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ON NON–SYMMETRIC t–CONVEX FUNCTIONS

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Abstract. Let $I \subseteq \mathbb{R}$ be an open interval. We consider a functions $f : I \to \mathbb{R}$ satisfying

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \quad (*)$$

for a fixed $t \in (0,1)$ and $x \leqslant y$. We discuss the relations between the class of functions satisfying inequality $(*)$ and the class of t-convex functions.

1. Introduction

Let $D \subseteq X$ be a convex subset of a real linear space X. A function $f : D \to \mathbb{R}$ is called convex iff the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \tag{1}$$

holds for all $t \in (0,1)$ and $x,y \in D$.

A class of convex functions is well investigated and has numerous applications (see A. W. Roberts and D. E. Varberg [5]). One of the most important directions in researching of this class of functions was a resignation from the assumption that the inequality (1) holds for each t taken from interval $(0,1)$. For example putting $t = \frac{1}{2}$ in (1) we get

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2}, \quad x,y \in D. \tag{2}$$

A function which satisfies the above inequality is called Jensen convex (J-convex) (see M. Kuczma [3] for more details). It is well known that there exist functions which are J-convex but not convex.

Now, fix $t \in (0,1)$. A function $f : D \to \mathbb{R}$ is called t-convex if it satisfies

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y), \quad x,y \in D.$$

It is rather simple calculation (see Z. Daróczy and Zs. Páles [1], N. Khun [2]) that each t-convex function is J-convex.

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Notice, that as points $x,y\in D$ can be taken arbitrary in the inequality (1), we trivially get that t-convex function has to be $(1-t)$-convex.

The notion of t-convexity is artificial from the view of the geometrical interpretation. To show this let $I\subseteq\mathbb{R}$ be an open interval and fix $x,y\in I$. The t-convexity of the function f means that the point $(tx+(1-t)y, f(tx+(1-t)y))$ lies below the line l passing through points $(x,f(x))$ and $(y,f(y))$. It is unnatural to assume that also the point $((1-t)x+ty, f((1-t)x+ty)$ lies below the line l.

The above remark gives a justification to the consideration of the following generalization (as far as we consider functions defined on interval) of the notion of t-convexity

DEFINITION 1.1. Let $t\in(0,1)$ be fixed. A function $f:I\to\mathbb{R}$ will be called non-symmetric t-convex iff the inequality

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y).$$

holds for all $x,y\in I$ such that $x\leqslant y$. Additionally, we say that f is strictly non-symmetric t-convex iff the above inequality is strict for all $x,y\in I$ such that $x<y$.

Moreover, we say that a function $f:I\to\mathbb{R}$ is non-symmetric t-affine iff the equation

$$f(tx+(1-t)y)=tf(x)+(1-t)f(y), \tag{3}$$

holds for all $x,y\in I$ such that $x\leqslant y$.

It is natural to ask whether there exists non – symmetric t-convex function which is not t-convex (Zs. Páles [4] posed this question with $t=\frac{1}{3}$).

In this paper we show first that non-symmetric t-affine functions are t-affine (Thm 2.1). This result suggests that using supporting technique (compare N. Khun [2]) one could give positive answer to the problem of Zs. Páles, and generally, to show that non-symmetric t-convex functions are in fact t-convex. However, in Example 3.1 we give negative answer to this general hypothesis in the case of transcendental t.

2. Non-symmetric t-affine functions

THEOREM 2.1. *Let $t\in(0,1)$ be fixed. If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is non-symmetric t-affine then it is t-affine.*

Proof. Define $\tilde{f}:\mathbb{R}\longrightarrow\mathbb{R}$ by $\tilde{f}:=f-f(0)$. Obviously $\tilde{f}$ is non-symmetric t-affine. Moreover $\tilde{f}(0)=0$. It suffices to show that $\tilde{f}$ is t-affine.

We began by proving that $\tilde{f}$ is t and $(1-t)$-homogeneous. Indeed, putting $y=0$ in (3) we get

$$\tilde{f}(tx)=t\tilde{f}(x),\quad x\leqslant 0. \tag{4}$$

Now, fix $u\geqslant 0$ we take $y(u)=-\frac{1-t}{t}u$. We have $y(u)\leqslant 0$. By virtue of (3) we get

$$0=\tilde{f}(0)=\tilde{f}(ty(u)+(1-t)u)=t\tilde{f}(y(u))+(1-t)\tilde{f}(u).$$

Thus

$$\tilde{f}(u) = -\frac{t}{1-t}\tilde{f}(y(u)), \quad u \geqslant 0. \tag{5}$$

Hence, for given $v \geqslant 0$, by (4) we get

$$\begin{aligned}\tilde{f}(tv) &= -\frac{t}{1-t}\tilde{f}(y(tv)) = -\frac{t}{1-t}\tilde{f}\left(-\frac{1-t}{t}(tv)\right) \\ &= -\frac{t}{1-t}\tilde{f}\left(t(-\frac{1-t}{t}v)\right) = -\frac{t}{1-t}\tilde{f}(ty(v)) = t\left(-\frac{t}{1-t}\tilde{f}(y(v))\right) = t\tilde{f}(v).\end{aligned}$$

Thus, in virtue of the above and (4) the function $\tilde{f}$ is t-homogenous.

Similarly we can show that $\tilde{f}$ is $(1-t)$-homogenous.

Now, we prove that $\tilde{f}$ is additive. Fix $x \in \mathbb{R}$. First we prove that

$$\tilde{f}(x+y) = \tilde{f}(x) + \tilde{f}(y), \tag{6}$$

for every $y \in \left(-\infty, \frac{t}{1-t}x\right] \cup \left[\frac{1-t}{t}x, +\infty\right)$.

Take $y \in \left(-\infty, \frac{t}{1-t}x\right]$. There exist $u, v \in \mathbb{R}$ such that $y = tu$ and $x = (1-t)v$. Since $y \leqslant \frac{t}{1-t}x$ we have $u \leqslant v$. Using non-symmetric t-affinity of $\tilde{f}$ together with t and $(1-t)$-homogeneity we get

$$\begin{aligned}\tilde{f}(x+y) &= \tilde{f}(tu + (1-t)v) = t\tilde{f}(u) + (1-t)\tilde{f}(v) \\ &= \tilde{f}(tu) + \tilde{f}((1-t)v) = \tilde{f}(x) + \tilde{f}(y).\end{aligned}$$

In the case $y \in \left[\frac{1-t}{t}x, +\infty\right)$ it is enough to change the role of x and y to get in the previous situation. This completes the proof of (6).

Now we show that $\tilde{f}$ is additive. We consider two cases.

Case 1. $t \in \left(0, \frac{1}{2}\right]$. Since for every $x \leqslant 0$ we have $\frac{1-t}{t}x \leqslant \frac{t}{1-t}x$, by (6)

$$\tilde{f}(x+y) = \tilde{f}(x) + \tilde{f}(y), \tag{7}$$

for $x \in (-\infty, 0]$ and $y \in \mathbb{R}$. Putting $y = -x$ in above equality we have

$$0 = \tilde{f}(x + (-x)) = \tilde{f}(x) + \tilde{f}(-x).$$

Hence the function $\tilde{f}$ is odd.

Fix $x \geqslant 0$ and $y \in \mathbb{R}$. We get

$$\begin{aligned}\tilde{f}(x+y) &= -\tilde{f}(-x-y) = -(\tilde{f}(-x) + \tilde{f}(-y)) \\ &= -\tilde{f}(-x) - \tilde{f}(-y) = \tilde{f}(x) + \tilde{f}(y).\end{aligned}$$

In virtue of (7) and the above equality the function $\tilde{f}$ is additive.

Case 2. $t \in \left[\frac{1}{2}, 1\right)$. The calculations are similar to this in the *Case 1* and thus are left to the reader.

Finally, by t-homogeneity and additivity of $\tilde{f}$ we have

$$\tilde{f}(tx + (1-t)y) = \tilde{f}(tx) + \tilde{f}((1-t)y) = t\tilde{f}(x) + (1-t)\tilde{f}(y).$$

This completes the proof. $\square$

3. Non-symmetric t-convex functions

Now, we give two examples. In Example 3.1 we show that the class of non-symmetric t-convex functions is greater than the class of t-convex functions, the second (see Example 3.2) shows that there exist non-symmetric t-convex functions which are J-concave. Our constructions are based on derivations. We start with reminding basic facts (for more information see M. Kuczma [3], Chapter 14).

DEFINITION 3.1. (see M. Kuczma [3]) Let $F \subset K$ be fields. A function $f : F \to K$ is called derivation iff it satisfies both the equations

$$f(x+y) = f(x) + f(y), \tag{8}$$

$$f(xy) = xf(y) + yf(x) \tag{9}$$

for all $x, y \in F$.

THEOREM 3.1. (see M. Kuczma [3], Thm 14.2.1) *Let $(K,+,\cdot)$ be a field of characteristic zero, let $(F,+,\cdot)$ be a subfield of $(K,+,\cdot)$, let S be an algebraic base of K over F, if it exists, and let $S = \emptyset$, otherwise. Let $f : F \to K$ be a derivation. Then, for every function $u : S \to K$, there exists a unique derivation $g : K \to K$ such that $g|_F = f$ and $g|_S = u$.*

EXAMPLE 3.1. Let $t \in (0,1)$ be a transcendental number $(over\,\mathbb{Q})$. There exists a strictly non-symmetric t-convex function $f : \mathbb{R} \to \mathbb{R}$ which is strictly non-symmetric $(1-t)$-concave.

Proof. By assumption $t \in \mathbb{R} \setminus algcl\mathbb{Q}$, so there exists an algebraic base S of $\mathbb{R}$ over $\mathbb{Q}$, such that $t \in S$. Let $u : S \to \mathbb{R}$ be an arbitrary function, such that $u(t) > 0$. If we take in Theorem 3.1 $F = \mathbb{Q}$, $K = \mathbb{R}$ then the trivial derivation $f_0 : \mathbb{Q} \to \mathbb{R}, f_0 \equiv 0$ can be uniquely extended onto $\mathbb{R}$ to a derivation $f : \mathbb{R} \to \mathbb{R}$ such that $f|_S = u$. Since f is a derivation we have

$$f(tx) = tf(x) + xf(t) = tf(x) + xu(t)$$

or, in the equivalent form,

$$f(tx) - tf(x) = xu(t), \qquad x \in \mathbb{R}.$$

By (8), (9) and the above we obtain

$$\begin{aligned} & f(tx+(1-t)y) - tf(x) - (1-t)f(y) \\ &= f(y) + f(t(x-y)) - f(y) - t(f(x) - f(y)) \\ &= f(t(x-y)) - tf(x-y) \\ &= (x-y)u(t). \end{aligned}$$

Now, if $x < y$, then $(x-y)u(t) < 0$. Consequently

$$f(tx+(1-t)y) < tf(x) + (1-t)f(y).$$

Thus f is strictly non-symmetric t-convex.
Similarly, if $x > y$, then $(x-y)u(t) > 0$, and

$$f(tx+(1-t)y) > tf(x)+(1-t)f(y),$$

Thus f is strictly non-symmetric $(1-t)$-concave. □

EXAMPLE 3.2. Let $t \in (0,1)$ be a transcendental number (over $\mathbb{Q}$). There exists a non-symmetric t-convex function $g : \mathbb{R} \to \mathbb{R}$ which is not Jensen convex.

Proof. Let $t \in (0,1)$ be a transcendental number (over $\mathbb{Q}$) and put

$$s : = 2max\{t, 1-t\}.$$

There exists a derivation $a\colon \mathbb{R} \to \mathbb{R}$ such that $a(t) = s$ (compare M. Kuczma [3], Lemma 14.2.2). Since the function a satisfies (9) we have

$$a(tx) = ta(x) + sx, \qquad x \in \mathbb{R}.$$

Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by the formula

$$g(x) := a(x) - |x|, \qquad x \in \mathbb{R}.$$

We will show that g is non-symmetric t-convex and Jensen concave function but not Jensen convex. Indeed, fix $x, y \in \mathbb{R}$, $x < y$. By the definition of g we get

$$\begin{aligned}
D_t g(x,y) : &= tg(x) + (1-t)g(y) - g(tx + (1-t)y) \\
&= t(a(x) - |x|) + (1-t)(a(y) - |y|) - (a(tx + (1-t)y) - |tx + (1-t)y|) \\
&= a(y) + ta(x-y) - a(y) - a(t(x-y)) + |tx + (1-t)y| - t|x| - (1-t)|y| \\
&= ta(x-y) - a(t(x-y)) + |tx + (1-t)y| - t|x| - (1-t)|y| \\
&= -s(x-y) + |tx + (1-t)y| - t|x| - (1-t)|y| \\
&= s(y-x) + |tx + (1-t)y| - t|x| - (1-t)|y|.
\end{aligned}$$

Put

$$L := |tx + (1-t)y|, \qquad R := t|x| + (1-t)|y|.$$

Consider three cases:
1° If $0 \leqslant x < y$ then $L - R = 0$ and hence $D_t g(x,y) = s(y-x) > 0$.
2° If $x < y < 0$ then $L - R = 0$ and hence $D_t g(x,y) = s(y-x) > 0$.
3° If $x < 0 \leqslant y$ then we have either
a) $x \geqslant \frac{t-1}{t} y$ then $L - R = 2tx$ and hence

$$D_t g(x,y) = s(y-x) + 2tx \geqslant 2t(y-x) + 2tx = 2ty \geqslant 0$$

or

b) $x < \frac{t-1}{t}y$ then $L-R = -2(1-t)y$ and hence

$$D_t g(x,y) = s(y-x) - 2(1-t)y \geqslant 2(1-t)(y-x) - 2(1-t)y$$
$$= -2(1-t)x > 0.$$

Due to the arbitrariness of $x,y \in \mathbb{R}$ $(x<y)$ we infer that g is non-symmetric t-convex. On the other hand, by convexity of function "$x \longrightarrow |x|$", we have

$$g\left(\frac{x+y}{2}\right) = a\left(\frac{x+y}{2}\right) - \left|\frac{x+y}{2}\right| \geqslant \frac{a(x)+a(y)}{2} - \frac{|x|+|y|}{2} = \frac{g(x)+g(y)}{2},$$

for $x,y \in \mathbb{R}$. This means that g is Jensen concave and not Jensen convex. □

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