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KERNEL FUNCTION BASED INTERIOR–POINT ALGORITHMS FOR SEMIDEFINITE OPTIMIZATION

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Abstract. We propose a primal-dual interior-point algorithm for semidefinite optimization(SDO) based on a class of kernel functions which are both eligible and self-regular. New search directions and proximity measures are defined based on these functions. We show that the algorithm has $\mathscr{O}(\sqrt{n}\log\frac{n}{\varepsilon})$ and $\mathscr{O}(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity results for small- and large-update methods, respectively. These are the best known complexity results for such methods. This is the first algorithm for SDO based on this kernel function, as far as we know.

1. Introduction

In this paper, we consider the standard SDO problem as follows:

$$\min\{C\bullet X : A_i\bullet X = b_i,\ 1\leqslant i\leqslant m,\ X\succeq 0\} \tag{1}$$

and its dual problem

$$\max\{b^Ty : \sum_{i=1}^{m} y_iA_i + S = C,\ S\succeq 0\}, \tag{2}$$

where $C,\ A_i\in\mathbf{S}^n,\ 1\leqslant i\leqslant m,\ b,\ y\in\mathbf{R}^m$ and $C\bullet X = \mathbf{Tr}(CX)$, where $\mathbf{Tr}$ denotes the trace.

Primal-dual interior-point method(IPM) is one of the most efficient numerical methods for solving large classes of optimization problems and highly efficient in both theory and practice. It is well known that the SDO has a variety of applications in engineering problems, such as optimal control, combinatorics, image processing, sensor networks, financial mathematics and statistics([16]).

Many researchers have proposed kernel function based interior-point algorithms for various optimization problems. Most of the polynomial-time interior-point algorithms are based on the logarithmic kernel function with $\mathscr{O}(\sqrt{n}\log\frac{n}{\varepsilon})$ and $\mathscr{O}(n\log\frac{n}{\varepsilon})$ iteration complexity for small- and large-update methods, respectively([13]). Recently,

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Peng et al.([12]) first defined a class of self-regular kernel functions, proposed primal-dual IPM for linear optimization(LO) and generalized to second-order cone optimization(SOCO) and SDO. Roos et al.([2]) first defined eligible kernel functions which were defined by 4 conditions on the function and proposed a primal-dual IPM for LO and simplified the complexity analysis of Peng et al.'s in [12]. Ghami et al.([5]) proposed a primal-dual interior-point algorithm for LO based on a kernel function with a trigonometric barrier term but they didn't obtain the best known complexity result for large-update method.

Several interior-point methods(IPMs) for LO have been successfully extended to SDO. Wang et al.([14]) proposed a primal-dual IPM for SDO based on a generalized version of the kernel function in [2] and obtained $\mathscr{O}(q^2\sqrt{n}\log\frac{n}{\varepsilon})$, $q > 1$, and $\mathscr{O}(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity results for small- and large-update, respectively. Ghami et al.([4]) extended the IPM for LO in [2] to SDO and obtained the similar iteration bounds as analog of LO. Ghami et al.([6]) proposed a primal-dual IPM for SDO based on a generalized version of the kernel function in [1] and obtained $\mathscr{O}(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ for large-update method. Lee et al.([10]) defined a new class of kernel functions and obtained the best known complexity results of the small- and large-update IPMs based on the kernel function for LO and SDO.

Motivated by their works, we extend a primal-dual interior-point algorithm for LO in [3] to SDO and obtain the best known compexity results of small- and large-update methods. Ghami et al.([4] proposed primal-dual IPMs for SDO based on eligible kernel functions which is first defined in [2]. They are using 4 eligible conditions for complexity analysis. However, when we analyze the algorithm, we only use three conditions of four eligible conditions. Furthermore, we summarize kernel functions which obtained the best known complexity results of small- and large-update methods for SDO up to date.

This paper is organized as follows: In section 2, we introduce fundamental concepts and describe the generic interior-point algorithm for SDO. In section 3, we define a class of kernel functions and give its essential properties for the complexity analysis. In section 4, we give some basic definitions of matrix-valued functions and the complexity results of small- and large-update algorithms for SDO. Moreover we review the kernel functions which obtain the best known complexity results for SDO up to date.

We will make use of the following notations throughout the paper: $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of real, nonnegative real and positive real vectors with n components, respectively. $\mathbf{S}^n$, $\mathbf{S}^n_+$ and $\mathbf{S}^n_{++}$ denote the set of symmetric, symmetric positive semidefinite and symmetric positive definite $n \times n$ matrices, respectively. $\|\cdot\|$ denotes the Frobenius norm for matrices. $\succeq$ denotes the nonnegativity in the Löwner partial order for symmetric matrices, i.e., $A \succeq B (A \succ B)$ if $A - B$ is symmetric positive semidefinite(positive definite). For $Q \in \mathbf{S}^n_{++}$, $Q^{1/2}$ denotes the symmetric square root of Q. For any $V \in \mathbf{S}^n$, we denote by $\lambda(V)$ the vector of eigenvalues of V arranged in non-increasing order, that is, $\lambda_1(V) \geqslant \lambda_2(V) \geqslant \cdots \geqslant \lambda_n(V)$ and $\Lambda := \mathrm{diag}(\lambda(V))$, i.e., the diagonal matrix from a vector $\lambda(V)$. $\mathbf{E}$ denotes an $n \times n$ identity matrix. For $a \in \mathbf{R}$, $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. For $f(x)$, $g(x) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(x) = \mathscr{O}(g(x))$ if $f(x) \leqslant c_1 g(x)$ for some positive constant c_1 and $f(x) = \Theta(g(x))$ if $c_2 g(x) \leqslant f(x) \leqslant c_3 g(x)$ for some positive constants c_2 and c_3.

2. Pimal-dual IPM for SDO

In this section we introduce the generic IPM for SDO problem (1) and (2). Throughout the paper we assume that the matrix A_i, $1 \leqslant i \leqslant m$, are linearly independent and SDO (1) and (2) satisfy the interior-point condition(IPC), i.e., there exists $X \in \mathscr{F}_P$, $S \in \mathscr{F}_D$ with $X \succ 0$, $S \succ 0$, where $\mathscr{F}_P$ and $\mathscr{F}_D$ denote the feasible sets of the problem (1) and (2), respectively.

Finding an optimal solution of the problem (1) and (2) is equivalent to solving the following system:

$$\begin{cases} A_i \bullet X = b_i, \ 1 \leqslant i \leqslant m, \ X \succeq 0, \\ \sum_{i=1}^{m} y_i A_i + S = C, \ S \succeq 0, \\ XS = 0. \end{cases} \tag{3}$$

The basic idea of primal-dual IPMs is to replace the complementarity condition of (3), $XS = 0$, by the parameterized equation $XS = \mu \mathbf{E}$ with $X, S \succ 0$ and $\mu > 0$. So we consider the following system:

$$\begin{cases} A_i \bullet X = b_i, \ 1 \leqslant i \leqslant m, \ X \succ 0, \\ \sum_{i=1}^{m} y_i A_i + S = C, \ S \succ 0, \\ XS = \mu \mathbf{E}, \end{cases} \tag{4}$$

where $\mu > 0$. Under the assumptions, the system (4) has a unique solution $(X(\mu), y(\mu), S(\mu))$ for $\mu > 0$ and $(X(\mu), y(\mu), S(\mu))$ converges to the optimal solution of the problem (1) and (2) as μ goes to zero([16]). We call the set $\{(X(\mu), y(\mu), S(\mu)) \mid \mu > 0\}$ the central path of the problems (1) and (2). Applying Newton's method to the system (4), we obtain the Newton-system as follows:

$$\begin{cases} A_i \bullet \Delta X = 0, \ 1 \leqslant i \leqslant m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ X \Delta S + \Delta X S = \mu \mathbf{E} - XS. \end{cases} \tag{5}$$

Under the assumptions, the system (5) has a unique search direction $(\Delta X, \Delta y, \Delta S)$. Note that ΔS is symmetric from the second equation of (5), but ΔX may not be symmetric. Various methods of symmetrizing the third equation of (5) are proposed so that the new system has a unique symmetric solution. In this paper, we use the NT symmetrizing scheme([11]).

Let $P := X^{1/2}(X^{1/2}SX^{1/2})^{-1/2}X^{1/2} = S^{-1/2}(S^{1/2}XS^{1/2})^{1/2}S^{-1/2}$ and $D := P^{1/2}$.
Define

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1}XSD)^{1/2}. \tag{6}$$

Then matrices D and V are symmetric positive definite.

Define for $\mu > 0,\ 1 \leqslant i \leqslant m$,

$$\bar{A}_i := \frac{1}{\sqrt{\mu}} D A_i D, \quad D_X := \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \quad D_S := \frac{1}{\sqrt{\mu}} D \Delta S D. \tag{7}$$

Then the third equation of (5) becomes

$$V(D_S + D_X) + (D_X + D_S)V = 2E - 2V^2. \tag{8}$$

For $V \in \mathbf{S}^n_{++}$, by the spectral theorem, there exists an orthogonal matrix Q such that $V = Q^T \Lambda Q$. Then (8) becomes

$$\Lambda Q(D_S + D_X)Q^T + Q(D_X + D_S)Q^T \Lambda = 2E - 2\Lambda^2.$$

Since $Q(D_X + D_S)Q^T$ is a diagonal matrix, we have $D_X + D_S = V^{-1} - V$. Then the Newton-system (5) can be rewritten as follows:

$$\begin{cases} \bar{A}_i \bullet D_X = 0,\ 1 \leqslant i \leqslant m, \\ \sum_{i=1}^m \Delta y_i \bar{A}_i + D_S = 0, \\ D_X + D_S = V^{-1} - V. \end{cases} \tag{9}$$

The solution $(D_X, \Delta y, D_S)$ of the system (9) is called the scaled NT search direction. We call $\psi : \mathbf{R}_{++} \to \mathbf{R}_+$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\psi'(1) = \psi(1) = 0, \quad \psi''(t) > 0,\ t > 0, \quad \lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty. \tag{10}$$

For $V = Q^T diag(\lambda_1(V), \lambda_2(V), \cdots, \lambda_n(V))Q$, the spectral decomposition of $V \in \mathbf{S}^n_{++}$, we generalize a function $\psi(t) : \mathbf{R}_{++} \to \mathbf{R}_+$ to the matrix function $\psi(V) : \mathbf{S}^n_{++} \to \mathbf{S}^n$ as follows:

$$\begin{aligned} \psi(V) &:= Q^T \mathrm{diag}\left(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \cdots, \psi(\lambda_n(V))\right) Q, \\ \psi'(V) &:= Q^T \mathrm{diag}\left(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), \cdots, \psi'(\lambda_n(V))\right) Q. \end{aligned} \tag{11}$$

Note that the right-hand side $V^{-1} - V$ of the third equation of (9) is $-\psi_c'(V)$, $\psi_c(t) := \frac{t^2-1}{2} - \log t$. Here $\psi_c(V)$ is called the classical kernel function.

In this paper, we replace $-\psi_c'(V)$ by $-\psi'(V)$, where a kernel function $\psi(t)$ will be defined in (14) and $\psi'(V)$ denotes the second equation of (11). Then we have

$$\begin{cases} \bar{A}_i \bullet D_X = 0,\ 1 \leqslant i \leqslant m, \\ \sum_{i=1}^m \Delta y_i \bar{A}_i + D_S = 0, \\ D_X + D_S = -\psi'(V). \end{cases} \tag{12}$$

For any kernel function $\psi(t)$, we define $\Psi(V) : \mathbf{S}^n_{++} \to \mathbf{R}_+$ by

$$\Psi(V) := \mathbf{Tr}\left(\psi(V)\right) = \sum_{i=1}^n \psi(\lambda_i(V)). \tag{13}$$

Then $\Psi(V)$ is strictly convex with respect to $V \succ 0$ and vanishes at its global minimal point $V = E$ and $\Psi(E) = 0$. Since D_X and D_S are orthogonal, for $\mu > 0$,

$$\Psi(V) = 0 \Leftrightarrow V = \mathbf{E} \Leftrightarrow D_X = D_S = 0 \Leftrightarrow X = X(\mu),\ S = S(\mu).$$

Hence we can use $\Psi(V)$ as a proximity function to measure the distance between the current iteration and the corresponding μ-center.
The primal-dual interior-point algorithm for SDO works as follows: Assume that $\tau \geqslant 1$ and there is a strictly feasible point (X,y,S) which is in a τ-neighborhood of the given μ-center([9]). We update μ to $\mu_+ := (1-\theta)\mu$, for some fixed $\theta \in (0,1)$, and then solve the system (12) and (7) to obtain the NT search direction. The positivity condition of a new iteration is ensured with the right choice of the step size α. This procedure is repeated until we find a new iteration (X_+, y_+, S_+) which is in a τ-neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(X,y,S) := (X_+, y_+, S_+)$. We repeat the process until $n\mu < \varepsilon$.

Primal-Dual Algorithm for SDO

Input:
  A threshold parameter $\tau \geqslant 1$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter θ, $0 < \theta < 1$;
  a strictly feasible (X^0, S^0) and $\mu^0 = 1$ such that $\Psi(X^0, S^0, \mu^0) \leqslant \tau$;
begin
  $X := X^0$; $S := S^0$; $\mu := \mu^0$;
  while $n\mu \geqslant \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu$;
    while $\Psi(X,S,\mu) > \tau$ do
    begin
      solve the system (12) and (7) for ΔX, Δy, ΔS;
      determine a step size α;
      $X := X + \alpha\Delta X$;
      $S := S + \alpha\Delta S$;
      $y := y + \alpha\Delta y$;
    end
  end
end

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3. Kernel function and its properties

In this section, we define a class of kernel functions and give its essential properties for complexity analysis.

We consider a function $\psi(t)$ as follows:

$$\psi(t) := \frac{p(t^2-1)}{2} + \frac{t^{-pq}-1}{q(q+1)} - \frac{pq(t-1)}{q+1},\ p \geqslant 1,\ q > 0,\ t > 0. \tag{14}$$

DEFINITION 1. ([2]) Let $\phi : \mathbf{R}_{++} \to \mathbf{R}_+$ be of $\mathscr{C}^3$. ϕ is eligible if it satisfies the following conditions:
(i) $t\phi''(t) + \phi'(t) \geqslant 0,\ \ 0 < t < 1,$
(ii) $t\phi''(t) - \phi'(t) \geqslant 0,\ \ t > 1,$
(iii) $\phi^{(3)}(t) < 0,\ \ t > 0,$
(iv) $2(\phi''(t))^2 - \phi'(t)\phi^{(3)}(t) > 0,\ \ 0 < t < 1.$

For $\psi(t)$, the first three derivatives are as follows:

$$\begin{aligned} \psi'(t) &= pt - \frac{p}{(q+1)t^{pq+1}} - \frac{pq}{q+1}, \\ \psi''(t) &= p + \frac{p(pq+1)}{(q+1)t^{pq+2}}, \\ \psi^{(3)}(t) &= -\frac{p(pq+1)(pq+2)}{(q+1)t^{pq+3}}. \end{aligned} \tag{15}$$

From (15), we have

$$\psi''(t) > p,\ \ p \geqslant 1,\ \ t > 0. \tag{16}$$

Note that the kernel function $\psi(t)$ defined as in (14) is eligible and self-regular([8]).

LEMMA 1. *Let $\psi(t)$ be defined as in (14). Then we have for $p \geqslant 1,\ q > 0,$*
(i) $t\psi''(t) + \psi'(t) \geqslant 0,\ \ 0 < t < 1,$
(ii) $t\psi''(t) - \psi'(t) \geqslant 0,\ \ t > 1,$
(iii) $\psi^{(3)}(t) < 0,\ \ t > 0.$

REMARK 1. (Lemma 2.4 in [2]) If $\psi(t)$ satisfy (ii) and (iii) in Lemma 1, then

$$\psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) > 0,\ \ t > 0,\ \ \beta > 1.$$

LEMMA 2. *For $\psi(t)$, we have for $p \geqslant 1,\ q > 0,$*
(i) $\frac{p}{2}(t-1)^2 \ \leqslant\ \psi(t)\ \leqslant\ \frac{1}{2p}(\psi'(t))^2,\ \ t > 0,$
(ii) $\psi(t)\ \leqslant\ \frac{1}{2}\psi''(1)(t-1)^2,\ \ t \geqslant 1.$

Proof. For (i), using the first condition of (10) and (16), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi\ \geqslant\ p\int_1^t \int_1^\xi d\zeta d\xi = \frac{p}{2}(t-1)^2$$

which proves the first inequality. From (16),

$$\psi(t)=\int_1^t\int_1^\xi \psi''(\zeta)d\zeta d\xi \leqslant \frac{1}{p}\int_1^t\int_1^\xi \psi''(\xi)\psi''(\zeta)d\zeta d\xi$$
$$=\frac{1}{p}\int_1^t \psi''(\xi)\psi'(\xi)d\xi=\frac{1}{p}\int_1^t \psi'(\xi)d\psi'(\xi)=\frac{1}{2p}(\psi'(t))^2.$$

For (ii), using Taylor's theorem, the first condition of (10) and Lemma 1 (iii), we have

$$\psi(t)=\psi(1)+\psi'(1)(t-1)+\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{3!}\psi^{(3)}(\xi)(t-1)^3$$
$$=\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{3!}\psi^{(3)}(\xi)(t-1)^3$$
$$<\frac{1}{2}\psi''(1)(t-1)^2,$$

for some ξ, $1\leqslant\xi\leqslant t$. This completes the proof. □

REMARK 2. Let $\psi_b(t)$ be the barrier term of the kernel function $\psi(t)$, i.e., $\psi(t) := \frac{p(t^2-1)}{2}+\psi_b(t)$. Since $\psi_b'(t)=-\frac{p}{q+1}t^{-pq-1}-\frac{pq}{q+1}<0$, $\psi_b(t)$ is monotonically decreasing with respect to $t>0$.

LEMMA 3. *Let* $\underline{\rho}:[0,\infty)\to[1,\infty)$ *be the inverse function of* $\psi(t)$ *for* $t\geqslant 1$. *Then we have*

$$\underline{\rho}(u)\ \leqslant\ 1+\sqrt{\frac{2u}{p}},\quad p\geqslant 1,\quad u\geqslant 0.$$

Proof. Let $u=\psi(t)$, $t\geqslant 1$. Then $u=\frac{p(t^2-1)}{2}+\psi_b(t)$. Using the first inequality of Lemma 2 (i), $t=\underline{\rho}(u)\leqslant 1+\sqrt{\frac{2u}{p}}$. This completes the proof. □

LEMMA 4. *Let* $\rho:[0,\infty)\to(0,1]$ *be the inverse function of* $-\frac{1}{2}\psi'(t)$ *for* $0<t\leqslant 1$. *Then we have*

$$\rho(z)\geqslant\left(\frac{p}{p+2(q+1)z}\right)^{\frac{1}{pq+1}},\ \ p\geqslant 1,\ \ q>0,\ \ z\geqslant 0.$$

Proof. Let $z:=-\frac{1}{2}\psi'(t)$ for $0<t\leqslant 1$. By the definition of ρ, $\rho(z)=t$, for $z\geqslant 0$. From (15), $-pt+\frac{p(t^{-pq-1})+q}{q+1}=2z$ and $\frac{p}{q+1}t^{-pq-1}=2z+pt-\frac{pq}{q+1}\leqslant 2z+p-\frac{pq}{q+1}$, $0<t\leqslant 1$. This implies $t^{-pq-1}\leqslant\frac{2(q+1)z+p}{p}$. Hence, $\rho(z)=t\geqslant(\frac{p}{p+2(q+1)z})^{\frac{1}{pq+1}}$. This completes the proof. □

LEMMA 5. *Let* $\beta \geqslant 1$ *and* $p \geqslant 1$. *Then* $\psi(\beta t) \leqslant \psi(t) + \frac{p(\beta^2-1)t^2}{2}$, $t > 0$.

Proof. By Remark 2, $\psi_b(\beta t) - \psi_b(t) \leqslant 0$ for $\beta \geqslant 1$. Hence we have

$$\begin{aligned}\psi(\beta t) &= \frac{p((\beta t)^2 - 1)}{2} + \psi_b(\beta t)\\ &= \frac{p(t^2-1)}{2} + \psi_b(t) + \frac{p}{2}(\beta^2 t^2 - t^2) + \psi_b(\beta t) - \psi_b(t)\\ &= \psi(t) + \frac{p(\beta^2-1)t^2}{2} + \psi_b(\beta t) - \psi_b(t)\\ &\leqslant \psi(t) + \frac{p(\beta^2-1)t^2}{2}.\end{aligned}$$

This completes the proof. □

4. Complexity analysis

In this section, we analyze the complexity of small- and large-update interior-point algorithms for SDO. For the analysis of the algorithm we also use the norm-based proximity measure $\delta(V)$ as follows:

$$\delta(V) := \frac{1}{2}\|\psi'(V)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2} = \frac{1}{2}\|D_X + D_S\|, \;\; V \in \mathbf{S}^n_{++}. \tag{17}$$

In the following lemma, we give a relationship between two proximity measures.

LEMMA 6. *Let* $\delta(V)$ *and* $\Psi(V)$ *be defined as in* (17) *and* (13)*, respectively. Then we have*

$$\delta(V) \geqslant \sqrt{\frac{p}{2}\Psi(V)}, \quad V \in \mathbf{S}^n_{++}, \quad p \geqslant 1.$$

Proof. Using (17) and the second inequality of Lemma 2 (i),

$$\delta^2(V) = \frac{1}{4}\sum_{i=1}^{n}(\psi'(\lambda_i(V)))^2 \geqslant \frac{p}{2}\sum_{i=1}^{n}\psi(\lambda_i(V)) = \frac{p}{2}\Psi(V).$$

Hence we have $\delta(V) \geqslant \sqrt{\frac{p}{2}\Psi(V)}$. This completes the lemma. □

In the following, using Remark 1, we estimate the effect of a μ-update on the value of $\Psi(V)$.

LEMMA 7. (Lemma 4.16 in [15]) *Let* $\underline{\rho}$ *be defined as in* Lemma 3. *Then we have*

$$\Psi(\beta V) \leqslant n\psi\left(\beta\underline{\rho}\left(\frac{\Psi(V)}{n}\right)\right), \; V \in \mathbf{S}^n_{++}, \; \beta \geqslant 1.$$

LEMMA 8. *Let* $0<\theta<1$ *and* $V_+ := \frac{V}{\sqrt{1-\theta}}$. *If* $\Psi(V)\leqslant\tau$, *then for* $p\geqslant 1$ *and* $q>0$, *we have*

(*i*) $\Psi(V_+)\leqslant \frac{p(pq+q+2)}{2(q+1)(1-\theta)}\left(\sqrt{n}\theta+\sqrt{\frac{2\tau}{p}}\right)^2$,

(*ii*) $\Psi(V_+)\leqslant \frac{1}{2(1-\theta)}\left(2\tau+np\theta+\theta\sqrt{8np\tau}\right)$.

Proof. For (i), since $\frac{1}{\sqrt{1-\theta}}\geqslant 1$ and $\underline{\rho}\left(\frac{\Psi(V)}{n}\right)\geqslant 1$, we have $\frac{\underline{\rho}\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}\geqslant 1$. Using Lemma 7 with $\beta=\frac{1}{\sqrt{1-\theta}}$, Lemma 2 (ii), Lemma 3 and $\Psi(V)\leqslant\tau$, we have

$$\Psi(V_+)\leqslant n\psi\left(\frac{1}{\sqrt{1-\theta}}\underline{\rho}\left(\frac{\Psi(V)}{n}\right)\right)\leqslant\frac{np(pq+q+2)}{2(q+1)}\left(\frac{\underline{\rho}\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}-1\right)^2$$
$$\leqslant\frac{np(pq+q+2)}{2(q+1)}\left(\frac{1+\sqrt{\frac{2\tau}{np}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2$$
$$\leqslant\frac{np(pq+q+2)}{2(q+1)}\left(\frac{\theta+\sqrt{\frac{2\tau}{np}}}{\sqrt{1-\theta}}\right)^2=\frac{p(pq+q+2)}{2(q+1)(1-\theta)}\left(\sqrt{n}\theta+\sqrt{\frac{2\tau}{p}}\right)^2,$$

where the last inequality holds from $1-\sqrt{1-\theta}=\frac{\theta}{1+\sqrt{1-\theta}}\leqslant\theta,\ 0\leqslant\theta<1$.

For (ii), using Lemma 7 with $\beta=\frac{1}{\sqrt{1-\theta}}$, Lemma 5, $\Psi(V)\leqslant\tau$ and Lemma 3, we obtain the other upper bound of $\Psi(V)$ as follows:

$$\Psi(V_+)\leqslant n\psi\left(\frac{\underline{\rho}(\frac{\Psi(V)}{n})}{\sqrt{1-\theta}}\right)\leqslant n\left(\psi\left(\underline{\rho}\left(\frac{\Psi(V)}{n}\right)\right)+\frac{p\underline{\rho}^2(\frac{\Psi(V)}{n})}{2}\left(\frac{1}{1-\theta}-1\right)\right)$$
$$=\Psi(V)+\frac{np\theta}{2(1-\theta)}\underline{\rho}^2\left(\frac{\Psi(V)}{n}\right)\leqslant\tau+\frac{np\theta}{2(1-\theta)}\left(1+\sqrt{\frac{2\tau}{np}}\right)^2$$
$$=\frac{2\tau+np\theta+\theta\sqrt{8np\tau}}{2(1-\theta)}.$$

This completes the proof. □

Define for $p\geqslant 1$, $q>0$ and $0<\theta<1$,

$$\tilde{\Psi}_0:=\frac{p(pq+q+2)}{2(q+1)(1-\theta)}\left(\sqrt{n}\theta+\sqrt{\frac{2\tau}{p}}\right)^2,\quad \bar{\Psi}_0:=\frac{2\tau+np\theta+\theta\sqrt{8np\tau}}{2(1-\theta)}. \tag{18}$$

We will use $\tilde{\Psi}_0$ and $\bar{\Psi}_0$ for the upper bounds of $\Psi(V)$ for small- and large-update methods, respectively.

REMARK 3. For small-update method with $\tau = \mathscr{O}(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, $\tilde{\Psi}_0 = \mathscr{O}(\frac{p(pq+q+2)}{q+1})$, and for large-update method with $\tau = \mathscr{O}(n)$ and $\theta = \Theta(1)$, $\bar{\Psi}_0 = \mathscr{O}(pn)$.

LEMMA 9. (Lemma 3.3.14(d) in [7]) *Let $M, N \in \mathbf{S}^n$ be two nonsingular matrices and $f(t)$ a real-valued function such that $f(e^t)$ is a convex function. Then*

$$\sum_{i=1}^{n} f(\sigma_i(MN)) \leqslant \sum_{i=1}^{n} f(\sigma_i(M)\sigma_i(N)),$$

where $\sigma_i(M)$, $i = 1, 2, \cdots, n$, denote the singular values of M.

Using Lemma 1 (i) and Lemma 9, we obtain the following lemma.

LEMMA 10. (Proposition 5.2.6 in [12]) *Let $V_1, V_2 \in \mathbf{S}^n_{++}$. Then we have*

$$\Psi\left(\left[V_1^{1/2} V_2 V_1^{1/2}\right]^{1/2}\right) \leqslant \frac{1}{2}\left(\Psi(V_1)+\Psi(V_2)\right).$$

Now we compute a feasible step size α and the decrement of the proximity function during an inner iteration. For fixed μ, if we take a step size α along the search direction $(\Delta X, \Delta y, \Delta S)$, we obtain a new iteration (X_+, y_+, S_+), where

$$X_+ := X + \alpha \Delta X, \;\; y_+ := y + \alpha \Delta y, \;\; S_+ := S + \alpha \Delta S, \;\; \alpha > 0.$$

Using (7), we can rewrite X_+ and S_+ as follows:

$$X_+ = \sqrt{\mu} D(V + \alpha D_X) D, \;\; S_+ = \sqrt{\mu} D^{-1}(V + \alpha D_S) D^{-1}. \tag{19}$$

From (6), we have $V_+ = \frac{1}{\sqrt{\mu}}\left(D^{-1} X_+ S_+ D\right)^{1/2}$. From (19), we have

$$V_+^2 = (V + \alpha D_X)(V + \alpha D_S).$$

Since $V + \alpha D_X \in \mathbf{S}^n_{++}$, $V + \alpha D_S \in \mathbf{S}^n_{++}$, V_+^2 is similar to $(V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}$. This implies that the eigenvalues of V_+ are the same as those of $\left((V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}\right)^{1/2}$. From (13), we have

$$\Psi(V_+) = \Psi\Big(\big((V + \alpha D_X)^{1/2}(V + \alpha D_S)(V + \alpha D_X)^{1/2}\big)^{1/2}\Big).$$

By Lemma 10, we obtain

$$\Psi(V_+) \leqslant \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right). \tag{20}$$

Define for $\alpha > 0$,

$$f(\alpha) := \Psi(V_+) - \Psi(V), \;\; f_1(\alpha) := \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right) - \Psi(V).$$

From (20), $f(\alpha) \leqslant f_1(\alpha)$ and $f(0) = f_1(0) = 0$.
Recall that a matrix $M(t)$ is said to be a matrix of functions if each entry of $M(t)$ is a function of t, i.e., $M(t) = [M_{ij}(t)]$, $1 \leqslant i, j \leqslant n$([7]). $M(t)$ is said to be differentiable if and only if $M_{ij}(t)$, $1 \leqslant i, j \leqslant n$, is differentiable. Let $M(t) := [M_{ij}(t)]$, $1 \leqslant i, j \leqslant n$, be a matrix of functions which are differentiable and diagonalizable at t. Then we have

$$\frac{d}{dt}M(t) = \left[\frac{d}{dt}M_{ij}(t)\right].$$

For notational convenience, we denote by $M'(t)$ the derivative of the matrix of functions $M(t)$. Furthermore, we have

$$\frac{d}{dt}\mathbf{Tr}(M(t)) = \mathbf{Tr}\left(\frac{d}{dt}M(t)\right) = \mathbf{Tr}(M'(t)), \tag{21}$$

$$\frac{d}{dt}\mathbf{Tr}(\psi(M(t))) = \mathbf{Tr}\left(\psi'(M(t))M'(t)\right). \tag{22}$$

Using (21), (22) and (13), we have

$$f_1'(\alpha) = \frac{1}{2}\mathbf{Tr}\left(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S\right),$$

$$f_1''(\alpha) = \frac{1}{2}\mathbf{Tr}\left(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2\right).$$

From the third equation of the system (12) and (17), we have

$$f_1'(0) = \frac{1}{2}\mathbf{Tr}\left(\psi'(V)(D_X+D_S)\right) = \frac{1}{2}\mathbf{Tr}\left(-(\psi'(V))^2\right) = -2\delta^2(V). \tag{23}$$

For notational convenience, let $\delta := \delta(V)$ and $\Psi := \Psi(V)$. To find the default step size, we need the following lemmas.
Using D_X and D_S are orthogonal and Lemma 1 (iii), we have the following lemma.

LEMMA 11. (Lemma 5.19 in [15]) Let δ be defined as in (17). Then we have

$$f_1''(\alpha) \leqslant 2\delta^2\psi''(\lambda_n(V) - 2\alpha\delta).$$

Using Lemma 11 and (23), we have the following lemma.

LEMMA 12. (Lemma 4.2 in [2]) If the step size α satisfies

$$-\psi'(\lambda_n(V) - 2\alpha\delta) + \psi'(\lambda_n(V)) \leqslant 2\delta, \tag{24}$$

then

$$f_1'(\alpha) \leqslant 0.$$

Using Lemma 1 (iii), we have following lemma.

LEMMA 13. (Lemma 4.3 in [2]) Let ρ be defined as in Lemma 4. Then the largest step size α which satisfies (24) is given by

$$\bar{\alpha} := \frac{1}{2}(\rho(\delta) - \rho(2\delta)).$$

Using Lemma 13 and Lemma 1 (iii), we have the following lemma.

LEMMA 14. (Lemma 4.4 in [2]) Let ρ and $\bar{\alpha}$ be defined as in Lemma 4 and Lemma 13, respectively. Then

$$\bar{\alpha} \geqslant \frac{1}{\psi''(\rho(2\delta))}.$$

We define the default step size $\tilde{\alpha}$ as follows:

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\delta))}. \tag{25}$$

Since $f_1(0) = 0$, $f_1'(0) = -2\delta^2$ and Lemma 1 (iii), using Lemma 1.3.3 in [12], we have the following lemma.

LEMMA 15. *If the step size* α *is such that* $\alpha \leqslant \bar{\alpha}$, *then*

$$f(\alpha) \leqslant -\alpha\delta^2.$$

Combining Lemma 15 and (25), we have the following lemma.

LEMMA 16. *Let* $\tilde{\alpha}$ *be defined as in* (25). *Then we have*

$$f(\tilde{\alpha}) \leqslant -\frac{\delta^2}{\psi''\Big(\rho(2\delta)\Big)}.$$

LEMMA 17. *Let* $\tau \geqslant 1$ *and* $\tilde{\alpha}$ *be defined as in* (25). *Then we have*

$$f(\tilde{\alpha}) \leqslant -\frac{\Psi^{\frac{pq}{2(pq+1)}}}{32\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}},\ p \geqslant 1,\ q > 0.$$

Proof. Since $\psi(t)$ is eligible, using Lemma 1 (iii), Lemma 4, Lemma 6 and the assumption $\tau \geqslant 1$, we have

$$\begin{aligned}
\psi''(\rho(2\delta)) &\leqslant \psi''\left(\left(\frac{p}{p+4(q+1)\delta}\right)^{\frac{1}{pq+1}}\right) \\
&= p + \frac{p(pq+1)}{q+1}\left(1 + \frac{4(q+1)\delta}{p}\right)^{\frac{pq+2}{pq+1}} \\
&\leqslant p\left(2 + \frac{pq+1}{q+1}\left(\sqrt{2} + \frac{4(q+1)}{p}\right)^{\frac{pq+2}{pq+1}}\right)\delta^{\frac{pq+2}{pq+1}}
\end{aligned} \tag{26}$$

Using (26), Lemma 16, $q>0$, $p\geqslant 1$ and Lemma 6, we have

$$\begin{aligned}
f(\tilde{\alpha}) &\leqslant -\frac{\delta^2}{p\left(2+\frac{pq+1}{q+1}\left(\sqrt{2}+\frac{4(q+1)}{p}\right)^{\frac{pq+2}{pq+1}}\right)\delta^{\frac{pq+2}{pq+1}}}\\
&\leqslant -\frac{\delta^{\frac{pq}{pq+1}}}{p\left(2+(pq+1)(q+1)^{\frac{1}{pq+1}}\left(\sqrt{2}+\frac{4}{p}\right)^{\frac{pq+2}{pq+1}}\right)}\\
&\leqslant -\frac{\delta^{\frac{pq}{pq+1}}}{32p(pq+1)(q+1)^{\frac{1}{pq+1}}} \leqslant -\frac{\Psi^{\frac{pq}{2(pq+1)}}}{32\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}}.
\end{aligned}$$

This completes the proof. □

LEMMA 18. (Proposition 1.3.2 in [12]) Suppose that a sequence $\{t_k>0,\ k=0,1,2,\cdots,\bar{K}\}$ is satisfying the following inequality:

$$t_{k+1}\leqslant t_k-\eta t_k^{\gamma},\ \eta>0,\ \gamma\in[0,1),\ k=0,1,2,\cdots,\bar{K}.$$

Then

$$\bar{K}\leqslant\left\lceil\frac{t_0^{1-\gamma}}{\eta(1-\gamma)}\right\rceil.$$

We denote the value of Ψ after μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as Ψ_l, $l=0,1,2,3,\cdots,K$, where K denotes the total number of inner iterations per an outer iteration. Then we have $\Psi_0\leqslant\tilde{\Psi}_0$ and $\Psi_0\leqslant\bar{\Psi}_0$, where $\tilde{\Psi}_0$ and $\bar{\Psi}_0$ are defined in (18). Then we have $\Psi_{K-1}>\tau$ and $0\leqslant\Psi_K\leqslant\tau$.

THEOREM 1. *Let $\tilde{\Psi}_0$ and $\bar{\Psi}_0$ be defined as in (18) and let K_1 and K_2 be the total numbers of inner iterations in the outer iteration for small- and large-update methods, respectively. Then for $p\geqslant 1$ and $q>0$, we have*

(i) $K_1 \leqslant \left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\tilde{\Psi}_0^{\frac{pq+2}{2(pq+1)}}\right\rceil$,

(ii) $K_2 \leqslant \left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\bar{\Psi}_0^{\frac{pq+2}{2(pq+1)}}\right\rceil$.

Proof. For (i), combining Lemma 17 and Lemma 18 with $\eta:=\frac{1}{32\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}}$ and $\gamma:=\frac{pq}{2(pq+1)}$, we have

$$K_1\leqslant\left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\tilde{\Psi}_0^{\frac{pq+2}{2(pq+1)}}\right\rceil.$$

For (ii), by the same way, we have

$$K_2 \leqslant \left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\bar{\Psi}_0^{\frac{pq+2}{2(pq+1)}} \right\rceil.$$

This completes the proof. □

THEOREM 2. *Let a SDO* (1) *and* (2) *be given and* $p \geqslant 1$, $q > 0$, $0 < \theta < 1$, *and* $\tau \geqslant 1$. *If there is a strictly feasible starting point* (X^0, S^0) *s.t.* $\Psi(X^0, S^0, \mu^0 := 1) \leqslant \tau$, *then the total number of iterations required by the algorithm to have an approximate solution s.t.* $n\mu < \varepsilon$ *is bounded by*

$$\left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\tilde{\Psi}_0^{\frac{pq+2}{2(pq+1)}} \frac{1}{\theta}\log\frac{n}{\varepsilon} \right\rceil$$

and

$$\left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\bar{\Psi}_0^{\frac{pq+2}{2(pq+1)}} \frac{1}{\theta}\log\frac{n}{\varepsilon} \right\rceil$$

for small- and large-update methods, respectively.

Proof. If the central path parameter μ has the initial value $\mu^0 = 1$ and is updated by multiplying $1-\theta$ with $0 < \theta < 1$, then after at most $\lceil \frac{1}{\theta}\log\frac{n}{\varepsilon} \rceil$ iterations we have $n\mu < \varepsilon$ ([13]). Hence the total number of iterations for small- and large-update methods are bounded by

$$\left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\tilde{\Psi}_0^{\frac{pq+2}{2(pq+1)}} \frac{1}{\theta}\log\frac{n}{\varepsilon} \right\rceil$$

and

$$\left\lceil 64\sqrt{2}p(pq+1)(q+1)^{\frac{1}{pq+1}}\bar{\Psi}_0^{\frac{pq+2}{2(pq+1)}} \frac{1}{\theta}\log\frac{n}{\varepsilon} \right\rceil$$

respectively. This completes the proof. □

REMARK 4. Using Remark 3 and Theorem 2, we obtain $\mathscr{O}(p(pq+1)(q+1)^{\frac{1}{pq+1}} (\frac{p(pq+q+2)}{q+1})^{\frac{pq+2}{2(pq+1)}}\sqrt{n}\log\frac{n}{\varepsilon})$ and $\mathscr{O}(p(pq+1)(q+1)^{\frac{1}{pq+1}}(pn)^{\frac{pq+2}{2(pq+1)}}\log\frac{n}{\varepsilon})$ iteration complexity for small- and large-update methods, respectively. Let $\tilde{h}(p,q) := p(pq+1)(q+1)^{\frac{1}{pq+1}}(\frac{p(pq+q+2)}{q+1})^{\frac{pq+2}{2(pq+1)}}$. Then for $p=1$, $\tilde{h}(1,q) = 2^{\frac{q+2}{2(q+1)}}(q+1)^{\frac{q+2}{q+1}}$ is increasing for $q > 0$. Choosing $p=1$, $q = \textit{any constant}$, we have $\mathscr{O}(\sqrt{n}\log\frac{n}{\varepsilon})$ iteration complexity result for small-update method. Similarly, let $\bar{h}(p,q) := p(pq+1)(q+1)^{\frac{1}{pq+1}}(pn)^{\frac{pq+2}{2(pq+1)}}$. Then for $p=1$, $\bar{h}(1,q) = n^{\frac{q+2}{2(q+1)}}(q+1)^{\frac{q+2}{q+1}}$ is increasing if $q > \frac{1}{2}\log n - 2$. Choosing $p=1$ and $q = \frac{1}{2}\log n - 1$, we have $\mathscr{O}(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ iteration complexity for large-update method. These are the best known complexity results for such methods.

In the following Table 1, we summarize the kernel functions which obtained the best known complexity results for small- and large-update methods for SDO until now.

Table1. Kernel functions which obtained the best known complexity for SDO

No.	kernel function	Reference
1	$\frac{t^{p+1}-1}{p(p+1)}+\frac{t^{1-q}-1}{q(q-1)}+\frac{p-q}{pq}(t-1), p\geqslant 1,\ q>1$	[12]
2	$\frac{t^2-1}{2}+\frac{t^{1-q}-1}{q-1},\ q>1$	[4]
3	$\frac{t^2-1}{2}-\frac{t^{1-q}-1}{q(q-1)}-\frac{q-1}{q}(t-1),\ q>1$	[4]
4	$\frac{t^{1+p}-1}{1+p}+\frac{t^{1-q}-1}{q-1},\ 0\leqslant p\leqslant 1,\ q>1$	[14]
5	$\frac{t^{1+p}-1}{1+p}+\frac{e^{\sigma(1-t)}-1}{\sigma},\ 0\leqslant p\leqslant 1,\ \sigma\geqslant 1$	[6]
6	$\frac{p(t^2-1)}{2}+\frac{t^{-pq}-1}{q},\ p\geqslant 1,\ q>0$	[10]
7	$\frac{p(t^2-1)}{2}+\frac{t^{-pq}-1}{q(q+1)}+\frac{pq(t-1)}{q+1},\ p\geqslant 1,\ q>0$	new

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