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INEQUALITIES OF HERMITE–HADAMARD TYPE FOR CONVEX FUNCTIONS WHICH ARE n–TIMES DIFFERENTIABLE

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Abstract. In the paper, by creating an integral identity and using Hölder's inequality, the authors establish some new inequalities of Hermite-Hadamard type for n-times differentiable convex functions.

1. Introduction

Throughout this paper, we use the notations $I \subseteq \mathbb{R} = (-\infty, \infty)$, I°, and $\mathbb{N}$ to denote an interval, the interior of I, and the set of all positive integers respectively.

A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) \tag{1.1}$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If the inequality (1.1) reverses, then f is said to be concave on I.

It is well known that the most important inequality in the theory of convex functions is Hermite-Hadamard's inequality below. If f is a convex function on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}. \tag{1.2}$$

If f is concave on $[a,b]$, then the inequality (1.2) is reversed.

The inequality (1.2) has been generalized in many articles for a long time. Some of them may be recited as follows.

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THEOREM 1.1. ([2, Theorems 1 and 2]) *If f is differentiable on $[a,b]$ such that $|f'|^q$ for $q \geqslant 1$ is a convex function on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q} \tag{1.3}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}. \tag{1.4}$$

THEOREM 1.2. ([1, Theorems 2.3 and 2.4]) *Let $f: I \to \mathbb{R}$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, and $p>1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\leqslant \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{1-1/p} \\ &\quad+\left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{1-1/p}\Big\}\end{aligned}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left[|f'(a)|+|f'(b)|\right]. \tag{1.5}$$

THEOREM 1.3. ([4]) *Let $0 \leqslant \lambda \leqslant 1$ and $a,b \in I^\circ$ with $a<b$. If $f: I^\circ \to \mathbb{R}$ is a twice differentiable mapping such that $f''(x)$ is integrable and $|f''(x)|$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\int_a^b f(x)\,\mathrm{d}x\right| \\ &\leqslant \begin{cases}\dfrac{(b-a)^2}{24}\Big\{\left[\lambda^4+(1+\lambda)(1-\lambda)^3+\dfrac{5\lambda-3}{5}\right]|f''(a)| \\ \qquad +\left[\lambda^4+(2-\lambda)\lambda^3+\dfrac{1-3\lambda}{4}\right]|f''(b)|\Big\}, & 0\leqslant\lambda\leqslant\dfrac{1}{2}, \\ \dfrac{(b-a)^2}{48}(3\lambda-1)\left[|f''(a)|+|f''(b)|\right], & \dfrac{1}{2}\leqslant\lambda\leqslant 1.\end{cases}\end{aligned} \tag{1.6}$$

THEOREM 1.4. ([5]) *Let $f: I \to \mathbb{R}$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geqslant 1$ is convex on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(x)\,\mathrm{d}x\right| \\ &\leqslant \frac{b-a}{12}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{1/q}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right]\end{aligned} \tag{1.7}$$

and

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leqslant \frac{5(b-a)}{72}\left[\left(\frac{61|f'(a)|^q+29|f'(b)|^q}{90}\right)^{1/q}+\left(\frac{29|f'(a)|^q+61|f'(b)|^q}{90}\right)^{1/q}\right]. \quad (1.8)$$

THEOREM 1.5. ([3, Theorem 3.1]) *Let $f: I \to \mathbb{R}$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, $0 \leqslant \lambda \leqslant \frac{1}{2} \leqslant \mu \leqslant 1$, and $f' \in L[a,b]$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\leqslant \frac{b-a}{24}\big[\big(10-3\lambda+8\lambda^3-15\mu+8\mu^3\big)|f'(a)| \quad (1.9)$$
$$+\big(8-9\lambda+24\lambda^2-8\lambda^3-21\mu+24\mu^2-8\mu^3\big)|f'(b)|\big].$$

In this paper, by creating an integral identity for n-times differentiable functions and using Hölder's integral inequality, we will establish some new inequalities of Hermite-Hadamard type for convex functions.

2. Lemmas

In order to obtain our main results, we need to create an integral identity below.

LEMMA 2.1. *Let $n \in \mathbb{N}$, $\xi_x = \frac{b-x}{b-a}$, and $f:[a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous. If $f^{(n)}(x)$ exists on $[a,b]$, then for $\lambda,\mu \in \mathbb{R}$ and $x \in [a,b]$,*

$$\begin{aligned}
S(x;\lambda,\mu) &\triangleq (1-\mu)f(a)+\lambda f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\
&\quad+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!(b-a)}\big\{(x-a)^k\big[x-a-(k+1)(1-\mu)(b-a)\big] \\
&\quad-(x-b)^k\big[x-b+(k+1)\lambda(b-a)\big]\big\}f^{(k)}(x) \qquad (2.1)\\
&=\frac{(b-a)^n}{n!}\bigg\{\int_0^{\xi_x} z^{n-1}(n\lambda-z)f^{(n)}[za+(1-z)b]\,\mathrm{d}z \\
&\quad+\int_{\xi_x}^1 (z-1)^{n-1}[1-z-n(1-\mu)]f^{(n)}[za+(1-z)b]\,\mathrm{d}z\bigg\}.
\end{aligned}$$

Proof. When $n=1$, integrating by part in the right-hand side of (2.1) gives

$$
\begin{aligned}
&(b-a)\bigg[\int_0^{\xi_x}(\lambda-z)f'(za+(1-z)b)\,\mathrm{d}z\\
&\quad+\int_{\xi_x}^1(1-z-(1-\mu))f'(za+(1-z)b)\,\mathrm{d}z\bigg]\\
&=\lambda f(b)-(\lambda-\xi_x)f(x)-\int_0^{\xi_x}f(za+(1-z)b)\,\mathrm{d}z\\
&\quad+(1-\mu)f(a)+(\mu-\xi_x)f(x)-\int_{\xi_x}^1 f(za+(1-z)b)\,\mathrm{d}z\\
&=(1-\mu)f(a)+\lambda f(b)+(\mu-\lambda)f(x)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.
\end{aligned}
$$

When $n=m-1$ and $m\geqslant 2$, suppose that the identity (2.1) holds. Then, when $n=m$,

$$
\begin{aligned}
&\frac{(b-a)^m}{m!}\bigg[\int_0^{\xi_x}z^{m-1}(m\lambda-z)f^{(m)}(za+(1-z)b)\,\mathrm{d}z\\
&+\int_{\xi_x}^1(z-1)^{m-1}(1-z-m(1-\mu))f^{(m)}(za+(1-z)b)\,\mathrm{d}z\bigg]\\
&=-\frac{(b-a)^{m-1}}{m!}\bigg[\xi_x^{m-1}(m\lambda-\xi_x)f^{(m-1)}(x)-(\xi_x-1)^{m-1}(1-\xi_x\\
&-m(1-\mu))f^{(m-1)}(x)-m\int_0^{\xi_x}z^{m-2}((m-1)\lambda-z)f^{(m-1)}(za+(1-z)b)\,\mathrm{d}z\\
&-m\int_{\xi_x}^1(z-1)^{m-2}(1-z-(m-1)(1-\mu))f^{(m-1)}(za+(1-z)b)\,\mathrm{d}z\bigg]\\
&=\frac{(-1)^{m-1}}{m!(b-a)}\big[(x-a)^{m-1}(x-a-m(1-\mu)(b-a))\\
&-(x-b)^{m-1}(x-b+m\lambda(b-a))\big]f^{(m-1)}(x)\\
&+\frac{(b-a)^{m-1}}{(m-1)!}\bigg[\int_0^{\xi_x}z^{m-2}((m-1)\lambda-z)f^{(m-1)}(za+(1-z)b)\,\mathrm{d}z\\
&+\int_{\xi_x}^1(z-1)^{m-2}(1-z-(m-1)(1-\mu))f^{(m-1)}(za+(1-z)b)\,\mathrm{d}z\bigg]\\
&=S(x;\lambda,\mu).
\end{aligned}
$$

This means that, when $n=m$, the integral identity (2.1) holds. By induction, the proof of Lemma 2.1 is complete. □

LEMMA 2.2. *Let* $\alpha,\beta\in\mathbb{R}$, $\xi,c\geqslant 0$, *and* $r>-1$. *Then*

$$
\int_0^c u^r|\xi-u|\,\mathrm{d}u=\frac{1}{(r+1)(r+2)}\begin{cases}[(r+2)\xi-(r+1)c]c^{r+1}, & \xi\geqslant c,\\ (r+1)c^{r+2}-(r+2)c^{r+1}\xi+2\xi^{r+2}, & 0\leqslant\xi<c\end{cases}
$$

and

$$\int_0^c (\alpha u+\beta)|\xi-u|^r \,\mathrm{d}u = \frac{1}{(r+1)(r+2)} \times \begin{cases} [(r+2)\beta+\alpha\xi]\xi^{r+1} - [\alpha c(r+1)+\beta(r+2)+\alpha\xi](\xi-c)^{r+1}, & \xi \geqslant c, \\ [(r+2)\beta+\alpha\xi]\xi^{r+1} + [\beta(r+2)+\alpha(c+cr+\xi)](c-\xi)^{r+1}, & 0 \leqslant \xi < c. \end{cases}$$

Proof. This follows from standard arguments. □

3. Some new inequalities of Hermite-Hadamard type for convex functions

Now we start out to establish some new inequalities of Hermite-Hadamard type.

THEOREM 3.1. *Let $n\in\mathbb{N}$ and $f: I\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a convex function on $[a,b]$ for $q\geqslant 1$, then for all $x\in[a,b]$ and $\lambda,\mu\in[0,1]$, we have*

$$\begin{aligned} \left|S(x;\lambda,\mu)\right| \leqslant \frac{(b-a)^n}{n!}\Big\{&\big[A(\lambda,\xi_x;n)\big]^{1-1/q}\big[A(\lambda,\xi_x;n+1)\big(\left|f^{(n)}(a)\right|^q-\left|f^{(n)}(b)\right|^q\big) \\ &+A(\lambda,\xi_x;n)\left|f^{(n)}(b)\right|^q\big]^{1/q} \\ &+\big[A(1-\mu,1-\xi_x;n)\big]^{1-1/q}\big[A(1-\mu,1-\xi_x;n)\left|f^{(n)}(a)\right|^q \\ &+A(1-\mu,1-\xi_x;n+1)\big(\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q\big)\big]^{1/q}\Big\}, \end{aligned} \tag{3.1}$$

where ξ_x is defined as in Lemma 2.1 and, for $c\geqslant 0$ and $r>-1$,

$$A(\lambda,c;r+1) = \frac{1}{(r+1)(r+2)} \times \begin{cases} [(r+2)(n\lambda)-(r+1)c]c^{r+1}, & n\lambda \geqslant c, \\ (r+1)c^{r+2}-(r+2)(n\lambda)c^{r+1}+2(n\lambda)^{r+2}, & 0\leqslant n\lambda \leqslant c. \end{cases} \tag{3.2}$$

Proof. By Lemma 2.1, Hölder's inequality, and the convexity of $\left|f^{(n)}\right|^q$, we have

$$\begin{aligned} &\left|S(x;\lambda,\mu)\right| \\ &\leqslant \frac{(b-a)^n}{n!}\bigg[\int_0^{\xi_x} z^{n-1}|n\lambda-z|\left|f^{(n)}(za+(1-z)b)\right|\mathrm{d}z \\ &\quad+\int_{\xi_x}^1 (1-z)^{n-1}|1-z-n(1-\mu)|\left|f^{(n)}(za+(1-z)b)\right|\mathrm{d}z\bigg] \\ &\leqslant \frac{(b-a)^n}{n!}\bigg\{\bigg(\int_0^{\xi_x} z^{n-1}|n\lambda-z|\,\mathrm{d}z\bigg)^{1-1/q} \\ &\quad\times\bigg[\int_0^{\xi_x} z^{n-1}|n\lambda-z|\big(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\big)\,\mathrm{d}z\bigg]^{1/q} \end{aligned} \tag{3.3}$$

$$+\left(\int_{\xi_x}^{1}(1-z)^{n-1}|1-z-n(1-\mu)|\,\mathrm{d}z\right)^{1-1/q}$$
$$\times\left[\int_{\xi_x}^{1}(1-z)^{n-1}|1-z-n(1-\mu)|\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\right]^{1/q}\Bigg\}.$$

By Lemma 2.2, a straightforward computation gives

$$\int_0^{\xi_x} z^{n-1}|n\lambda-z|\,\mathrm{d}z=A(\lambda,\xi_x;n),$$
$$\int_{\xi_x}^{1}(1-z)^{n-1}|1-z-n(1-\mu)|\,\mathrm{d}z=A(1-\mu,1-\xi_x;n),$$
$$\int_0^{\xi_x} z^{n-1}|n\lambda-z|\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z$$
$$=A(\lambda,\xi_x;n+1)\left(\left|f^{(n)}(a)\right|^q-\left|f^{(n)}(b)\right|^q\right)+A(\lambda,\xi_x;n)\left|f^{(n)}(b)\right|^q,$$

and

$$\int_{\xi_x}^{1}(1-z)^{n-1}|1-z-n(1-\mu)|\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z$$
$$=A(1-\mu,1-\xi_x;n)\left|f^{(n)}(a)\right|^q+A(1-\mu,1-\xi_x;n+1)\left(\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q\right).$$

Substituting the above equations into (3.3) leads to the inequality (3.1). □

REMARK 3.1. Taking $n=1$, $x=\frac{a+b}{2}$, $q=1$, and $0\leqslant\lambda\leqslant\frac{1}{2}\leqslant\mu\leqslant 1$ in Theorem 3.1 may derive [3, Theorem 3.1] which has been recited as the above Theorem 1.5.

THEOREM 3.2. *Let $n\in\mathbb{N}$ and $f:I\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ and $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a convex function on $[a,b]$ for $q>1$ and $q(n-1)\geqslant r\geqslant 0$, then for $x\in[a,b]$ and $\lambda,\mu\in[0,1]$, we have*

$$\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}\left\{\left[A\left(\lambda,\xi_x;\frac{nq-r-1}{q-1}\right)\right]^{1-1/q}\right.$$
$$\times\left[A(\lambda,\xi_x;r+2)\left(\left|f^{(n)}(a)\right|^q-\left|f^{(n)}(b)\right|^q\right)+A(\lambda,\xi_x;r+1)\left|f^{(n)}(b)\right|^q\right]^{1/q}$$
$$+\left[A\left(1-\mu,1-\xi_x;\frac{nq-r-1}{q-1}\right)\right]^{1-1/q}\left[A(1-\mu,1-\xi_x;r+1)\left|f^{(n)}(a)\right|^q\right.$$
$$\left.\left.+A(1-\mu,1-\xi_x;r+2)\left(\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q\right)\right]^{1/q}\right\},\tag{3.4}$$

where ξ_x and $A(\lambda,c;r+1)$ are defined as in Lemma 2.1 and (3.2) respectively.

Proof. Similar to the proof of Theorem 3.1, by the convexity, Lemma 2.1, and Hölder's inequality, we have

$$\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}\left\{\left(\int_0^{\xi_x} z^{[(n-1)q-r]/(q-1)}|n\lambda-z|\,\mathrm{d}z\right)^{1-1/q}\right.$$

$$\times \left[\int_0^{\xi_x} z^r |n\lambda - z| \left(z \left| f^{(n)}(a) \right|^q + (1-z) \left| f^{(n)}(b) \right|^q \right) \mathrm{d}z \right]^{1/q}$$
$$+ \left(\int_{\xi_x}^1 (1-z)^{[(n-1)q-r]/(q-1)} |1 - z - n(1-\mu)| \,\mathrm{d}z \right)^{1-1/q}$$
$$\times \left[\int_{\xi_x}^1 (1-z)^r |1 - z - n(1-\mu)| \left(z \left| f^{(n)}(a) \right|^q + (1-z) \left| f^{(n)}(b) \right|^q \right) \mathrm{d}z \right]^{1/q} \Bigg\}.$$

The rest is the same as in the proof of Theorem 3.1. □

COROLLARY 3.1. *Under the conditions of Theorem 3.2,*

1. *if $r = 0$, then*

$$\begin{aligned} \left| S(x;\lambda,\mu) \right| \leqslant \frac{(b-a)^n}{n!} \Bigg\{ & \left[A\left(\lambda, \xi_x; \frac{nq-1}{q-1} \right) \right]^{1-1/q} \Big[A(\lambda,\xi_x;2) \left(\left| f^{(n)}(a) \right|^q \right. \\ & \left. - \left| f^{(n)}(b) \right|^q \right) + A(\lambda,\xi_x;1) \left| f^{(n)}(b) \right|^q \Big]^{1/q} \\ & + \left[A\left(1-\mu, 1-\xi_x; \frac{nq-1}{q-1} \right) \right]^{1-1/q} \Big[A(1-\mu,1-\xi_x;1) \left| f^{(n)}(a) \right|^q \\ & + A(1-\mu,1-\xi_x;2) \left(\left| f^{(n)}(b) \right|^q - \left| f^{(n)}(a) \right|^q \right) \Big]^{1/q} \Bigg\}; \end{aligned}$$

2. *if $r = (n-1)q$, then*

$$\begin{aligned} \left| S(x;\lambda,\mu) \right| \leqslant \frac{(b-a)^n}{n!} \Big\{ & \left[A(\lambda,\xi_x;1) \right]^{1-1/q} \Big[A(\lambda,\xi_x;(n-1)q+2) \\ & \times \left(\left| f^{(n)}(a) \right|^q - \left| f^{(n)}(b) \right|^q \right) + A(\lambda,\xi_x;(n-1)q+1) \left| f^{(n)}(b) \right|^q \Big]^{1/q} \\ & + \left[A(1-\mu,1-\xi_x;1 \right]^{1-1/q} \Big[A(1-\mu,1-\xi_x;(n-1)q+1) \left| f^{(n)}(a) \right|^q \\ & + A(1-\mu,1-\xi_x;(n-1)q+2) \left(\left| f^{(n)}(b) \right|^q - \left| f^{(n)}(a) \right|^q \right) \Big]^{1/q} \Big\}. \end{aligned}$$

THEOREM 3.3. *Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ and $a,b \in I^\circ$ with $a < b$. If $\left| f^{(n)} \right|^q$ is a convex function on $[a,b]$ for $q > 1$, then for $x \in [a,b]$ and $\lambda,\mu \in [0,1]$, we have*

$$\begin{aligned} & \left| S(x;\lambda,\mu) \right| \\ \leqslant & \frac{(b-a)^n}{n!} \Bigg\{ \left[B\left(\lambda, 0, 1, \xi_x; \frac{2q-1}{q-1} \right) \right]^{1-1/q} \Big[A(0,\xi_x;(n-1)q+1) \left| f^{(n)}(a) \right|^q \\ & + B(0,-1,1,\xi_x;(n-1)q+1) \left| f^{(n)}(b) \right|^q \Big]^{1/q} + \left[B\left(1-\mu, 0, 1, 1-\xi_x; \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\ & \times \Big[B(0,-1,1,1-\xi_x;(n-1)q+1) \left| f^{(n)}(a) \right|^q + A(0,1-\xi_x;(n-1)q+1) \left| f^{(n)}(b) \right|^q \Big]^{1/q} \Bigg\}, \end{aligned}$$

where ξ_x *and* $A(\lambda,c;r+1)$ *are defined as in Lemma* 2.1 *and* (3.2) *respectively,* $\alpha,\beta\in\mathbb{R}$, $c\geqslant 0$, $r>-1$, *and*

$$B(\lambda,\alpha,\beta,c;r+1)=\frac{1}{(r+1)(r+2)}\times\begin{cases}[(r+2)\beta+\alpha n\lambda](n\lambda)^{r+1}\\ \qquad -[\alpha c(r+1)+\beta(r+2)+\alpha n\lambda](n\lambda-c)^{r+1}, & n\lambda\geqslant c,\\ [(r+2)\beta+\alpha n\lambda](n\lambda)^{r+1}\\ \qquad +[\beta(r+2)+\alpha(c+cr+n\lambda)](c-n\lambda)^{r+1}, & 0\leqslant n\lambda\leqslant c.\end{cases}\tag{3.5}$$

Proof. Similar to the proof of Theorem 3.1, applying Lemma 2.1, the convexity, and Hölder's inequality results in

$$\begin{aligned}\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}\Bigg\{&\left(\int_0^{\xi_x}|n\lambda-z|^{q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}\left[\int_0^{\xi_x}z^{(n-1)q}\left(z\left|f^{(n)}(a)\right|^q\right.\right.\\&\left.\left.+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\right]^{1/q}+\left(\int_{\xi_x}^1|1-z-n(1-\mu)|^{q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}\\&\times\left[\int_{\xi_x}^1(1-z)^{(n-1)q}\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\right]^{1/q}\Bigg\}.\end{aligned}$$

The rest is also similar to the proof of Theorem 3.1. □

THEOREM 3.4. *Let* $n\in\mathbb{N}$ *and* $f:I\to\mathbb{R}$ *be a mapping such that* $f^{(n-1)}(x)$ *is absolutely continuous on* $[a,b]$ *such that* $a,b\in I^\circ$ *with* $a<b$. *If* $\left|f^{(n)}\right|^q$ *for* $q>1$ *is a convex function on* $[a,b]$, *then for* $x\in[a,b]$ *and* $\lambda,\mu\in[0,1]$, *we have*

$$\begin{aligned}&\left|S(x;\lambda,\mu)\right|\\&\leqslant\frac{(b-a)^n}{n!}\Bigg\{\left[B\left(\lambda,1,0,\xi_x;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\Big[A(0,\xi_x;(n-2)q+2)\left|f^{(n)}(a)\right|^q\\&+B(0,-1,1,\xi_x;(n-2)q+2)\left|f^{(n)}(b)\right|^q\Big]^{1/q}+\left[B\left(1-\mu,1,0,1-\xi_x;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\\&\times\Big[B(0,-1,1,1-\xi_x;(n-2)q+2)\left|f^{(n)}(a)\right|^q+A(0,1-\xi_x;(n-2)q+2)\left|f^{(n)}(b)\right|^q\Big]^{1/q}\Bigg\},\end{aligned}$$

where ξ_x *and* $A(\lambda,c;r+1)$ *are defined as in Lemma* 2.1 *and* (3.2) *respectively, and* $B(\lambda,\alpha,\beta,c;r+1)$ *is defined as in* (3.5) *for* $\alpha,\beta\in\mathbb{R}$, $c\geqslant 0$, *and* $r>-1$.

Proof. Similar to the proof of Theorem 3.1, applying Lemma 2.1, the convexity, and Hölder's inequality results in

$$\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}\Bigg\{\left(\int_0^{\xi_x}z|n\lambda-z|^{q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}\left[\int_0^{\xi_x}z^{(n-2)q+1}\left(z\left|f^{(n)}(a)\right|^q\right.\right.$$

$$+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\Bigg]^{1/q}+\left(\int_{\xi_x}^1(1-z)|1-z-n(1-\mu)|^{q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}$$

$$\times\left[\int_{\xi_x}^1(1-z)^{(n-2)q+1}\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\right]^{1/q}\Bigg\}.$$

The rest is similar to the proof of Theorem 3.1. □

THEOREM 3.5. *Let $n\in\mathbb{N}$ and $f:I\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,b]$ such that $a,b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is a convex function on $[a,b]$ for $q>1$, then for $x\in[a,b]$ and $\lambda,\mu\in[0,1]$, we have*

$$\begin{aligned}\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q}&\Big\{\xi_x^{(nq-1)/q}\big[B(\lambda,1,0,\xi_x;q+1)\left|f^{(n)}(a)\right|^q\\&+B(\lambda,-1,1,\xi_x;q+1)\left|f^{(n)}(b)\right|^q\big]^{1/q}\\&+\left(1-\xi_x\right)^{(nq-1)/q}\big[B(1-\mu,-1,1,1-\xi_x;q+1)\left|f^{(n)}(a)\right|^q\\&+B(1-\mu,1,0,1-\xi_x;q+1)\left|f^{(n)}(b)\right|^q\big]^{1/q}\Big\},\end{aligned}$$

where ξ_x and $B(\lambda,\alpha,\beta,c;r+1)$ are defined as in Lemma 2.1 and (3.5) respectively.

Proof. Similar to the proof of Theorem 3.1, employing the convexity, Lemma 2.1, and Hölder's inequality produces

$$\begin{aligned}\left|S(x;\lambda,\mu)\right|\leqslant\frac{(b-a)^n}{n!}&\Bigg\{\left(\int_0^{\xi_x}z^{(n-1)q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}\Bigg[\int_0^{\xi_x}|n\lambda-z|^q\left(z\left|f^{(n)}(a)\right|^q\right.\\&\left.+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\Bigg]^{1/q}+\left(\int_{\xi_x}^1(1-z)^{(n-1)q/(q-1)}\,\mathrm{d}z\right)^{1-1/q}\\&\times\left[\int_{\xi_x}^1|1-z-n(1-\mu)|^q\left(z\left|f^{(n)}(a)\right|^q+(1-z)\left|f^{(n)}(b)\right|^q\right)\mathrm{d}z\right]^{1/q}\Bigg\}.\end{aligned}$$

The rest is also similar to the proof of Theorem 3.1. □

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