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GENERALIZATION OF JENSEN'S INEQUALITY BY LIDSTONE'S POLYNOMIAL AND RELATED RESULTS

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Abstract. In this paper we consider $(2n)$-convex functions and completely convex functions. Using Lidstone's interpolating polynomials and conditions on Green's functions we present results for Jensen's inequality and converses of Jensen's inequality for signed measure. By using the obtained inequalities, we produce new exponentially convex functions. Finally, we give several examples of the families of functions for which the obtained results can be applied.

1. Introduction

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. The well-known Jensen inequality asserts that

$$f\left(\frac{\int_\Omega pg d\mu}{\int_\Omega pd\mu(x)}\right) \leqslant \frac{\int_\Omega pf(g)d\mu}{\int_\Omega pd\mu}, \tag{1}$$

holds if f is a convex function on interval $I \subseteq \mathbb{R}$, where $g : \Omega \to I$ be a function from $L^\infty(\mu)$ and $p : \Omega \to \mathbb{R}$ be a nonnegative function from $L^1(\mu)$, such that $\int_\Omega pd\mu \neq 0$.

If $I = [\alpha, \beta]$, where $-\infty < \alpha < \beta < +\infty$, and function f is continuous, then the converse of the integral Jensen's inequality states

$$\frac{\int_\Omega pf(g)d\mu}{\int_\Omega pd\mu} \leqslant f(\alpha)\frac{\beta - \bar{g}}{\beta - \alpha} + f(\beta)\frac{\bar{g} - \alpha}{\beta - \alpha}, \tag{2}$$

where $\bar{g} = \frac{\int_\Omega pg\,d\mu}{\int_\Omega pd\mu}$ (see [9] or [14]).

In [4], [11] and [12] authors obtained the generalization for real (signed) measure:

THEOREM 1. *Let $f : [\alpha, \beta] \to \mathbb{R}$ be convex, and $g : [a,b] \to [\alpha, \beta]$ integrable with respect to a real (signed) measure μ. If μ is such that $\int_a^b d\mu(t) = 1$ and*

$$\int_a^b G_L(g(t), s)\, d\mu(t) \leqslant 0, \qquad \forall s \in [\alpha, \beta], \tag{3}$$

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then the inequality

$$\int_a^b f(g(t))d\mu(t) \leqslant f(\alpha)\frac{\beta - \int_a^b g(t)d\mu(t)}{\beta-\alpha} + f(\beta)\frac{\int_a^b g(t)d\mu(t) - \alpha}{\beta-\alpha}, \tag{4}$$

holds, where the Lagrange Green's function on $[\alpha,\beta]\times[\alpha,\beta]$ *is defined by*

$$G_L(t,s) = \begin{cases} \frac{(\alpha-s)(t-\beta)}{\alpha-\beta}, & s \leqslant t \\ -\frac{(\beta-s)(t-\alpha)}{\beta-\alpha}, & s \geqslant t. \end{cases} \tag{5}$$

The reverse inequality in (3) implies the reverse inequality in (4).

Recently, in [13] the following generalization for $m, M \in [\alpha,\beta]$ and Stieltjes measure $d\lambda$ is done:

THEOREM 2. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous function and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $m, M \in [\alpha,\beta]$ $(m \neq M)$ *be such that* $m \leqslant g(t) \leqslant M$ *for all* $t \in [a,b]$. *Let* $\lambda : [a,b] \to \mathbb{R}$ *be a continuous function or a function of bounded variation, and* $\lambda(a) \neq \lambda(b)$. *Then the following two statements are equivalent:*

(1) For every continuous convex function $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$\frac{\int_a^b f(g(t))\,d\lambda(t)}{\int_a^b d\lambda(t)} \leqslant \frac{M-\overline{g}}{M-m}f(m) + \frac{\overline{g}-m}{M-m}f(M).$$

(2) For all $s \in [\alpha,\beta]$ *it holds*

$$\frac{\int_a^b G_L(g(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)} \leqslant \frac{M-\overline{g}}{M-m}G_L(m,s) + \frac{\overline{g}-m}{M-m}G_L(M,s),$$

where G_L *is Green's function defined on* $[\alpha,\beta]\times[\alpha,\beta]$ *by (5), and*

$$\overline{g} = \frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}.$$

Also, in [13] authors established the following generalization of Jensen inequality for Stieltjes measure $d\lambda$:

THEOREM 3. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous function and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $\lambda : [a,b]\to\mathbb{R}$ *be a continuous function or a function of bounded variation,* $\lambda(a)\neq\lambda(b)$ *and* $\overline{g}\in[\alpha,\beta]$. *Then the following two statements are equivalent:*

(1) For every continuous convex function $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$f(\overline{g}) \leqslant \frac{\int_a^b f(g(t))\,d\lambda(t)}{\int_a^b d\lambda(t)}.$$

(2) For all $s \in [\alpha,\beta]$ it holds

$$G_L(\overline{g},s) \leqslant \frac{\int_a^b G_L(g(t),s)\,d\lambda(t)}{\int_a^b d\lambda(t)}.$$

Lidstone series is a generalization of Taylor's series. It approximates a given function in the neighborhood of two points (instead of one). Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others.

DEFINITION 1. Let $f \in C^{\infty}([0,1])$, then the Lidstone series has the form

$$\sum_{k=0}^{\infty}\left(f^{(2k)}(0)\Lambda_k(1-x)+f^{(2k)}(1)\Lambda_k(x)\right),$$

where Λ_n is a polynomial of degree $2n+1$ defined by the relations

$$\begin{aligned}&\Lambda_0(t)=t,\\ &\Lambda_n''(t)=\Lambda_{n-1}(t),\\ &\Lambda_n(0)=\Lambda_n(1)=0, \qquad n\geqslant 1.\end{aligned} \tag{6}$$

Other explicit representations of the Lidstone polynomial are given by [1] and [17],

$$\Lambda_n(t) = (-1)^n\frac{2}{\pi^{2n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2n+1}}\sin k\pi t,$$

$$\Lambda_n(t) = \frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!}-\frac{t^{2n-1}}{(2n-1)!}\right]-\sum_{k=0}^{n-2}\frac{2(2^{2k+3}-1)}{(2k+4)!}B_{2k+4}\frac{t^{2n-2k-3}}{(2n-2k-3)!}, \quad n=1,2,\ldots,$$

$$\Lambda_n(t) = \frac{2^{2n+1}}{(2n+1)!}B_{2n+1}\left(\frac{1+t}{2}\right), \quad n=1,2\ldots,$$

where B_{2k+4} is the $(2k+4)$-th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is the Bernoulli polynomial.

In [19], Widder proved the fundamental lemma:

LEMMA 1. *If $f \in C^{(2n)}([0,1])$, then*

$$f(t)=\sum_{k=0}^{n-1}\left[f^{(2k)}(0)\Lambda_k(1-t)+f^{(2k)}(1)\Lambda_k(t)\right]+\int_0^1 G_n(t,s)f^{(2n)}(s)ds, \tag{7}$$

where

$$G_1(t,s)=G(t,s)=\begin{cases}(t-1)s, & \text{if } s\leqslant t,\\ (s-1)t, & \text{if } t\leqslant s,\end{cases} \tag{8}$$

is homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0,1]$, and with the successive iterates of $G(t,s)$

$$G_n(t,s)=\int_0^1 G_1(t,p)G_{n-1}(p,s)dp, \qquad n\geqslant 2. \tag{9}$$

The Lidstone polynomial can be expressed in terms of $G_n(t,s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t,s)s\,ds.$$

DEFINITION 2. Let f be a real-valued function defined on the segment $[a,b]$. The *divided difference* of order n of the function f at distinct points $x_0,...,x_n \in [a,b]$ is defined recursively (see [2], [14]) by

$$f[x_i] = f(x_i), \quad (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-\text{times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{10}$$

The notion of n*-convexity* goes back to Popoviciu ([16]). We follow the definition given by Karlin ([7]):

DEFINITION 3. A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the n-th order divided difference of f satisfies

$$f[x_0,...,x_n] \geqslant 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[a,b]$ is the uniform limit of the sequence of n-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [8].

Bernstein in [3] introduced the term absolutely monotonic function and completely monotonic function. A function is absolutely monotonic on $[a,b]$ if

$$f^{(k)}(x) \geqslant 0, \quad k = 0,1,\dots$$

and completely monotonic function on $[a,b]$ if

$$(-1)^k f^{(k)}(x) \geqslant 0, \quad k = 0,1,\dots$$

Many studies were made on the influence of the sign of the derivatives of a functions on its analytic character.

Widder in [18] introduces the term of completely convex function:

DEFINITION 4. A real function f is completely convex on the interval $[a,b]$ if it has derivatives of all orders and

$$(-1)^k f^{(2k)}(x) \geqslant 0, \;\; a \leqslant x \leqslant b; \;\; k = 0,1,\ldots$$

For example, the functions $\sin x$ and $\cos x$ are completely convex on the intervals $[0,\pi]$ and $[-\frac{\pi}{2},\frac{\pi}{2}]$, respectively.

In [19], Widder showed that if f is a completely convex function on (a,b) then f can be analytically extended to an entire function of exponential type at most π.

Further, he showed the close connection of completely convex function with Lidstone series, similar to the one that exists between the completely monotonic function and the Taylor's series:

THEOREM 4. *If a real function f is completely convex on $[\alpha,\beta]$ with $(\beta-\alpha)\geqslant 1$ then*

(i) there exists a positive number $p<\pi$ such that $f^{(n)}(x) = O(p^n)$ uniformly on $\alpha \leqslant x \leqslant c$, where $c \in [\alpha,\beta]$, $(\beta - c) > 1$,

(ii) and the equation

$$f(x) = \sum_{k=0}^{\infty} (\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right) + f^{(2k)}(\beta)\Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right)\right] \tag{11}$$

holds.

We will also consider $(2n)$-completely convex function, $(2n)$-absolutely convex function and $(2n)$-convex function:

DEFINITION 5. A real function f is $(2n)$-completely convex on the interval $[\alpha,\beta]$ if it has derivatives of orders $i = 1,2,\ldots,2n$ and if

$$(-1)^k f^{(2k)}(x) \geqslant 0, \;\; \alpha \leqslant x \leqslant \beta, \;\; k = 0,1,\ldots,n.$$

DEFINITION 6. A real function f is $(2n)$-absolutely convex on the interval $[\alpha,\beta]$ if it has derivatives of orders $i = 1,2,\ldots,2n$ and if

$$f^{(2k)}(x) \geqslant 0, \;\; \alpha \leqslant x \leqslant \beta, \;\; k = 0,1,\ldots,n.$$

In this paper, we will obtain some new identities by using the Lidstone polynomials. From these identities we find sufficient conditions on the corresponding Green's functions, on a real signed measure μ, under which generalizations of Jensen's type and conversed Jensen's type inequality are valid. We will give a results for Theorem 2 and Theorem 3 for $(2n)$-completely convex function, $(2n)$-absolutely convex function and $(2n)$-convex function. Also, we give mean value theorems for the obtained inequalities. We will deduce a method of producing n-exponentially convex functions by using some known families of functions.

2. Generalization of Jensen's inequality by Lidstone's polynomial

LEMMA 2. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be of class $C^{(2n)}$ on $[\alpha,\beta]$. Let μ be a regular, real (signed) Borel measure and let $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ such that $g([a,b])\subseteq[\alpha,\beta]$ and $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}\in[\alpha,\beta]$. Then*

$$f(\overline{g})-\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \tag{12}$$
$$=\sum_{k=1}^{n-1}\left\{f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)-\frac{\int_a^b\Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right.$$
$$\left.+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b\Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right\}$$
$$+(\beta-\alpha)^{2n-1}\int_\alpha^\beta f^{(2n)}(s)\left[G_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]ds.$$

Proof. By Widder's lemma we can represent every function $f\in C^{(2n)}([\alpha,\beta])$ in the form:

$$f(x)=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right)+f^{(2k)}(\beta)\Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right)\right]$$
$$+(\beta-\alpha)^{2n-1}\int_\alpha^\beta G_n\left(\frac{x-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)f^{(2n)}(s)ds. \tag{13}$$

The integration of the composition $f\circ g$ for the real measure μ on $[a,b]$ gives

$$\int_a^b f(g(t))\,d\mu(t) \tag{14}$$
$$=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\int_a^b\Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)+f^{(2k)}(\beta)\int_a^b\Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)d\mu(t)\right]$$
$$+(\beta-\alpha)^{2n-1}\int_\alpha^\beta f^{(2n)}(s)\left[\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)\right]ds.$$

By using (13) we calculate $f(\overline{g})$

$$f(\overline{g})=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)+f^{(2k)}(\beta)\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)\right]$$
$$+(\beta-\alpha)^{2n-1}\int_\alpha^\beta G_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)f^{(2n)}(s)\,ds.$$

By easy calculation we obtain the difference

$$
\begin{aligned}
& f(\overline{g})-\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \\
&=\sum_{k=0}^{n-1}\left\{f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)-\frac{\int_a^b \Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right. \\
&\quad \left.+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b \Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right\} \\
&\quad +(\beta-\alpha)^{2n-1}\int_\alpha^\beta f^{(2n)}(s)\left[G_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]ds.
\end{aligned}
$$

Since

$$
\frac{\int_a^b \Lambda_0\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}=\Lambda_0\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right),
$$

we obtain identity (12). □

Using Lemma 2 we can get the following generalization of Jensen's inequality for $(2n)$-convenx function:

THEOREM 5. *Let $n\in\mathbb{N}$, μ be a regular, real (signed)Borel measure and $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ such that $g([a,b])\subseteq[\alpha,\beta]$ and $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}\in[\alpha,\beta]$.*

If for all $s\in[\alpha,\beta]$

$$
G_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)\leqslant\frac{\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}, \tag{15}
$$

then for every $(2n)$-convex function $f:[\alpha,\beta]\to\mathbb{R}$

$$
\begin{aligned}
f(\overline{g}) &\leqslant \frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \\
&\quad+\sum_{k=1}^{n-1}\left\{f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)-\frac{\int_a^b \Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right. \\
&\quad\left.+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b \Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right\}.
\end{aligned} \tag{16}
$$

If the reverse inequality in (15) holds, then also the reverse inequality in (16) holds.

REMARK 1. For $n=1$ in Theorem 5 we obtain Theorem 3 for real (signed) measure. The following two statements are equivalent:

(1) For every continuous convex function $f:[\alpha,\beta]\to\mathbb{R}$ it holds

$$f(\overline{g})\leqslant\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}.$$

(2) For all $s\in[\alpha,\beta]$ it holds

$$G\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)\leqslant\frac{\int_a^b G\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}.$$

We use the fact that the function $G(\cdot,\tilde{s})$, $\tilde{s}\in[0,1]$ is continuous and convex on $[0,1]$.

Furthermore, the statements (1) and (2) are also equivalent if we change the sign of both inequalities.

Using Lemma 2 we get the following result for $(2n)$-completely convex functions:

COROLLARY 1. *Let $n\in\mathbb{N}$, μ be a regular, real (signed) Borel measure and $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ such that $g([a,b])\subseteq[\alpha,\beta]$ $((\beta-\alpha)>1)$ and $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}\in[\alpha,\beta]$.*

If for all $k=1,\ldots,n-1$ and for all $s\in[\alpha,\beta]$

$$(-1)^k\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)\leqslant(-1)^k\frac{\int_a^b\Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)},\tag{17}$$

$$(-1)^k\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)\leqslant(-1)^k\frac{\int_a^b\Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)}{\int_a^b d\mu(t)}$$

and

$$(-1)^nG_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)\leqslant(-1)^n\frac{\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)},$$

then for every $(2n)$-completely convex function $f:[\alpha,\beta]\to\mathbb{R}$ it holds

$$f(\overline{g})\leqslant\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}.\tag{18}$$

If the reverse inequalities in (17) hold, then also the reverse inequality in (18) holds.

We can get similar result for $(2n)$-absolutely convex functions:

COROLLARY 2. *Let n,μ,g be as in Theorem 1. If for all $k=1,\dots,n-1$ and for all $s\in[\alpha,\beta]$*

$$\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right)\leqslant\frac{\int_a^b\Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)},\tag{19}$$

$$\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)\leqslant\frac{\int_a^b\Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)}{\int_a^b d\mu(t)}$$

and

$$G_n\left(\frac{\overline{g}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)\leqslant\frac{\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)},$$

then for every $(2n)$-absolutely convex function $f:[\alpha,\beta]\to\mathbb{R}$ it holds

$$f(\overline{g})\leqslant\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}.\tag{20}$$

If the reverse inequalities in (19) hold, then also the reverse inequality in (20) holds.

As a consequences of the above results, the following results for the left-hand side of the generalized Hermite-Hadamard inequality hold:

COROLLARY 3. *Let $n\in\mathbb{N}$, μ be a regular, real (signed) Borel measure on the interval $[a,b]\subseteq[\alpha,\beta]$ and $\overline{x}=\frac{\int_a^b x\,d\mu(x)}{\int_a^b d\mu(x)}\in[\alpha,\beta]$.*

If for all $s\in[\alpha,\beta]$

$$G_n\left(\frac{\overline{x}-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)\leqslant\frac{\int_a^b G_n\left(\frac{x-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)d\mu(x)}{\int_a^b d\mu(x)},\tag{21}$$

then for every $(2n)$-convex function $f:[\alpha,\beta]\to\mathbb{R}$

$$\begin{aligned}f(\overline{x})\leqslant&\frac{\int_a^b f(x)\,d\mu(x)}{\int_a^b d\mu(x)}\\&+\sum_{k=1}^{n-1}\left\{f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\beta-\overline{x}}{\beta-\alpha}\right)-\frac{\int_a^b\Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right)d\mu(x)}{\int_a^b d\mu(x)}\right]\right.\\&\left.+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\overline{x}-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b\Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right)d\mu(x)}{\int_a^b d\mu(x)}\right]\right\}.\end{aligned}\tag{22}$$

If the reverse inequality in (21) holds, then also the reverse inequality in (22) holds.

COROLLARY 4. *Let $n \in \mathbb{N}$, μ be a regular, real (signed) Borel measure on the interval $[a,b] \subseteq [\alpha,\beta]$ and $\overline{x} = \frac{\int_a^b x\, d\mu(x)}{\int_a^b d\mu(x)} \in [\alpha,\beta]$.*

If for all $k = 1,\ldots,n-1$ and for all $s \in [\alpha,\beta]$

$$(-1)^k \Lambda_k\left(\frac{\beta-\overline{x}}{\beta-\alpha}\right) \leqslant (-1)^k \frac{\int_a^b \Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)}, \tag{23}$$

$$(-1)^k \Lambda_k\left(\frac{\overline{x}-\alpha}{\beta-\alpha}\right) \leqslant (-1)^k \frac{\int_a^b \Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)}$$

and

$$(-1)^n G_n\left(\frac{\overline{x}-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \leqslant (-1)^n \frac{\int_a^b G_n\left(\frac{x-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)},$$

then for every $(2n)$-completely convex function $f : [\alpha,\beta] \to \mathbb{R}$ it holds

$$f(\overline{x}) \leqslant \frac{\int_a^b f(x)\, d\mu(x)}{\int_a^b d\mu(x)}. \tag{24}$$

If the reverse inequalities in (23) hold, then also the reverse inequality in (24) holds.

COROLLARY 5. *Let $n \in \mathbb{N}$, μ be a regular, real (signed) Borel measure on the interval $[a,b] \subseteq [\alpha,\beta]$ and $\overline{x} = \frac{\int_a^b x\, d\mu(x)}{\int_a^b d\mu(x)} \in [\alpha,\beta]$.*

If for all $k = 1,\ldots,n-1$ and for all $s \in [\alpha,\beta]$

$$\Lambda_k\left(\frac{\beta-\overline{x}}{\beta-\alpha}\right) \leqslant \frac{\int_a^b \Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)}, \tag{25}$$

$$\Lambda_k\left(\frac{\overline{x}-\alpha}{\beta-\alpha}\right) \leqslant \frac{\int_a^b \Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)}$$

and

$$G_n\left(\frac{\overline{x}-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) \leqslant \frac{\int_a^b G_n\left(\frac{x-\alpha}{\beta-\alpha}, \frac{s-\alpha}{\beta-\alpha}\right) d\mu(x)}{\int_a^b d\mu(x)},$$

then for every $(2n)$-absolutely convex function $f : [\alpha,\beta] \to \mathbb{R}$ it holds

$$f(\overline{x}) \leqslant \frac{\int_a^b f(x)\, d\mu(x)}{\int_a^b d\mu(x)}. \tag{26}$$

If the reverse inequalities in (25) hold, then also the reverse inequality in (26) holds.

3. Generalization of converses of Jensen's inequality by Lidstone's polynomial

We will use the following notation for composition of functions:

$$\Lambda_k\left(\frac{x-\alpha}{\beta-\alpha}\right)=\tilde{\Lambda}_k(x),\ \ x\in[\alpha,\beta],\ \ k=0,1,\ldots,n-1, \tag{27}$$

$$\Lambda_k\left(\frac{\beta-x}{\beta-\alpha}\right)=\hat{\Lambda}_k(x),\ \ x\in[\alpha,\beta],\ \ k=0,1,\ldots,n-1, \tag{28}$$

$$G_n\left(\frac{x-\alpha}{\beta-\alpha},\frac{s-\alpha}{\beta-\alpha}\right)=\tilde{G}_n(x,s),\ \ x,s\in[\alpha,\beta],\ \ n\geqslant 1. \tag{29}$$

Let μ be a regular, real (signed) Borel measure and let $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ such that $g([a,b])\subseteq[m,M]\subseteq[\alpha,\beta]$. For a function $F:[\alpha,\beta]\to\mathbb{R}$ denote by $LR(F,g,m,M,\mu)$

$$LR(F,g,m,M,\mu)=\frac{\int_a^b F(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}-\frac{M-\overline{g}}{M-m}F(m)-\frac{\overline{g}-m}{M-m}F(M) \tag{30}$$

where $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}$.

LEMMA 3. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be of class $C^{(2n)}$ on $[\alpha,\beta]$. Let μ be a regular, real (signed) Borel measure and let $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ and such that $g([a,b])\subseteq[m,M]\subseteq[\alpha,\beta]$. Then*

$$\begin{aligned}&LR(f,g,m,M,\mu) \qquad\qquad (31)\\ &=\sum_{k=1}^{n-1}\left[f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot LR(\hat{\Lambda}_k,g,m,M,\mu)+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot LR(\tilde{\Lambda}_k,g,m,M,\mu)\right]\\ &\quad+(\beta-\alpha)^{2n-1}\int_\alpha^\beta f^{(2n)}(s)[LR(\tilde{G}_n(\cdot,s),g,m,M,\mu)]\,ds\end{aligned}$$

Proof. We use Widder's Lemma for representation of function in the form:

$$\begin{aligned}f(x)&=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\hat{\Lambda}_k(x)+f^{(2k)}(\beta)\tilde{\Lambda}_k(x)\right]\\ &\quad+(\beta-\alpha)^{2n-1}\int_\alpha^\beta \tilde{G}_n(x,s)f^{(2n)}(s)ds.\end{aligned} \tag{32}$$

Using the above representation, for $f\in C^{(2n)}[\alpha,\beta]$ we can calculate $f(m)$ and $f(M)$:

$$\begin{aligned}f(m)&=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\hat{\Lambda}_k(m)+f^{(2k)}(\beta)\tilde{\Lambda}_k(m)\right]\\ &\quad+(\beta-\alpha)^{2n-1}\int_\alpha^\beta \tilde{G}_n(m,s)f^{(2n)}(s)ds,\end{aligned}$$

$$f(M)=\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}\left[f^{(2k)}(\alpha)\hat{\Lambda}_k(M)+f^{(2k)}(\beta)\tilde{\Lambda}_k(M)\right]$$
$$+(\beta-\alpha)^{2n-1}\int_\alpha^\beta \tilde{G}_n(M,s)f^{(2n)}(s)ds.$$

We can easily calculate the difference $LR(f,g,m,M,\mu)$ defined by (30):

$$\begin{aligned}&LR(f,g,m,M,\mu)\\&=\sum_{k=0}^{n-1}\left[f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot LR(\hat{\Lambda}_k,g,m,M,\mu)+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot LR(\tilde{\Lambda}_k,g,m,M,\mu)\right]\\&\quad+(\beta-\alpha)^{2n-1}\int_\alpha^\beta f^{(2n)}(s)[LR(\tilde{G}_n(\cdot,s),g,m,M,\mu)]\,ds.\end{aligned}$$

Since $LR(\hat{\Lambda}_0,g,m,M,\mu)=0$, $LR(\tilde{\Lambda}_0,g,m,M,\mu)=0$ we obtain (31). □

REMARK 2. For $n=1$ in Lemma 3 we obtain the identity from [13] for regular real measure μ :

$$\begin{aligned}&\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}-\frac{M-\overline{g}}{M-m}f(m)-\frac{\overline{g}-m}{M-m}f(M)\\&=(\beta-\alpha)\int_\alpha^\beta f''(s)\left[\frac{\int_a^b \tilde{G}(g(t),s)\,d\mu(t)}{\int_a^b d\mu(t)}-\frac{M-\overline{g}}{M-m}\tilde{G}(m,s)-\frac{\overline{g}-m}{M-m}\tilde{G}(M,s)\right]ds.\end{aligned}$$

Using Lemma 3 we can get the following generalization of converse of Jensen's inequality for $(2n)$-convex function:

THEOREM 6. *Let $n\in\mathbb{N}$, μ be a regular, real (signed) Borel measure and $g:[a,b]\to\mathbb{R}$ be integrable function with respect to μ such that $g([a,b])\subseteq[m,M]\subseteq[\alpha,\beta]$. Let $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}$.*

If for all $s\in[\alpha,\beta]$

$$\frac{\int_a^b \tilde{G}_n(g(t),s)\,d\mu(t)}{\int_a^b d\mu(t)}\leqslant\frac{M-\overline{g}}{M-m}\tilde{G}_n(m,s)+\frac{\overline{g}-m}{M-m}\tilde{G}_n(M,s), \tag{33}$$

then for every $(2n)$-convex function $f:[\alpha,\beta]\to\mathbb{R}$ it holds

$$\begin{aligned}&\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}\leqslant\frac{M-\overline{g}}{M-m}f(m)+\frac{\overline{g}-m}{M-m}f(M)\\&+\sum_{k=1}^{n-1}\left[f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot LR(\hat{\Lambda}_k,g,m,M,\mu)+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot LR(\tilde{\Lambda}_k,g,m,M,\mu)\right].\end{aligned} \tag{34}$$

If the reverse inequalities in (33) hold, then also the reverse inequality in (34) holds.

REMARK 3. For $n = 1$ in Theorem 6 we obtain result in Theorem 2 for real (signed) measure μ. The following two statements are equivalent:

(1) For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$ it holds

$$\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m} f(m) + \frac{\overline{g}-m}{M-m} f(M),$$

(2) for all $s \in [\alpha,\beta]$ it holds

$$\frac{\int_a^b \tilde{G}(g(t),s)\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}\tilde{G}(m,s) + \frac{\overline{g}-m}{M-m}\tilde{G}(M,s).$$

Furthermore, the statements (1) and (2) are also equivalent if we change the sign of both inequalities.

We use the fact that the function $\tilde{G}(\cdot,s),\ s \in [\alpha,\beta]$ is continuous and convex on $[\alpha,\beta]$.

Setting $m = \alpha$ and $M = \beta$ and $\int_a^b d\mu(t) = 1$ in Theorem 6 we got the corollary as in [11]:

COROLLARY 6. *Let $f : [\alpha,\beta] \to \mathbb{R}$ be $(2n)$-convex function on $[\alpha,\beta]$. Let μ be a regular, real (signed) Borel measure and $g : [a,b] \to [\alpha,\beta]$ integrable with respect to μ. If*

$$\int_a^b G_n\left(\frac{g(t)-\alpha}{\beta-\alpha}, s\right) d\mu(t) \leqslant 0, \forall s \in [0,1], \tag{35}$$

then

$$\begin{aligned}\int_a^b f(g(t))\,d\mu(t) \leqslant \sum_{k=0}^{n-1} (\beta-\alpha)^{2k} \Bigg[f^{(2k)}(\alpha) \int_a^b \Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right) d\mu(t) \\ + f^{(2k)}(\beta) \int_a^b \Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right) d\mu(t)\Bigg].\end{aligned} \tag{36}$$

If the reverse inequality in (35) holds, then also the reverse inequality in (36) holds.

Using Lemma 3 we get the following generalization for $(2n)$-completely convex functions:

COROLLARY 7. *Let $n \in \mathbb{N}$, μ be a regular, real (signed) Borel measure and $g : [a,b] \to \mathbb{R}$ be an integrable function with respect to measure μ such that $g([a,b]) \subseteq [m,M] \subset [\alpha,\beta]$ and $((\beta-\alpha) > 1)$. Let $\overline{g} = \frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}$.*

If for all $k = 1,\ldots,n-1$ and for all $s \in [\alpha,\beta]$

$$(-1)^k \frac{\int_a^b \hat{\Lambda}_k(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant (-1)^k \left[\frac{M-\overline{g}}{M-m}\hat{\Lambda}_k(m) + \frac{\overline{g}-m}{M-m}\hat{\Lambda}_k(M)\right], \tag{37}$$

$$(-1)^k \frac{\int_a^b \tilde{\Lambda}_k(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant (-1)^k \left[\frac{M-\overline{g}}{M-m}\tilde{\Lambda}_k(m) + \frac{\overline{g}-m}{M-m}\tilde{\Lambda}_k(M)\right]$$

and

$$(-1)^k \frac{\int_a^b \tilde{G}_n(g(t),s)\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant (-1)^k \left[\frac{M-\overline{g}}{M-m}\tilde{G}_n(m,s) + \frac{\overline{g}-m}{M-m}\tilde{G}_n(M,s)\right],$$

then for every $(2n)$-completely convex function $f:[\alpha,\beta]\to\mathbb{R}$ it holds

$$\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}f(m) + \frac{\overline{g}-m}{M-m}f(M). \tag{38}$$

If the reverse inequalities in (37) hold, then also the reverse inequality in (38) hold.

Proof. For any $n \geqslant 1$, using the representation (32) for $(2n)$-completely convex in a form

$$\begin{aligned} f(x) = &\sum_{k=0}^{n-1}(\beta-\alpha)^{2k}[(-1)^k f^{(2k)}(\alpha)(-1)^k\hat{\Lambda}_k(x) + (-1)^k f^{(2k)}(\beta)(-1)^k\tilde{\Lambda}_k(x)] \\ &+(\beta-\alpha)^{2n-1}\int_\alpha^\beta (-1)^n\tilde{G}_n(x,s)(-1)^n f^{(2n)}(s)ds, \end{aligned}$$

we can calculate the difference

$$\begin{aligned} &\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} - \frac{M-\overline{g}}{M-m}f(m) - \frac{\overline{g}-m}{M-m}f(M) \\ =&\sum_{k=1}^{n-1}(-1)^k f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot(-1)^k LR(\hat{\Lambda}_k,g,m,M,\mu) \\ &+(-1)^k f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot(-1)^k LR(\tilde{\Lambda}_k,g,m,M,\mu) \\ &+(\beta-\alpha)^{2n-1}\int_\alpha^\beta (-1)^n f^{(2n)}(s)[(-1)^n LR(\tilde{G}_n(\cdot,s),g,m,M,\mu)]\,ds. \end{aligned}$$

By definition of $(2n)$-completely convex function the proof is done. □

Using Lemma 3 we get the following generalization for $(2n)$-absolutely completely convex function:

COROLLARY 8. *Let $n\in\mathbb{N}$ and μ be a regular, real (signed) Borel measure and $g:[a,b]\to\mathbb{R}$ be integrable function with respect to measure μ such that $g([a,b])\subseteq [m,M]\subset[\alpha,\beta]$ and $((\beta-\alpha)>1)$. Let $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}$.*

If for all $k=1,\dots,n-1$ and for all $s\in[\alpha,\beta]$

$$\frac{\int_a^b \hat{\Lambda}_k(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}\hat{\Lambda}_k(m) + \frac{\overline{g}-m}{M-m}\hat{\Lambda}_k(M), \tag{39}$$

$$\frac{\int_a^b \tilde{\Lambda}_k(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}\tilde{\Lambda}_k(m) + \frac{\overline{g}-m}{M-m}\tilde{\Lambda}_k(M)$$

and

$$\frac{\int_a^b \tilde{G}_n(g(t),s)\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}\tilde{G}_n(m,s) + \frac{\overline{g}-m}{M-m}\tilde{G}_n(M,s),$$

then for every $(2n)$*-absolutely convex function* $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \leqslant \frac{M-\overline{g}}{M-m}f(m) + \frac{\overline{g}-m}{M-m}f(M). \tag{40}$$

If the reverse inequalities in (39) hold, then also the reverse inequality in (40) hold.

As a consequence of the above results, the following results for the right-hand side of generalized Hermite-Hadamard inequality hold:

COROLLARY 9. *Let* $n\in\mathbb{N}$, μ *be a regular, real (signed) Borel measure on interval* $[a,b]\subseteq[\alpha,\beta]$ *and* $\overline{x}=\frac{\int_a^b x\,d\mu(x)}{\int_a^b d\mu(x)}$. *If for all* $s\in[\alpha,\beta]$

$$\frac{\int_a^b \tilde{G}_n(x,s)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}\tilde{G}_n(a,s) + \frac{\overline{x}-a}{b-a}\tilde{G}_n(b,s), \tag{41}$$

then for every $(2n)$*-convex function* $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$\frac{\int_a^b f(x)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}f(a) + \frac{\overline{x}-a}{b-a}f(b) \tag{42}$$

$$+\sum_{k=1}^{n-1}\left[f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot LR(\hat{\Lambda}_k,id,a,b,\mu)+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot LR(\tilde{\Lambda}_k,id,a,b,\mu)\right].$$

If the reverse inequalities in (41) hold, then also the reverse inequality in (43) holds.

Using Lemma 3 we get the following generalization for $(2n)$-completely convex functions:

COROLLARY 10. *Let* $n\in\mathbf{N}$, μ *be a regular, real (signed) Borel measure on interval* $[a,b]\subseteq[\alpha,\beta]$ *and* $\overline{x}=\frac{\int_a^b x\,d\mu(x)}{\int_a^b d\mu(x)}$.

If for all $k=1,\dots,n-1$ *and for all* $s\in[\alpha,\beta]$

$$(-1)^k\frac{\int_a^b \hat{\Lambda}_k(x)\,d\mu(t)}{\int_a^b d\mu(x)} \leqslant (-1)^k\left[\frac{b-\overline{x}}{b-a}\hat{\Lambda}_k(m) + \frac{\overline{x}-a}{b-a}\hat{\Lambda}_k(b)\right], \tag{43}$$

$$(-1)^k\frac{\int_a^b \tilde{\Lambda}_k(x)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant (-1)^k\left[\frac{b-\overline{x}}{b-a}\tilde{\Lambda}_k(a) + \frac{\overline{x}-a}{b-a}\tilde{\Lambda}_k(b)\right]$$

and

$$(-1)^k \frac{\int_a^b \tilde{G}_n(x,s)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant (-1)^k \left[\frac{b-\overline{x}}{b-a}\tilde{G}_n(a,s) + \frac{\overline{x}-a}{b-a}\tilde{G}_n(b,s)\right],$$

then for every $(2n)$*-completely convex function* $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$\frac{\int_a^b f(x)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}f(a) + \frac{\overline{x}-a}{b-a}f(b). \tag{44}$$

If the reverse inequalities in (43) hold, then also the reverse inequality in (44) holds.

Using Lemma 3 we get the following generalization for $(2n)$-absolutely convex functions:

COROLLARY 11. *Let* $n\in\mathbb{N}$, μ *be a regular, real (signed) Borel measure on interval* $[a,b]\subseteq[\alpha,\beta]$ *and* $\overline{x} = \frac{\int_a^b x\,d\mu(x)}{\int_a^b d\mu(x)}$.

If for all $k=1,\ldots,n-1$ *and for all* $s\in[\alpha,\beta]$

$$\frac{\int_a^b \hat{\Lambda}_k(x)\,d\mu(t)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}\hat{\Lambda}_k(m) + \frac{\overline{x}-a}{b-a}\hat{\Lambda}_k(b), \tag{45}$$

$$\frac{\int_a^b \tilde{\Lambda}_k(x)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}\tilde{\Lambda}_k(a) + \frac{\overline{x}-a}{b-a}\tilde{\Lambda}_k(b)$$

and

$$\frac{\int_a^b \tilde{G}_n(x,s)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}\tilde{G}_n(a,s) + \frac{\overline{x}-a}{b-a}\tilde{G}_n(b,s),$$

then for every $(2n)$*-absolutely convex function* $f:[\alpha,\beta]\to\mathbb{R}$ *it holds*

$$\frac{\int_a^b f(x)\,d\mu(x)}{\int_a^b d\mu(x)} \leqslant \frac{b-\overline{x}}{b-a}f(a) + \frac{\overline{x}-a}{b-a}f(b). \tag{46}$$

If the reverse inequalities in (45) hold, then also the reverse inequality in (46) holds.

4. n-exponential convexity of Jensen's inequality by Lidstone's polynomial

Motivated by the inequalities (16) and (34), we define functionals $\Phi_1(f)$ and $\Phi_2(f)$ by

$$\Phi_1(f) = f(\overline{g}) - \frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)} \tag{47}$$

$$-\sum_{k=1}^{n-1}\left\{ f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\beta-\overline{g}}{\beta-\alpha}\right) - \frac{\int_a^b \Lambda_k\left(\frac{\beta-g(t)}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\right.$$

$$+ f^{(2k)}(\beta)(\beta-\alpha)^{2k}\left[\Lambda_k\left(\frac{\overline{g}-\alpha}{\beta-\alpha}\right)-\frac{\int_a^b \Lambda_k\left(\frac{g(t)-\alpha}{\beta-\alpha}\right)d\mu(t)}{\int_a^b d\mu(t)}\right]\Bigg\}$$

and

$$\Phi_2(f)=\frac{\int_a^b f(g(t))\,d\mu(t)}{\int_a^b d\mu(t)}-\frac{M-\overline{g}}{M-m}f(m)-\frac{\overline{g}-m}{M-m}f(M) \tag{48}$$

$$-\sum_{k=1}^{n-1}\left[f^{(2k)}(\alpha)(\beta-\alpha)^{2k}\cdot LR(\hat{\Lambda}_k,g,m,M,\mu)+f^{(2k)}(\beta)(\beta-\alpha)^{2k}\cdot LR(\tilde{\Lambda}_k,g,m,M,\mu)\right].$$

THEOREM 7. *Let μ be a regular, real (signed) Borel measure and $f:[\alpha,\beta]\to\mathbb{R}$, $f\in C^{2n}([\alpha,\beta])$. Let $g:[a,b]\to\mathbb{R}$ be integrable with respect to μ such that $g([a,b])\subseteq[\alpha,\beta]$ and $\overline{g}=\frac{\int_a^b g(t)\,d\mu(t)}{\int_a^b d\mu(t)}\in[\alpha,\beta]$. If for all $s\in[\alpha,\beta]$ the reverse inequalities in (15) and (33) hold, then there exists $\xi\in[\alpha,\beta]$ such that*

$$\Phi_i(f)=f^{(2n)}(\xi)\Phi_i(\varphi),\; i=1,2, \tag{49}$$

where $\varphi(x)=\frac{x^{2n}}{(2n)!}$.

Proof. Let us denote $m=\min f^{(2n)}$ and $M=\max f^{(2n)}$. We first consider the following function $\phi_1(x)=\frac{Mx^{2n}}{(2n)!}-f(x)$. Then $\phi_1^{(2n)}(x)=M-f^{(2n)}(x)\geqslant 0$, $x\in[\alpha,\beta]$, so ϕ_1 is a $(2n)$-convex function. Similarly, a function $\phi_2(x)=f(x)-\frac{mx^{2n}}{(2n)!}$ is a $(2n)$-convex function. Now, we use inequalities from Theorem 5 and Theorem 7 for $(2n)$-convex functions ϕ_1 and ϕ_2. So, we can conclude that there exists $\xi\in[\alpha,\beta]$ that we are looking for in (49). □

COROLLARY 12. *Let $f,h:[\alpha,\beta]\to\mathbb{R}$ such that $f,h\in C^{2n}([\alpha,\beta])$. If for all $s\in[\alpha,\beta]$ the reverse inequalities in (15) and (33) hold, then there exists $\xi\in[\alpha,\beta]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(h)}=\frac{f^{(2n)}(\xi)}{h^{(2n)}(\xi)},\;\; i=1,2, \tag{50}$$

provided that the denominator of the left-hand side is non-zero.

Proof. We use the following standard technique: Let us define the linear functional $L(\chi)=\Phi_i(\chi)$, $i=1,2$. Next, we define $\chi(t)=f(t)L(h)-h(t)L(f)$. According to Theorem 7, applied on χ, there exists $\xi\in(\alpha,\beta)$ so that

$$L(\chi)=\chi^{(2n)}(\xi)\Phi_i(\varphi),\;\; \varphi(x)=\frac{x^{2n}}{(2n)!},\; i=1,2.$$

From $L(\chi)=0$, it follows $f^{(2n)}(\xi)L(h)-h^{(2n)}(\xi)L(f)=0$ and (50) is proved. □

Now, let us recall some definitions and facts about exponentially convex functions (see [6]):

DEFINITION 7. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0,$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 4. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition:

PROPOSITION 1. If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly, $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ for all $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 8. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 5. It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

We use an idea from [6] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [15]):

THEOREM 8. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha, \beta]$ in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \ldots, z_{2l}]$ is n-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $z_0, \ldots, z_{2l} \in [\alpha, \beta]$. Let $\Phi_i(f)$, $i = 1, 2$ be linear functional defined as in (47) and (48). Then $s \mapsto \Phi_i(f_s)$ is an n-exponentially convex function in the Jensen sense* *on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i = 1,\dots,n$ and $s_i \in J$, $i = 1,\dots,n$, we define the function

$$h(z) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(z).$$

Using the assumption that the function $s \mapsto f_s[z_0,\dots,z_{2l}]$ is n-exponentially convex in the Jensen sense, we have

$$h[z_0,\dots,z_{2l}] = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}[z_0,\dots,z_{2l}] \geqslant 0,$$

which in turn implies that h is a $(2l)$-convex function on J, so it is $\Phi_k(h) \geqslant 0$, hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_k\left(f_{\frac{s_i+s_j}{2}}\right) \geqslant 0.$$

We conclude that the function $s \mapsto \Phi_k(f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto \Phi_k(f_s)$ is also continuous on J, then $s \mapsto \Phi_k(f_s)$ is n-exponentially convex by definition. $\square$

The following corollaries are an immediate consequences of the above theorem:

COROLLARY 13. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0,\dots,z_{2l}]$ is exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $z_0,\dots,z_{2l} \in [\alpha,\beta]$. Let $\Phi_i(f)$, $i = 1,2$ be linear functional defined as in (47) and (48). Then $s \mapsto \Phi_i(f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is exponentially convex on J.*

COROLLARY 14. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval $[\alpha,\beta]$ in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0,\dots,z_{2l}]$ is 2-exponentially convex in the Jensen sense on J for every $(2l+1)$ mutually different points $z_0,\dots,z_{2l} \in [\alpha,\beta]$. Let $\Phi_i(f), i = 1,2$ be linear functional defined as in (47) and (48). Then the following statements hold:*

(i) If the function $s \mapsto \Phi_i(f_s)$ is continuous on J, then it is 2-exponentially convex function on J. If $s \mapsto \Phi_i(f_s)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true:

$$[\Phi_i(f_s)]^{t-r} \leqslant [\Phi_i(f_r)]^{t-s} [\Phi_i(f_t)]^{s-r} \tag{51}$$

for every choice $r,s,t \in J$, such that $r < s < t$.

(ii) If the function $s \mapsto \Phi_i(f_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v \in J$, such that $s \leqslant u$ and $q \leqslant v$, we have

$$\mu_{s,q}(g,\Phi_i,\Upsilon) \leqslant \mu_{u,v}(g,\Phi_i,\Upsilon), \tag{52}$$

where

$$\mu_{s,q}(g,\Phi_i,\Upsilon) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_q)}\right), & s = q, \end{cases} \tag{53}$$

for $f_s, f_q \in \Upsilon$.

Proof.

(i) This is an immediate consequence of Theorem 8 and Remark 5.

(ii) Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on J, that is, the function $s \mapsto \log\Phi_i(f_s)$ is convex on J. So, we get

$$\frac{\log\Phi_i(f_s) - \log\Phi_i(f_q)}{s-q} \leqslant \frac{\log\Phi_i(f_u) - \log\Phi_i(f_v)}{u-v} \tag{54}$$

for $s \leqslant u, q \leqslant v, s \neq q, u \neq v$, and there form conclude that

$$\mu_{s,q}(g,\Phi_i,\Upsilon) \leqslant \mu_{u,v}(g,\Phi_i,\Upsilon).$$

Cases $s = q$ and $u = v$ follows from (54) as limit cases. □

REMARK 6. Note that the results from above theorem and corollaries still hold when two of the points $z_0,\ldots,z_{2l} \in [\alpha,\beta]$ coincide, say $z_1 = z_0$, for a family of differentiable functions f_s such that the function $s \mapsto f_s[z_0,\ldots,z_{2l}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(2l+1)$ points coincide for a family of $2l$ differentiable functions with the same property. The proofs are obtained by (10) and suitable characterization of convexity.

5. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 8, Corollary 13, Corollary 14 and Remark 6. This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [5].

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{l_s : \mathbb{R} \to [0,\infty) : s \in \mathbb{R}\}$$

defined by

$$l_s(x) = \begin{cases} \frac{e^{sx}}{s^{2n}}, & s \neq 0, \\ \frac{x^{2n}}{(2n)!}, & s = 0. \end{cases}$$

We have $\frac{d^{2n}l_s}{dx^{2n}}(x) = e^{sx} > 0$ which shows that l_s is $(2n)$-convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^{2n}l_s}{dx^{2n}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8 we also have that $s \mapsto l_s[z_0, \dots, z_{2n}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 13 we conclude that $s \mapsto \Phi_i(l_s), i = 1,2$ are exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto l_s$ is not continuous for $s = 0$), so it is exponentially convex.

For this family of functions, $\mu_{s,q}(g, \Phi_i, \Omega_1), i = 1,2$ from (53), becomes

$$\mu_{s,q}(g, \Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(l_s)}{\Phi_i(l_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot l_s)}{\Phi_i(l_s)} - \frac{2n}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{1}{2n+1} \frac{\Phi_i(id \cdot l_0)}{\Phi_i(l_0)}\right), & s = q = 0. \end{cases}$$

Now, using (52) it is monotonous function in parameters s and q.

We observe here that $\left(\frac{\frac{d^{2n}l_s}{dx^{2n}}}{\frac{d^{2n}l_q}{dx^{2n}}}\right)^{\frac{1}{s-q}} (\ln x) = x$ so using Corollary 12 it follows that:

$$M_{s,q}(g, \Phi_i, \Omega_1) = \ln \mu_{s,q}(g, \Phi_i, \Omega_1), \quad i = 1,2$$

satisfy

$$\alpha \leqslant M_{s,q}(g, \Phi_i, \Omega_1) \leqslant \beta, \quad i = 1,2.$$

If we set that the image of the function g is $[\alpha, \beta]$, we have that

$$\alpha = \min_{t \in [a,b]} \{g(t)\} \leqslant M_{s,q}(g, \Phi_i, \Omega_1) \leqslant \max_{t \in [a,b]} \{g(t)\} = \beta, \quad i = 1,2,$$

which shows that $M_{s,q}(g, \Phi_i, \Omega_1)$ are means of $g(t)$ for $i = 1,2$. Because of above inequality, this mean is also monotonic.

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{f_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)\cdots(s-2n+1)}, & s \notin \{0,1,\dots,2n-1\} \\ \frac{x^j \ln x}{(-1)^{2n-1-j} j!(2n-1-j)!}, & s = j \in \{0,1,\dots,2n-1\}. \end{cases}$$

Here, $\frac{d^{2n}f_s}{dx^{2n}}(x) = x^{s-2n} = e^{(s-2n)\ln x} > 0$ which shows that f_s is $(2n)$-convex for $x > 0$ and $s \mapsto \frac{d^{2n}f_s}{dx^{2n}}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $s \mapsto \Phi_i(f_s)$, $i = 1,2$ are exponentially convex. In this case we assume that $[\alpha,\beta] \in \mathbb{R}^+$. Function (53) now is equal to:

$$\mu_{s,q}(g,\Phi_i,\Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-(2n-1)!\frac{\Phi_i(f_0 f_s)}{\Phi_i(f_s)} + \sum_{k=0}^{2n-1}\frac{1}{k-s}\right), & s = q \notin \{0,1,\dots,2n-1\}, \\ \exp\left(-(2n-1)!\frac{\Phi_i(f_0 f_s)}{2\Phi_i(f_s)} + \sum_{\substack{k=0 \\ k\neq s}}^{2n-1}\frac{1}{k-s}\right), & s = q \in \{0,1,\dots,2n-1\}. \end{cases}$$

We observe that $\left(\frac{\frac{d^{2n}f_s}{dx^{2n}}}{\frac{d^{2n}f_q}{dx^{2n}}}\right)^{\frac{1}{s-q}}(x) = x$, so if $\Phi_i (i = 1,2)$ are positive, then Corollary 12 yield that there exist some $\xi_i \in [\alpha,\beta], i = 1,2$ such that

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \quad i = 1,2.$$

Since the function $\xi \to \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$\alpha \leqslant \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leqslant \beta, \quad i = 1,2. \tag{55}$$

As in the previous example, if we set that the image of the function g is $[\alpha,\beta]$, in that case we have that

$$\alpha = \min_{t\in[a,b]} g(t) \leqslant \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leqslant \max_{t\in[a,b]} g(t) = \beta, \quad i = 1,2, \tag{56}$$

which shows that $\mu_{s,q}(g,\Phi_i,\Omega_2)$, $i = 1,2$ is mean.

Now, we impose one additional parameter r. For $r \neq 0$ by substituting $g \to g^r, s \to \frac{s}{r}$ and $q \to \frac{q}{r}$ in (56), we get the following:

$$\min_{t\in[a,b]} (g(t))^r \leqslant \left(\frac{\Phi_i(g^r,\cdot,f_s)}{\Phi_i(g^r,\cdot,f_q)}\right)^{\frac{r}{s-q}} \leqslant \max_{t\in[a,b]} (g(t))^r, \quad i = 1,2. \tag{57}$$

We define new generalized mean as follows:

$$\mu_{s,q;r}(g,\Phi_i,\Omega_2) = \begin{cases} \left(\mu_{\frac{s}{r},\frac{q}{r}}(g^r,\Phi_i,\Omega_2)\right)^{\frac{1}{r}}, & r \neq 0, \\ \mu_{s,q}(\ln g,\Phi_i,\Omega_2), & r = 0. \end{cases} \tag{58}$$

This new generalized mean is also monotonic. If $s,q,u,v \in \mathbb{R}, r \neq 0$ such that $s \leqslant u, q \leqslant v$, then we have

$$\mu_{s,q;r}(g,\Phi_i,\Omega_2) \leqslant \mu_{u,v;r}(g,\Phi_i,\Omega_2), \quad i = 1,2.$$

The above result follows from the following inequality:

$$\mu_{\frac{s}{r},\frac{q}{r}}(g^r,\Phi_i,\Omega_2) = \left(\frac{\Phi_i(g^r,\cdot,f_s)}{\Phi_i(g^r,\cdot,f_q)}\right)^{\frac{r}{s-q}} \leqslant \left(\frac{\Phi_i(g^r,\cdot,f_s)}{\Phi_i(g^r,\cdot,f_q)}\right)^{\frac{r}{u-v}} = \mu_{\frac{u}{r},\frac{v}{r}}(g^r,\Phi_i,\Omega_2),$$

for $s,q,u,v \in \mathbb{R}, r \neq 0$, such that $\frac{s}{r} \leqslant \frac{u}{r}, \frac{q}{r} \leqslant \frac{v}{r}$, and the fact that $\mu_{s,q}(g,\Phi_i,\Omega_2)$ for $i = 1,2$ is monotonous in both parameters. For $r = 0$, we obtain the required result by taking the limit $r \to 0$.

EXAMPLE 3. Consider a family of functions

$$\Omega_3 = \{h_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{(\ln s)^{2n}}, & s \neq 1 \\ \frac{x^{2n}}{(2n)!}, & s = 1. \end{cases}$$

Since $\frac{d^{2n}h_s}{dx^{2n}}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [20]) it is exponentially convex. Obviously h_s are $(2n)$-convex functions for every $s > 0$. For this family of functions, $\mu_{s,q}(g,\Phi_i,\Omega_3)$, $i = 1,2$, in this case for $[\alpha,\beta] \in \mathbb{R}^+$, from (53) becomes

$$\mu_{s,q}(g,\Phi_i,\Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)} - \frac{2n}{s\ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{1}{2n+1}\frac{\Phi_i(id\cdot h_1)}{\Phi_i(h_1)}\right), & s = q = 1. \end{cases}$$

This is monotonous function in parameters s and q by (52).

Using Corollary 12 it follows that

$$M_{s,q}(\Phi_i,\Omega_3) = -L(s,q)\ln \mu_{s,q}(\Phi_i,\Omega_3), \quad i = 1,2$$

satisfy

$$\alpha \leqslant M_{s,q}(\Phi_i,\Omega_3) \leqslant \beta, \quad i = 1,2.$$

As in the previous examples, if we set that the image of the function g is $[\alpha,\beta]$, in that case we have that

$$\alpha = \min_{t\in[a,b]} g(t) \leqslant M_{s,q}(\Phi_i,\Omega_3) \leqslant \max_{t\in[a,b]} g(t) = \beta, \quad i = 1,2.$$

So $M_{s,q}(\Phi_i,\Omega_3)$ is mean of $g(t)$ for $i = 1,2$ and also monotonic. $L(s,q)$ is logarithmic mean defined by

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q \\ s, & s = q. \end{cases}$$

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s^n}.$$

Since $\frac{d^{2n}k_s}{dx^{2n}}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [20]) it is exponentially convex. Obviously k_s are $(2n)$-convex functions for every $s > 0$. For this family of functions, $\mu_{s,q}(g,\Phi_i,\Omega_4), i = 1,2$, in this case for $[\alpha,\beta] \in \mathbb{R}^+$, from (53) becomes

$$\mu_{s,q}(g,\Phi_i,\Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{n}{s}\right), & s = q. \end{cases}$$

This is monotonous function in parameters s and q by (52).

Using Corollary 12 it follows that

$$M_{s,q}(\Phi_i,\Omega_4) = -(\sqrt{s}+\sqrt{q})\ln \mu_{s,q}(\Phi_i,\Omega_4), \quad i = 1,2$$

satisfy

$$\alpha \leqslant M_{s,q}(\Phi_i,\Omega_4) \leqslant \beta, \ \ i = 1,2.$$

As in the previous examples, if we set that the image of the function g is $[\alpha,\beta]$, in that case we have that

$$\alpha = \min_{t\in[a,b]} g(t) \leqslant M_{s,q}(\Phi_i,\Omega_4) \leqslant \max_{t\in[a,b]} g(t) = \beta, \ \ i = 1,2.$$

So $M_{s,q}(\Phi_i,\Omega_4)$ is mean of $g(t)$ for $i = 1,2$ and also monotonic.

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