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ON INEQUALITIES FOR QUERMASSINTEGRALS AND DUAL QUERMASSINTEGRALS OF DIFFERENCE BODIES

WEIDONG WANG

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Abstract. In this paper, inequalities for quermassintegrals and dual quermassintegrals of difference bodies are given. In particular, an extension of the Rogers-Shephard inequality is obtained.

1. Introduction

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$ and let $\mathscr{K}_o^n$ denote the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. Denote by $\mathscr{S}_o^n$ the set of star bodies (about the origin) in $\mathbb{R}^n$ and by S^{n-1} the unit sphere in $\mathbb{R}^n$. By $V(K)$ we denote the n-dimensional volume of body K and for the standard unit ball B in $\mathbb{R}^n$ we set $\omega_n = V(B)$.

If $K \in \mathscr{K}^n$, then its support function $h_K = h(K,\cdot):\ \mathbb{R}^n \to (-\infty,\infty)$ is defined by (see [2, 8])

$$h(K,x) = \max\{x\cdot y : y \in K\}, \qquad x \in \mathbb{R}^n,$$

where $x\cdot y$ denotes the standard inner product of x and y.

For each $K, L \in \mathscr{K}^n$ and $\lambda, \mu \geqslant 0$ (not both zero), the Minkowski linear combination $\lambda K + \mu L \in \mathscr{K}^n$ is defined by (see [2, 8])

$$h(\lambda K + \mu L, \cdot) = \lambda h(K,\cdot) + \mu h(L,\cdot). \tag{1.1}$$

If $K \in \mathscr{K}^n$, the difference body DK of K is defined by (see [8])

$$DK = K + (-K), \tag{1.2}$$

For the difference body DK, the affine invariant $V(DK)/V(K)$ is estimated by (see [8])

$$2^n \leqslant \frac{V(DK)}{V(K)} \leqslant \binom{2n}{n}. \tag{1.3}$$

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Equality in the first inequality in (1.3) holds if and only if K is a centrally symmetric convex body, and in the second inequality in (1.3) if and only if K is a simplex. The right-hand side inequality of (1.3) is known as Rogers-Shephard inequality (see [6, 8]). For further study on difference bodies one may see [1, 3, 6, 7] or book [8].

For $K \in \mathscr{K}^n$, the quermassintegrals $W_i(K)$ $(i=0,1,\cdots,n)$ of K are given by (see [2, 8])

$$W_i(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS_i(K,u), \tag{1.4}$$

where $S_i(K,\cdot)(i=0,1,\cdots,n-1)$ are positive Borel measures called mixed surface area measures on S^{n-1}. $S_{n-1}(K,\cdot)$ is the Lebesgue measure on S^{n-1}.

From (1.4), we easily see that

$$W_0(K) = V(K). \tag{1.5}$$

For $K \in \mathscr{S}_o^n$ and any real i, the dual quermassintegrals $\tilde{W}_i(K)$ of K are defined by (see [2, 8])

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}du. \tag{1.6}$$

Obviously,

$$\tilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n}du = V(K). \tag{1.7}$$

In this paper, we continue to study difference bodies by quermassintegrals and dual quermassintegrals. First, we extend the left-hand side inequality of (1.3) to quermassintegrals of difference bodies as follows:

THEOREM 1.1. *If $K \in \mathscr{K}^n$, $i=0,1,\cdots,n-1$, then*

$$\frac{W_i(DK)}{W_i(K)} \geqslant 2^{n-i}, \tag{1.8}$$

with equality if and only if K is a centrally symmetric convex body.

By letting $i=0$ in (1.8) and combining it with (1.5), the left-hand side inequality of (1.3) is obtained.

Further, for the polar of difference body, we establish the following inequality of dual quermassintegrals.

THEOREM 1.2. *If $K \in \mathscr{K}_o^n$, i is real and $i \neq n$, then for $i<n$ or $n<i<n+1$,*

$$\frac{\tilde{W}_i(D^*K)}{\tilde{W}_i(K^*)} \leqslant \frac{1}{2^{n-i}}, \tag{1.9}$$

with equality if and only if K is a centered convex body; for $i>n+1$, the inequality sign in (1.9) is reversed.

From Theorem 1.2 we also get

THEOREM 1.3. *If $K \in \mathscr{K}_o^n$, i is real and $0 \leqslant i < n$, then*

$$\tilde{W}_i(D^*K)\tilde{W}_i(K) \leqslant \frac{\omega_n^2}{2^{n-i}}, \tag{1.10}$$

with equality if and only if $i=0$ and K is a centered ellipsoid or $0<i<n$ and K is a centered ball.

In particular, let $i=0$ in Theorem 1.3 and together with (1.7), we have:

COROLLARY 1.4. *If $K \in \mathscr{K}_o^n$, then*

$$V(D^*K)V(K) \leqslant \frac{\omega_n^2}{2^n},$$

with equality if and only if K is a centered ellipsoid.

Our second aim is to give an extension of the Rogers-Shephard inequality for the dual quermassintegrals of difference bodies.

THEOREM 1.5. *If $K \in \mathscr{K}^n$, i is integer and $i<n$, then there exits a point $x_0 \in K$ such that*

$$\tilde{W}_i(DK) \leqslant \binom{2n-i}{n}\tilde{W}_i(K-x_0), \tag{1.11}$$

with equality if and only if K is a simplex.

Obviously, taking $i=0$ in Theorem 1.5 and noting that $V(K-x_0)=V(K)$, we get from inequality (1.11) the Rogers-Shephard inequality.

2. Preliminaries

If K is compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function $\rho_K = \rho(K,\cdot)$ is defined by (see [2, 8])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda u \in K\}, \tag{2.1}$$

for all $u \in S^{n-1}$. If ρ_K is positive and continuous, K is called a star body (about the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If K is compact star-shaped with respect to $x \in \mathbb{R}^n$, its radial function $\rho_K(x,\cdot)$ with respect to x is defined, for all $u \in S^{n-1}$ such that the line through x parallel to u intersects K, by (see [3])

$$\rho_K(x,u) = \max\{\lambda \geqslant 0 : x+\lambda u \in K\}. \tag{2.2}$$

From (2.1) and (2.2), it easily follows

$$\rho_K(x,u) = \rho_{K-x}(u), \tag{2.3}$$

for $u \in S^{n-1}$. We call (2.2) the extended radial function of K with respect to x. If x is the origin o, then $\rho_K(x,u) = \rho_K(u)$ for $u \in S^{n-1}$.

From (2.3) and (1.6), the following can immediately be obtained:

$$\tilde{W}_i(K-x) = \frac{1}{n}\int_{S^{n-1}} \rho_K(x,u)^{n-i}du. \tag{2.4}$$

If $K \in \mathscr{K}_o^n$, the polar body K^* of K is defined by (see [2, 8])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}.$$

Obviously, we have $(K^*)^* = K$.

From (2.1), we also know that: if $K \in \mathscr{K}_o^n$, then the support and the radial functions of K^* are given respectively by (see [2, 8])

$$h_{K^*} = \frac{1}{\rho_K} \qquad \text{and} \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.5}$$

For $K, L \in \mathscr{S}_o^n$ and real $\lambda, \mu \geqslant 0$ (not both zero), the harmonic radial combination $\lambda \cdot K \tilde{+} \mu \cdot L \in \mathscr{S}_o^n$ is defined by (see [2])

$$\rho(\lambda \cdot K \tilde{+} \mu \cdot L, \cdot)^{-1} = \lambda \rho(K, \cdot)^{-1} + \mu \rho(KL, \cdot)^{-1}. \tag{2.6}$$

From (1.1), (2.5) and (2.6), we easily see that

$$(K+L)^* = K^* \tilde{+} L^*. \tag{2.7}$$

The notion of the radial pth mean body was given by Gardner and Zhang (see [3]). For $K \in \mathscr{K}^n$, the radial pth mean body R_pK of K is defined for nonzero $p > -1$ by

$$\rho_{R_pK}(u) = \left(\frac{1}{V(K)}\int_K \rho_K(x,u)^p dx\right)^{\frac{1}{p}}, \tag{2.8}$$

for each $u \in S^{n-1}$; for $p = 0$ define R_0K by

$$\rho_{R_0K}(u) = \exp\left(\frac{1}{V(K)}\int_K \log \rho_K(x,u)dx\right),$$

for each $u \in S^{n-1}$. In particular, they proved that:

If $K \in \mathscr{K}^n$ and $p > -1$, then

$$DK \subseteq c_{n,p}R_pK, \tag{2.9}$$

with equality if and only if K is a simplex. Here

$$c_{n,p} = (nB(p+1,n))^{-\frac{1}{p}}, \tag{2.10}$$

for nonzero $p > -1$, and

$$c_{n,0} = \lim_{p\to 0}(nB(p+1,n))^{-\frac{1}{p}} = \exp(\sum_{k=1}^{n} \frac{1}{k}),$$

where $B(p+1,n)$ is the Beta function.

3. Proofs of the Theorems

The proof of Theorem 1.1 requires the following lemma (see [8]):

LEMMA 3.1. *If* $K \in \mathscr{K}^n$, $0 \leqslant i < n$, *then*

$$W_i(K+L)^{\frac{1}{n-i}} \geqslant W_i(K)^{\frac{1}{n-i}} + W_i(L)^{\frac{1}{n-i}}, \tag{3.1}$$

with equality for $0 \leqslant i < n-1$ *if and only if* K *and* L *are homothetic; for* $i = n-1$, *(3.1) is an identity.*

Proof of Theorem 1.1. Using definition (1.2) of a difference body, inequality (3.1) and definition (1.4), we obtain

$$W_i(DK)^{\frac{1}{n-i}} = W_i(K+(-K))^{\frac{1}{n-i}} \geqslant W_i(K)^{\frac{1}{n-i}} + W_i(-K)^{\frac{1}{n-i}} = 2W_i(K)^{\frac{1}{n-i}}.$$

This yields (1.8). According to the condition of equality in (3.1), equality holds in (1.8) for $0 \leqslant i < n-1$ if and only if K and $-K$ are homothetic, i.e. K is a centrally symmetric convex body. For $i = n-1$, (1.8) is an identity. □

The proof of Theorem 1.2 requires the following lemma:

LEMMA 3.2. *If* $K, L \in \mathscr{S}_o^n$, i *is real and* $i \neq n$, *then for* $i < n$ *or* $n < i < n+1$,

$$\widetilde{W}_i(K \tilde{+} L)^{-\frac{1}{n-i}} \geqslant \widetilde{W}_i(K)^{-\frac{1}{n-i}} + \widetilde{W}_i(L)^{-\frac{1}{n-i}}; \tag{3.2}$$

with equality if and only if K *and* L *are dilates; for* $i > n+1$, *the inequality sign in (3.2) is reversed.*

Proof. For $i < n$ or $n < i < n+1$, since $i-n < 0$ or $0 < i-n < 1$, then using (1.6), (2.6) and the Minkowski integral inequality (see [4]), we know for $K, L \in \mathscr{S}_o^n$,

$$\begin{aligned}
\widetilde{W}_i(K \tilde{+} L)^{-\frac{1}{n-i}} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho(K \tilde{+} L, u)^{n-i} dS(u)\right]^{-\frac{1}{n-i}} \\
&= \left(\frac{1}{n}\right)^{-\frac{1}{n-i}} \left[\int_{S^{n-1}} (\rho(K \tilde{+} L, u)^{-1})^{-(n-i)} dS(u)\right]^{-\frac{1}{n-i}} \\
&= \left(\frac{1}{n}\right)^{-\frac{1}{n-i}} \left[\int_{S^{n-1}} (\rho(K,u)^{-1} + \rho(L,u)^{-1})^{-(n-i)} dS(u)\right]^{-\frac{1}{n-i}} \\
&\geqslant \left(\frac{1}{n}\right)^{-\frac{1}{n-i}} \left[\int_{S^{n-1}} \rho(K,u)^{n-i} dS(u) + \int_{S^{n-1}} \rho(L,u)^{n-i} dS(u)\right]^{-\frac{1}{n-i}} \\
&= \widetilde{W}_i(K)^{-\frac{1}{n-i}} + \widetilde{W}_i(L)^{-\frac{1}{n-i}}.
\end{aligned}$$

According to the equality condition in the Minkowski integral inequality, we see that equality holds in (3.2) if and only if K and L are dilates.

Similarly, the reversed inequality in (3.2) can be proved for the case $i > n+1$. □

Proof of Theorem 1.2. For $i<n$ or $n<i<n+1$, together with identity (2.7), definition (1.6) and inequality (3.2), we have that for $K\in\mathscr{K}_o^n$,

$$\begin{aligned}\tilde{W}_i(D^*K)^{-\frac{1}{n-i}} &= \tilde{W}_i((K+(-K))^*)^{-\frac{1}{n-i}}\\ &= \tilde{W}_i(K^*\tilde{+}(-K)^*)^{-\frac{1}{n-i}}\\ &\geqslant \tilde{W}_i(K^*)^{-\frac{1}{n-i}}+\tilde{W}_i((-K)^*)^{-\frac{1}{n-i}}\\ &= 2\tilde{W}_i(K^*)^{-\frac{1}{n-i}}.\end{aligned}$$

From this, inequality (1.9) is obtained. Equality holds in (1.9) if and only if K^* and $(-K)^*$ are dilates, i.e. K^* is a centered convex body, which means K is a centered convex body.

For $i>n+1$, according to (2.7), (1.6) and the reverse of (3.2), we easily prove that inequality (1.9) is reversed. □

For the proof of Theorem 1.3, the following two results are essential.

LEMMA 3.3. (see [5]) *If $K\in\mathscr{K}^n$, i is real and $0\leqslant i<n$, then*

$$\tilde{W}_i(K)\leqslant \omega_n^{\frac{i}{n}}V(K)^{\frac{n-i}{n}}, \tag{3.3}$$

with equality for $0<i<n$ if and only if K is a centered ball or for $i=0$.

LEMMA 3.4. (see [2]) *If K is a centered convex body, then*

$$V(K)V(K^*)\leqslant \omega_n^2, \tag{3.4}$$

with equality if and only if K is an ellipsoid.

Note that inequality (3.4) is the well-known Blaschke-Santalö inequality.

Proof of Theorem 1.3. From inequalities (1.9), (3.3) and (3.4), we get

$$\begin{aligned}\tilde{W}_i(D^*K)\tilde{W}_i(K) &\leqslant \frac{1}{2^{n-i}}\tilde{W}_i(K^*)\tilde{W}_i(K)\\ &\leqslant \frac{1}{2^{n-i}}\omega_n^{\frac{2i}{n}}\left[V(K)V(K^*)\right]^{\frac{n-i}{n}}\\ &\leqslant \frac{1}{2^{n-i}}\omega_n^2.\end{aligned}$$

This gives inequality (1.10).

According to the conditions of equality in (1.9), (3.3) and (3.4), equality holds in (1.10) if and only if $i=0$ and K is a centered ellipsoid or $0<i<n$ and K is a centered ball. □

In order to prove Theorem 1.5, we establish a lemma as follows:

LEMMA 3.5. *If $K \in \mathscr{K}^n$, i is real and $i < n$, then there exists a point $x_0 \in K$ such that*

$$\tilde{W}_i(R_{n-i}K) = \tilde{W}_i(K - x_0). \tag{3.5}$$

Proof. From formulas (1.6), (2.4) and definition (2.8), we have

$$\begin{aligned} \tilde{W}_i(R_{n-i}K) &= \frac{1}{n}\int_{S^{n-1}} \rho_{R_{n-i}K}(u)^{n-i}du \\ &= \frac{1}{nV(K)}\int_{S^{n-1}}\int_K \rho_K(x,u)^{n-i}dxdu \\ &= \frac{1}{V(K)}\int_K \left[\frac{1}{n}\int_{S^{n-1}} \rho_K(x,u)^{n-i}du\right]dx \\ &= \frac{1}{V(K)}\int_K \tilde{W}_i(K-x)dx. \end{aligned} \tag{3.6}$$

Integral mean value theorem implies that there exists a point $x_0 \in K$ such that

$$\int_K \tilde{W}_i(K-x)dx = \tilde{W}_i(K-x_0)\int_K dx = \tilde{W}_i(K-x_0)V(K).$$

This together with (3.6) gives

$$\tilde{W}_i(R_{n-i}K) = \frac{1}{V(K)}\tilde{W}_i(K-x_0)V(K) = \tilde{W}_i(K-x_0). \qquad \square$$

Proof of Theorem 1.5. Since $i < n$, by letting $p = n-i$ in (2.9), it follows that $DK \subseteq c_{n,n-i}R_{n-i}K$. Using definition (1.6), we have

$$\tilde{W}_i(DK) \leqslant (c_{n,n-i})^{n-i}\tilde{W}_i(R_{n-i}K). \tag{3.7}$$

Since i is an integer and $i < n$, (2.10) gives

$$(c_{n,n-i})^{n-i} = \tbinom{2n-i}{n}.$$

From this, (3.5) and (3.7), it follows that there exits $x_0 \in K$ such that

$$\tilde{W}_i(DK) \leqslant \tbinom{2n-i}{n}\tilde{W}_i(K-x_0).$$

According to the condition of equality in (2.9), equality holds in (1.11) if and only if K is a simplex. $\square$

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Wang Weidong
Department of Mathematics
China Three Gorges University
Yichang, 443002
China
e-mail: wdwxh722@163.com