximate solution of various types of variational and its generalizations were considered by using the projection method and its variant forms. Subsequently, the most problems for solving the existence and iterative approximations of variational inequalities problems have been considered underlying convex sets, which are considered by using the properties of projection operator, for guaranteeing the well definedness of the proposed iterative algorithm. In fact, for this purpose, the convexity assumption may not require because it may be well-defined even the considered set are nonconvexs (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of a Hilbert space, etc.). Motivated by these observations, in 2003, M. Bounkhel et al. [2] studied and analyzed a variational inequality problem on a class of nonconvex set, by considering the following problem: find a point $x^* \in K$ such that

$$-Tx^* \in N_K^P(x^*), \tag{4}$$

where K is a closed subset of a real Hilbert space H and $N_K^P(x)$ denotes for the proximal normal cone to K at x. In a such paper, they proposed some iterative algorithms for finding a solution of type (4), when K is belong to a class of nonconvex sets, namely uniformly prox-regular sets. Later, inspired by R. U. Verma [25] and M. Bounkhel et al. [2], N. Petrot [19] studied a system of variational inequality problem on a class of uniformly prox-regular sets. He considered the problem of finding $x^*, y^* \in K$ such that

$$\begin{aligned} y^* - x^* - \rho T y^* \in N_K^P(x^*), \\ x^* - y^* - \eta T x^* \in N_K^P(y^*), \end{aligned} \tag{5}$$

where ρ, η are two fixed positive real numbers, K is a closed uniformly prox-regular subset of H. Notice that the problem (5), and also the problem (3), are relatively more challenging than the usual variational inequalities, because it can be applied to problems arising, especially from complementarity problems, convex quadratic programming, and other variational problems, see also [11].

In this paper, we will continuing study the problem of type (5) by consider a such problem by using the concept of quasi-variational inequality problem. Some sufficient conditions for the existence solutions of a considered problem will be provided. Also, some remarks related to our main results are discussed.

2. Preliminaries

Now we will provide some basic concepts and results, which will be used in this paper.

Let H be a real Hilbert space equipped with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let 2^H be denoted for the class of all nonempty subsets of H, and K be a nonempty subset of H. We denote by $d(\cdot,K)$ for the usual distance function on H to the subset K, that is, $d(u,K) = \inf_{v\in K}\|u-v\|$ for all $u \in H$.

DEFINITION 1. Let $u \in H$ be a point not lying in K. A point $v \in K$ is called a closest point or a projection of u onto K if $d(u,K) = \|u-v\|$. The set of all such closest points is denoted by $Proj_K(u)$, that is,

$$Proj_K(u) = \{v \in K : d(u,K) = \|u-v\|\}.$$

DEFINITION 2. Let K be a subset of H. The proximal normal cone to K at x is given by

$$N_K^P(x) = \{z \in H : \exists\rho > 0 \quad \text{such that} \quad x \in Proj_K(x+\rho z)\}$$

The following characterization of $N_K^P(x)$ can be found in [6].

LEMMA 1. [2] *Let K be a closed subset of a Hilbert space H. Then*

$$z \in N_K^P(x) \Leftrightarrow \exists\sigma > 0; \quad \langle z, y-x\rangle \leqslant \sigma\|y-x\|^2, \quad \textit{for all} \quad y \in K.$$

Recall that the Clarke normal cone is given by

$$N(K,x) = \overline{co}[N_K^P(x)],$$

where $\overline{co}[S]$ means the closure of the convex hull of S, see [4]. It is clear that one always has $N_K^P(x) \subset N(K,x)$, however, the converse is not true in general. Note that $N(K,x)$ is always a closed and convex cone and that $N_K^P(x)$ is always a convex cone but may be nonclosed (see [4, 6]).

In 1995, Clarke et al. [8] has introduced and studied a new class of nonconvex sets, which are called proximally smooth sets. Notice that, in 2000, this concepts was also considered independently by Poliquin et al. [20]. Later, the proximally smooth sets have been studied by many researchers. In recent years, Bounkhel et al. [2], Cho et al. [5], Moudafi [10], Noor [14, 15], M. A. Noor et al. [17], N. Petrot [19] and Pang et al. [22] have considered both variational inequalities and equilibrium problems in the context of proximally smooth sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique. This class of proximally smooth sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. Next, we will take the following characterization proved in [6] as definition of the proximally smooth set. Note that the original definition was given in terms of the differentiability of the distance function.

DEFINITION 3. For a given $r \in (0,+\infty]$, a subset K of H is said to be *uniformly prox-regular with respect to* r, say, *uniformly* r*-prox-regular sets*, if for all $\overline{x} \in K$ and for all $0 \neq z \in N_K^P(\overline{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x-\overline{x} \right\rangle \leqslant \frac{1}{2r}\|x-\overline{x}\|^2, \quad \text{for all} \quad x \in K.$$

REMARK 1. For the case of $r = \infty$, the uniform r-prox-regularity K is equivalent to the convexity of K. Moreover, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [6, 20].

For each $r \in (0,+\infty]$, from now on, we put

$$K_r := \{x \in H : d(x,K) < r\}.$$

Next, we will recall some results which summarizes the important consequences of the uniformly prox-regularity sets which are needed in the sequel. The proof of this result can be found in [8, 20].

LEMMA 2. *Let* $r \in (0,+\infty]$ *and* K *be a nonempty closed subset of* H*. If* K *is uniformly* r*-prox-regular sets, then the following holds*

(i) For all $x \in K_r, Proj_K(x) \neq \emptyset$*;*

(ii) For all $s \in (0,r), Proj_K$ *is Lipschitz continuous with constant* $\frac{r}{r-s}$ *on* K_s*;*

(iii) The proximal normal cone is closed as a set-valued mapping.

As a direct consequence of Lemma 2 (iii), we obtain that $N(K,x) = N_K^P(x)$.

From now on, we will denote $[Cl(H)]_r$ for the class of all uniformly r-prox regular subset of H, where $r \in (0,+\infty]$ is a fixed positive real number.

Recall that, a set-valued mapping $C : H \to 2^H$ is said to be a κ*-Lipschitz continuous* if there exists $\kappa > 0$ such that

$$|d(y,C(x)) - d(y',C(x'))| \leqslant \|y-y'\| + \kappa\|x-x'\| \tag{6}$$

for all $x,x',y,y' \in H$.

The following result is also needed and important here.

LEMMA 3. [2] *Let* $r \in (0,+\infty]$ *and let* $C : H \to 2^H$ *be a* κ*-Lipschitz continuous set-valued mapping with uniformly* r*-prox-regular values then the following closedness property holds: "For any* $x_n \to x^*, y_n \to y^*$ *and* $u_n \to u^*$ *with* $y_n \in C(x_n)$ *and* $u_n \in N(C(x_n),y_n)$*, one has* $u^* \in N(C(x^*),y^*)$*."*

3. Existence theorem of the system of the quasi-variational inequality on the uniformly prox-regular set

Now, we introduce the main problem of this paper.

Let $T : H \to H$ and $C : H \to 2^H$ be mappings. Let ρ and η be two fixed positive real numbers, in this paper, we are mainly interesting in the following problem: find $x^* \in C(y^*)$ and $y^* \in C(x^*)$ such that

$$\begin{aligned} y^* - x^* - \rho T y^* &\in N^P_{C(y^*)}(x^*), \\ x^* - y^* - \eta T x^* &\in N^P_{C(x^*)}(y^*). \end{aligned} \tag{7}$$

In this case, we will write $(x^*, y^*) \in SQVIP(T, C, \rho, \eta)$. The problem (7) will be called the system of quasi-variational inequality problem.

Next, we collect some special cases of the problem (7).

REMARK 2. If K is a closed subset of H, and $C : H \to 2^H$ is defined by

$$C(x) = K, \quad \text{for all} \quad x \in H. \tag{8}$$

Then, the problem (7) is reduced to the problem of finding $x^*, y^* \in K$ such that

$$\begin{aligned} y^* - x^* - \rho T y^* &\in N^P_K(x^*), \\ x^* - y^* - \eta T x^* &\in N^P_K(y^*). \end{aligned} \tag{9}$$

The problem of type (9) was consider by N. Petrot [19], in 2010.

Further, if K is a closed convex subset of H, one can show that the problem (9) is equivalent to the following problem: find $x^*, y^* \in K$ such that

$$\begin{aligned} \langle \rho T y^* - y^* + x^*, x - x^* \rangle \geqslant 0 \quad &\text{for all} \quad x \in K, \\ \langle \eta T y^* - x^* + y^*, y - y^* \rangle \geqslant 0 \quad &\text{for all} \quad y \in K, \end{aligned}$$

which was introduced by R. U. Verma [25].

In this work, we are interested in the following classes of mappings.

DEFINITION 4. A single-valued mapping $T : H \to H$ is said to be β-*strongly monotone* if there exists $\beta > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geqslant \beta \|x - y\|^2,$$

for all $x, y \in H$.

DEFINITION 5. A single-valued mapping $T : H \to H$ is said to be ξ-*Lipschitz continuous* if there exists $\xi > 0$ such that

$$\|T(x) - T(y)\| \leqslant \xi \|x - y\|,$$

for all $x, y \in H$.

Here, for the sake of simplicity, let us propose the following assumption.

ASSUMPTION $(\mathscr{A})$. Let $T : H \to H$ and $C : H \to [Cl(H)]_r$ be mappings, which are satisfied the following conditions:

(a) T is a β- strongly monotone and a ξ- Lipschitz continuous mapping;

(b) C is a κ- Lipschitzian continuous mapping for some $\kappa \in [0,1)$;

(c) there is $\omega \in [0,1)$ such that

$$\|Proj_{C(x)}(z) - Proj_{C(y)}(z)\| \leqslant \omega\|x-y\|, \quad \text{for all} \quad x,y,z \in H.$$

In order to obtain our main result, We will use the algorithm below as an important tool.

ALGORITHM $(\mathscr{C})$. Let $T : H \to H$ and $C : H \to [Cl(H)]_r$ be mappings. Let x_0 be an element in H, we define the following two-step projection method:

$$\begin{aligned} y_n &= Proj_{C(x_n)}[x_n - \eta T x_n], \\ x_{n+1} &= Proj_{C(y_n)}[y_n - \rho T y_n], \end{aligned} \tag{10}$$

where ρ and η are fixed positive real numbers which were appeared in the problem (7).

REMARK 3. The algorithm (10) is well-defined provided the projection on C is not empty, and our adaptation of the projection algorithm will be base on Lemma 2.

To prove the well-definedness of the sequence $\{x_n\}$ and $\{y_n\}$, proposed in Algorithm $(\mathscr{C})$, we start with an observation.

REMARK 4. Let $T : H \to H$ and $C : H \to [Cl(H)]_r$ be mappings. Let s and η be positive real numbers such that $s \in (0,r)$. If C is a Lipschitz continuous mapping and there is $x_0 \in H$ such that $d(x_0, C(x_0)) \leqslant s - \eta\|Tx_0\|$, then $Proj_{C(x_0)}[x_0 - \eta T x_0] \neq \emptyset$. Indeed, by the Lipschitz continuous of C, we see that

$$\begin{aligned} d(x_0 - \eta T x_0, C(x_0)) &\leqslant d(x_0, C(x_0)) + \eta\|Tx_0\| \\ &\leqslant s - \eta\|Tx_0\| + \eta\|Tx_0\| \\ &< r. \end{aligned}$$

Thus, by Lemma 2 (i), the required result is followed immediately.

Using, Remark 4, we now show the well-definedness of Algorithm $(\mathscr{C})$.

LEMMA 4. *Let* $T : H \to H$ *and* $C : H \to [Cl(H)]_r$ *be mappings. Assume that Assumption* $(\mathscr{A})$ *holds and there are* $\mu > 1$ *and* $x_0 \in H$ *such that*

(i) $\rho, \eta \in \left(0, \frac{s^*}{\delta_T}\right)$, *where* $\delta_T = \sup\{\|Tx\| : x \in H\}$ *and* $s^* = \frac{r(1-\kappa)}{1+\mu\kappa}$, *and* (*ii*) $d(x_0,C(x_0)) \leqslant s^* - \eta\|Tx_0\|$ *and* $d(y_0,C(y_0)) \leqslant s^* - \rho\|Ty_0\|$, *where* $y_0 = Proj_{C(x_0)}[x_0 - \eta Tx_0]$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *in* (10) *are well-defined. Further,*

$$\frac{1}{\phi(\eta)}\|y_{n+1} - y_n\| \leqslant \|x_{n+1} - x_n\| \leqslant \phi(\eta)\phi(\rho)\|x_n - x_{n-1}\|,$$

where $\phi(t) = \frac{1}{\kappa(\mu-1)}\sqrt{1 - 2t\beta + t^2\xi^2} + \omega$.

Proof. Let us consider a positive real number s^* and $x_0 \in H$, which are satisfied condition (i) and (ii). By using the Lipchitz continuity of C and the condition (ii), we see that

$$\begin{aligned} d(y_0 - \rho Ty_0, C(y_0)) &\leqslant d(y_0, C(y_0)) + \rho\|Ty_0\| \\ &\leqslant s^* - \rho\|Ty_0\| + \rho\|Ty_0\| \\ &= s^*. \end{aligned} \tag{11}$$

This means $y_0 - \rho Ty_0 \in [C(y_0)]_{s^*}$. Thus, by Lemma 2 (i), we have $Proj_{C(y_0)}[y_0 - \rho Ty_0] \neq \emptyset$. Let $x_1 \in Proj_{C(y_0)}[y_0 - \rho Ty_0]$. Using the condition (i) and the κ-Lipschitz continuous of C, we obtain

$$\begin{aligned} d(x_1 - \eta Tx_1, C(x_1)) &\leqslant d(x_1, C(x_1)) + \eta\|Tx_1\| \\ &= d(x_1, C(x_1)) - d(x_1, C(y_0)) + \eta\|Tx_1\| \\ &< \kappa\|x_1 - y_0\| + s^*. \end{aligned} \tag{12}$$

Meanwhile, by a choice of x_1 and (11), we have

$$\begin{aligned} \|x_1 - y_0\| &\leqslant \|x_1 - (y_0 - \rho Ty_0)\| + \|(y_0 - \rho Ty_0) - y_0\| \\ &= d(y_0 - \rho Ty_0, C(y_0)) + \rho\|Ty_0\| \\ &< s^* + s^* \\ &= 2s^*. \end{aligned} \tag{13}$$

Replacing (13) into (12), we have

$$\begin{aligned} d(x_1 - \eta Tx_1, C(x_1)) &< 2\kappa s^* + s^* \\ &= s^*(2\kappa + 1) \\ &= s^*\left(\frac{1 + \kappa - 2\kappa^2}{1 - \kappa}\right) \\ &< r. \end{aligned}$$

Then, by Lemma 2 (i), we have $Proj_{C(x_1)}[x_1 - \eta Tx_1] \neq \emptyset$. Let $y_1 \in Proj_{C(x_1)}[x_1 - \eta Tx_1]$, we see that

$$\begin{aligned} d(y_1 - \rho Ty_1, C(y_1)) &\leqslant d(y_1, C(y_1)) + \rho\|Ty_1\| \\ &= d(y_1, C(y_1)) - d(y_1, C(x_1)) + \rho\|Ty_1\| \\ &< \kappa\|y_1 - x_1\| + s^*, \end{aligned} \tag{14}$$

and

$$\begin{aligned}\|y_1 - x_1\| &\leqslant \|y_1 - (x_1 - \eta T x_1)\| + \|x_1 - \eta T x_1 - x_1\| \\ &= d(x_1 - \eta T x_1, C(x_1)) + \eta \|T x_1\| \\ &< s^* \left(\frac{1+\kappa-2\kappa^2}{1-\kappa}\right) + s^*. \end{aligned} \tag{15}$$

Replacing (15) into (14), we obtain that

$$\begin{aligned} d(y_1 - \rho T y_1, C(y_1)) &< s^* \kappa \left(\frac{1+\kappa-2\kappa^2}{1-\kappa}\right) + s^*\kappa + s^* \\ &= s^* \left(\frac{1+\kappa-2\kappa^3}{1-\kappa}\right) \\ &< r. \end{aligned}$$

Hence, $Proj_{C(y_1)}[y_1 - \rho T y_1] \neq \emptyset$. Consequently, we can choose $x_2 \in Proj_{C(y_1)}[y_1 - \rho T y_1]$.

Now, let us assume that

$$d(x_j - \eta T x_j, C(x_j)) < s^* \left(\frac{1+\kappa-2\kappa^{2j}}{1-\kappa}\right)$$

and

$$d(y_j - \rho T y_j, C(y_j)) < s^* \left(\frac{1+\kappa-2\kappa^{2j+1}}{1-\kappa}\right),$$

where $y_j \in proj_{C(x_j)}[x_j - \eta T x_j]$. For $x_{j+1} \in Proj_{C(y_j)}[y_j - \rho T y_j]$, we see that

$$\begin{aligned} d(x_{j+1} - \eta T x_{j+1}, C(x_{j+1})) &\leqslant d(x_{j+1}, C(x_{j+1})) + \eta \|T x_{j+1}\| \\ &< d(x_{j+1}, C(x_{j+1})) - d(x_{j+1}, C(y_j)) + s^* \\ &< \kappa \|x_{j+1} - y_j\| + s^*. \end{aligned} \tag{16}$$

Further,

$$\begin{aligned} \|x_{j+1} - y_j\| &\leqslant \|x_{j+1} - (y_j - \rho T y_j)\| + \|y_j - \rho T y_j - y_j\| \\ &= d(y_j - \rho T y_j, C(y_j)) + \rho \|T y_j\| \\ &< s^* \left(\frac{1+\kappa-2\kappa^{2j+1}}{1-\kappa}\right) + s^* \\ &= 2s^* \left(\frac{1-\kappa^{2j+1}}{1-\kappa}\right). \end{aligned} \tag{17}$$

Replacing (17) into (16), we get

$$\begin{aligned} d(x_{j+1} - \eta T x_{j+1}, C(x_{j+1})) &< 2\kappa s^* \left(\frac{1-\kappa^{2j+1}}{1-\kappa}\right) + s^* \\ &= s^* \left(\frac{1+\kappa-2\kappa^{2(j+1)}}{1-\kappa}\right) \\ &< r. \end{aligned}$$

This means, $Proj_{C(x_{j+1})}[x_{j+1}-\eta Tx_{j+1}]\neq\emptyset$. Let $y_{j+1}\in Proj_{C(x_{j+1})}[x_{j+1}-\eta Tx_{j+1}]$, we have

$$\begin{aligned} d(y_{j+1}-\rho Ty_{j+1},C(y_{j+1})) &\leqslant d(y_{j+1},C(y_{j+1}))+\rho\|Ty_{j+1}\| \\ &< d(y_{j+1},C(y_{j+1}))-d(y_{j+1},C(x_{j+1}))+s^* \\ &< \kappa\|y_{j+1}-x_{j+1}\|+s^*. \end{aligned} \tag{18}$$

Also,

$$\begin{aligned} \|y_{j+1}-x_{j+1}\| &\leqslant \|y_{j+1}-(x_{j+1}-\eta Tx_{j+1})\|+\|x_{j+1}-\eta Tx_{j+1}-x_{j+1}\| \\ &= d(x_{j+1}-\eta Tx_{j+1},C(x_{j+1}))+\eta\|Tx_{j+1}\| \\ &< s^*\left(\frac{1+\kappa-2\kappa^{2(j+1)}}{1-\kappa}\right)+s^* \\ &= 2s^*\left(\frac{1-\kappa^{2(j+1)}}{1-\kappa}\right). \end{aligned} \tag{19}$$

Replacing (19) into (18), we get

$$\begin{aligned} d(y_{j+1}-\rho Ty_{j+1},C(y_{j+1})) &< 2\kappa s^*\left(\frac{1-\kappa^{2(j+1)}}{1-\kappa}\right)+s^* \\ &= s^*\left(\frac{1+\kappa-2\kappa^{2(j+1)+1}}{1-\kappa}\right) \\ &< r. \end{aligned}$$

Hence, from (17) and (19), we have a conclusion that the sequences $\{x_n\}$ and $\{y_n\}$ are well-defined.

Moreover, it is easy to see that

$$y_n-\rho Ty_n\in[C(y_n)]_{\frac{r(1+\kappa)}{1+\mu\kappa}} \quad \text{and} \quad x_n-\eta Tx_n\in[C(x_n)]_{\frac{r(1+\kappa)}{1+\mu\kappa}},$$

for all $n\in\mathbb{N}$. Consequently, from (10) and Lemma 2 (ii), we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|Proj_{C(y_n)}(y_n-\rho Ty_n)-Proj_{C(y_{n-1})}(y_{n-1}-\rho Ty_{n-1})\| \\ &\leqslant \|Proj_{C(y_n)}(y_n-\rho Ty_n)-Proj_{C(y_n)}(y_{n-1}-\rho Ty_{n-1})\| \\ &\quad +\|Proj_{C(y_n)}(y_{n-1}-\rho Ty_{n-1})-Proj_{C(y_{n-1})}(y_{n-1}-\rho Ty_{n-1})\| \\ &\leqslant \frac{1}{\kappa(\mu-1)}\|y_n-y_{n-1}-\rho[Ty_n-Ty_{n-1}]\|+\omega\|y_n-y_{n-1}\|. \end{aligned} \tag{20}$$

Since T is a β- strongly monotone and ξ-Lipschitz continuous mapping, we see that

$$\begin{aligned} &\|y_n-y_{n-1}-\rho[Ty_n-Ty_{n-1}]\|^2 \\ &= \|y_n-y_{n-1}\|^2-2\rho\langle Ty_n-Ty_{n-1},y_n-y_{n-1}\rangle+\rho^2\|Ty_n-Ty_{n-1}\|^2 \\ &\leqslant \|y_n-y_{n-1}\|^2-2\rho\beta\|y_n-y_{n-1}\|^2+\rho^2\xi^2\|y_n-y_{n-1}\|^2 \\ &= (1-2\rho\beta+\rho^2\xi^2)\|y_n-y_{n-1}\|^2. \end{aligned}$$

This means,

$$\|y_n - y_{n-1} - \rho[Ty_n - Ty_{n-1}]\| \leqslant \sqrt{1-2\rho\beta+\rho^2\xi^2}\|y_n - y_{n-1}\|. \tag{21}$$

Replacing (21) into (20), we get

$$\|x_{n+1}-x_n\| \leqslant \left[\frac{1}{\kappa(\mu-1)}\sqrt{1-2\rho\beta+\rho^2\xi^2}+\omega\right]\|y_n - y_{n-1}\|. \tag{22}$$

Similarly, in the same way as obtaining (22), we can show that

$$\|y_{n+1}-y_n\| \leqslant \left[\frac{1}{\kappa(\mu-1)}\sqrt{1-2\eta\beta+\eta^2\xi^2}+\omega\right]\|x_{n+1} - x_n\|. \tag{23}$$

Replacing (23) into (22), we get

$$\begin{aligned}\|x_{n+1}-x_n\| \leqslant &\left[\frac{1}{\kappa(\mu-1)}\sqrt{1-2\rho\beta+\rho^2\xi^2}+\omega\right]\\ &\times\left[\frac{1}{\kappa(\mu-1)}\sqrt{1-2\eta\beta+\eta^2\xi^2}+\omega\right]\|x_n - x_{n-1}\|.\end{aligned}$$

This completes the proof. □

Now, by using Lemma 4, we are in a position to present our main result.

THEOREM 1. (Main) *Let $T : H \to H$ and $C : H \to [Cl(H)]_r$ be mappings. Assume that all assumptions of Lemma 4 are satisfied, and*

(i') $\frac{\beta}{\xi^2} - f(t_{s^*}) < \rho,\ \eta < \min\left\{\frac{\beta}{\xi^2} + f(t_{s^*}), \frac{s^*}{\delta_T}\right\},$

where $f(t) = \frac{\sqrt{\beta^2t^2-\xi^2[t^2-(1-\omega)^2]}}{t\xi^2}$ *for all* $t \in \left[1, \frac{\xi(1-\omega)}{\sqrt{\xi^2-\beta^2}}\right]$, *and* $t_{s^*} = \frac{r}{r-s^*}$. *Then the problem (7) has a solution.*

Proof. Notice that, from condition (i'), one can check that $\phi(\rho)$ and $\phi(\eta)$ are belong to the open interval $(0,1)$. On the other hand, from Lemma 4, we know that

$$\|x_{n+1}-x_n\| \leqslant \phi\|x_n - x_{n-1}\|,$$

where $\phi = \phi(\rho)\phi(\eta)$. This implies that

$$\|x_{n+1}-x_n\| \leqslant \phi^n\|x_1 - x_0\|, \quad \text{for all} \quad n\in\mathbb{N}.$$

Hence, for any $m \geqslant n > 1$, we see that

$$\begin{aligned}\|x_m - x_n\| &\leqslant \Sigma_{i=n}^{m-1}\|x_{i+1}-x_i\|\\ &\leqslant \|x_1-x_0\|\Sigma_{i=n}^{m-1}\phi^i\\ &\leqslant \|x_1-x_0\|\Sigma_{i=n}^{\infty}\phi^i\\ &\leqslant \left(\frac{\phi^n}{1-\phi}\right)\|x_1-x_0\|.\end{aligned}$$

Since $\phi < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in H. Moreover, since $\|y_{n+1} - y_n\| \leqslant \phi(\eta)\|x_{n+1} - x_n\|$, we can show that $\{y_n\}$ is also a Cauchy sequence in H. Thus, by the completeness of H, there exist $x^*, y^* \in H$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

Next, by the κ- Lipschitz continuous of C, we have

$$\begin{aligned} d(x^*, C(y^*)) &= |d(x_{n+1}, C(y_n)) - d(x^*, C(y^*))| \\ &\leqslant \|x_{n+1} - x^*\| + \kappa\|y_n - y^*\|. \end{aligned} \tag{24}$$

It follows that $d(x^*, C(y^*)) = 0$. Similarly, we can show that $d(y^*, C(x^*)) = 0$. So, by the closedness of $C(x^*)$ and $C(y^*)$, we must have $y^* \in C(x^*)$ and $x^* \in C(y^*)$, respectively.

We now claim that $(x^*, y^*) \in SQVIP(T, C, \rho, \eta)$. In fact, from (10), we have

$$\begin{aligned} y_n &= Proj_{C(x_n)}[y_n + (x_n - y_n - \eta T x_n)], \\ x_{n+1} &= Proj_{C(y_n)}[x_{n+1} + (y_n - x_{n+1} - \rho T y_n)]. \end{aligned}$$

That is,

$$\begin{aligned} (x_n - y_n) - \eta T x_n &\in N(C(x_n), y_n), \\ (y_n - x_{n+1}) - \rho T y_n &\in N(C(y_n), x_{n+1}). \end{aligned}$$

Hence, by letting $n \to \infty$, then in view of Lemma 3, we have

$$\begin{aligned} x^* - y^* - \eta T x^* &\in N(C(x^*), y^*), \\ y^* - x^* - \rho T y^* &\in N(C(y^*), x^*). \end{aligned}$$

This means $(x^*, y^*) \in SQVIP(T, C, \rho, \eta)$, and the proof is completed. □

REMARK 5. Let us consider the proposed assumptions of Theorem 1. In the application point of view, one may ask for the best choice of the real number μ, and hence s^*. We would like to notice here that, the real number $\mu = \frac{\kappa\Delta - 1}{\kappa(1-\Delta)}$, where $\Delta = \frac{\xi(1-\omega)}{\sqrt{\xi^2 - \beta^2}}$, should provide the answer. This is because, by the following observation:

- the domain of function f is $\frac{\xi(1-\omega)}{\sqrt{\xi^2-\beta^2}}$,
- $s^* = \frac{r(1-\kappa)}{1+\mu\kappa} \Leftrightarrow t_s = \frac{1+\mu\kappa}{\kappa(1+\mu)}$,
- the function $\mu \mapsto \frac{1+\mu\kappa}{\kappa(1+\mu)}$ is an increasing function on its domain,
- $\frac{1+\mu\kappa}{\kappa(1+\mu)} = \Delta \Leftrightarrow \mu = \frac{\kappa\Delta-1}{\kappa(1-\Delta)}$, where $\Delta = \frac{\xi(1-\omega)}{\sqrt{\xi^2-\beta^2}}$.

Recall that a set-valued mapping $C : H \to 2^H$ is said to be a *Hausdorff Lipschitz continuous* if there exists a real number $\kappa > 0$ such that

$$\mathscr{H}(C(x), C(y)) \leqslant \kappa\|x - y\| \quad \text{for all} \quad x, y \in H,$$

where $\mathscr{H}$ stands for the Hausdorff distance relative to the norm associated with the Hilbert space H, that is,

$$\mathscr{H}(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad \text{for all} \quad A,B \in 2^H.$$

It is easy to check that the class of Lipschitz continuous mappings, which has defined in (6), is larger that the class of Hausdorff Lipschitz continuous mappings. Thus, Theorem 1 can also be applied when the Assumption ($\mathscr{A}$)(b) is replaced by "C is a κ-Hausdorff Lipschitzian continuous set-valued mapping". Moreover, it is well-known that, in this case, a set valued mapping C has a fixed point, see [12]. Thus, we have the following result.

COROLLARY 1. *Let $T : H \to H$ be a mapping and $C : H \to [Cl(H)]_r$ be a Hausdorff Lipschitz continuous mapping. If Assumption $(\mathscr{A})(a)$ and $(\mathscr{A})(c)$ hold and the condition (i') in Theorem 1 is satisfied, then the problem (7) has a solution.*

Proof. The required result is followed immediately from Theorem 1 and above observation. □

It is well known that if K is a closed convex set then it is r-prox-regular set for every $r > 0$ (see [8]). Using this fact, and by careful consideration the proof of Theorem 1, one can see that in this case a control condition (ii) of Lemma 4 can be omitted. That is, we have the following results.

COROLLARY 2. [23] *Let $T : H \to H$ be a single-valued mapping and $C : H \to \mathscr{CC}(H)$ be a set-valued mapping, where $\mathscr{CC}(H)$ is a class of nonempty closed convex subset of H. If Assumption $(\mathscr{A})$ holds and the condition (ii) in Theorem 1 is satisfied, then the problem (7) has a solution.*

If K is a convex subset of H, and a mapping $C : H \to 2^H$ is defined by

$$C(x) = K, \quad \text{for all} \quad x \in H. \tag{25}$$

Then we obtain as the following result immediately.

COROLLARY 3. [19] *Let K be a uniformly r-prox-regular closed subset of a Hilbert space H. Let $T : K \to H$ be a strongly monotone and a Lipschitz continuous mapping. If the condition (i') in Theorem 1 is satisfied, then the problem (7) has a solution.*

In view of Remark 1 and Corollary 3, we also obtain the following result, as a special case.

REMARK 6. In this paper, we not only give the conditions for the existence solutions of the considered problem, but also provide the algorithm to find such solutions.

COROLLARY 4. [25] *Let H be a real Hilbert space and K be a closed convex subset of H and $T : H \to H$ be a single-valued mapping. If T is a strongly monotone and a Lipschitz continuous single-valued mapping, then the problem* (7) *has a solution.*

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