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BOUNDS OF THE PERIMETER OF AN ELLIPSE USING ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

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Abstract. In this paper, we present several bounds for the perimeter of an ellipse in terms of arithmetic, geometric, and harmonic means, which improve some known results.

1. Introduction

Let a and b be the semiaxes of an ellipse with eccentricity $e=\sqrt{a^2-b^2}/a$, and $L(a,b)$ be the perimeter of the ellipse. Then

$$L(a,b)=4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2 t}dt=4a\mathscr{E}(e), \tag{1.1}$$

where $\mathscr{E}(r)(0\leqslant r\leqslant 1)$ is the complete elliptic integrals of the second kind. Elliptic integrals are so named because of their connection with $L(a,b)$. In turn, these are related to the Gaussian hypergeometric function ${}_2F_1$, defined by

$${}_2F_1(a,b;c;x)=F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \quad -1<x<1$$

with the Pochhammer symbol $(a)_n=a(a+1)(a+2)(a+3)\cdots(a+n-1)$ for $n\geqslant 1$ and $(a)_0=1$, $a\neq 0$. It is well known that the complete elliptic integrals of the first and second kinds can be expressed as

$$\mathscr{K}=\mathscr{K}(r)=\frac{\pi}{2}F(1/2,1/2;1;r^2)$$

and

$$\mathscr{E}=\mathscr{E}(r)=\frac{\pi}{2}F(-1/2,1/2;1;r^2),$$

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respectively. In particular, the Gaussian hypergeometric functions and complete elliptic integrals have many important applications in physics and geometric function theory. For these, and for their properties see [1, 4, 5, 15–17, 20].

During the past few centuries, many easily computable approximations to $L(a,b)$ have been suggested by a large number of mathematicians [6–9, 12, 19]. The Almkvist-Berndt survey article [2] has an extensive discussion of these approximations. These approximations and their historical and recent connections to the approximations of π can be found in the Borweins' book [10]. An excellent source for all the above ideas is the Anderson-Vamanamurthy-Vuorinen book *Conformal Invariants, Inequalities, and Quasiconformal Mappings* [5].

In 1883, it was proposed by Muir that $L(a,b)$ could be simply approximated by $2\pi[(a^{3/2}+b^{3/2})/2]^{2/3}$. In 1997, based on numerical experiments, Vuorinen [19] conjectured that

$$L(a,b) > 2\pi\left(\frac{a^{3/2}+b^{3/2}}{2}\right)^{2/3} \tag{1.2}$$

for all $a>b>0$. This conjecture was proved in [6].

In [18], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt.$$

Note that

$$L(a,b) = 2\pi T(a,b). \tag{1.3}$$

Let $A(a,b)=(a+b)/2$, $G(a,b)=\sqrt{ab}$, $H(a,b)=2ab/(a+b)$, $S(a,b)=\sqrt{(a^2+b^2)/2}$, and $M_p(a,b)=[(a^p+b^p)/2]^{1/p}$ $(p\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ be the arithmetic, geometric, harmonic, quadratic, and p-th power mean of two different positive numbers a and b, respectively. Then it is well known that

$$\begin{aligned}\min\{a,b\} &< H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < A(a,b) = M_1(a,b)\\ &< T(a,b) < S(a,b) = M_2(a,b) < \max\{a,b\}\end{aligned}$$

for all $a,b>0$ with $a\neq b$.

It is the aim of this paper to find the better bounds for the perimeter of an ellipse in terms of arithmetic, geometric, and harmonic means.

2. Lemmas

In order to establish our main results we need several formulas and lemmas, which we present in this section.

Throughout this paper, we denote by $r'=\sqrt{1-r^2}$ for $0<r<1$. For $0<r<1$, the following formulas were presented in [5, Appendix E, pp. 474–475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E}-r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E}-\mathcal{K}}{r},$$

$$\frac{d(\mathscr{E}-r'^2\mathscr{K})}{dr}=r\mathscr{K}, \qquad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}-r'^2\mathscr{K}}{1+r}.$$

LEMMA 2.1. [5, Theorem 1.25] *For $-\infty<a<b<\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following Lemma 2.2 can be found in [5, Theorem 3.21(1) and Exercise 3.43(10)].

LEMMA 2.2. (1) $(\mathscr{E}-r'^2\mathscr{K})/r^2$ *is strictly increasing from $(0,1)$ onto $(\pi/4,1)$;* (2) $[(\mathscr{E}-r'^2\mathscr{K})-r'^2(\mathscr{K}-\mathscr{E})]/r^4$ *is strictly increasing from $(0,1)$ onto $(3\pi/16,1)$.*

LEMMA 2.3. *If $k\in\mathbb{N}^*$, then the inequality*

$$\frac{3\sqrt{\pi}(k^2+15k+16)}{16(k+3)(2k+1)}>\frac{1}{\sqrt{k+1/4}}$$

holds for all $k\geqslant 5$.

Proof. Elementary computations lead to

$$\left[\frac{3\sqrt{\pi}(k^2+15k+16)}{16(k+3)(2k+1)}\right]^2-\left(\frac{1}{\sqrt{k+1/4}}\right)^2$$
$$=\frac{9\pi(4k+1)(k^2+15k+16)^2-1024(k+3)^2(2k+1)^2}{256(k+3)^2(2k+1)^2(4k+1)}. \tag{2.1}$$

Let

$$\begin{aligned} f(x)=&9\pi(4x+1)(x^2+15x+16)^2-1024(x+3)^2(2x+1)^2\\ =&36\pi x^5+(1089\pi-4096)x^4+(9522\pi-28672)x^3+(19593\pi-62464)x^2\\ &+(13536\pi-43008)x+2304\pi-9216. \end{aligned} \tag{2.2}$$

Then simple computations yield

$$f(5)=16(158949\pi-495616)>0, \tag{2.3}$$

$$\begin{aligned} f'(x)=&180\pi x^4+(4356\pi-16384)x^3+(28566\pi-86016)x^2\\ &+(39186\pi-124928)x+13536\pi-43008\\ >&500x^4-3000x^3+3500x^2-2000x-500\\ =&500[x^2(x^2-6x+6)+x^2-4x-1]\\ >&500(x^2+4)>0 \end{aligned} \tag{2.4}$$

for $x\in[5,\infty)$. From inequalities (2.3) and (2.4) we clearly see that $f(x)>0$ for $x\in[5,\infty)$. Then Lemma 2.3 follows from (2.1) and (2.2). $\square$

3. Main Results

THEOREM 3.1. *For $p,q\in(0,1)$, then the double inequality*

$$p\left(\frac{3}{2}A(a,b)-\frac{1}{2}G(a,b)\right)+(1-p)\left(\frac{5}{4}A(a,b)-\frac{1}{4}H(a,b)\right)<T(a,b)$$
$$<q\left(\frac{3}{2}A(a,b)-\frac{1}{2}G(a,b)\right)+(1-q)\left(\frac{5}{4}A(a,b)-\frac{1}{4}H(a,b)\right) \tag{3.1}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leqslant 16/\pi-5$ and $q\geqslant 1/4$.

Proof. Without loss of generality, we assume that $a>b$. Let $t=b/a\in(0,1)$ and $r=(1-t)/(1+t)$, then

$$\begin{aligned}&\frac{3A(a,b)/2-G(a,b)/2-T(a,b)}{3A(a,b)/2-G(a,b)/2-[5A(a,b)/4-H(a,b)/4]}\\ &=\frac{3(1+t)/4-\sqrt{t}/2-2\mathscr{E}(\sqrt{1-t^2})/\pi}{(1+t)/8-\sqrt{t}/2+t/[2(1+t)]}\\ &=\frac{6-2r'-8[2\mathscr{E}(r)-r'^2\mathscr{K}(r)]/\pi}{1-2r'+r'^2}.\end{aligned} \tag{3.2}$$

Let

$$J(r)=\frac{6-2r'-8[2\mathscr{E}(r)-r'^2\mathscr{K}(r)]/\pi}{1-2r'+r'^2}, \tag{3.3}$$

$J_1(r)=6-2r'-8[2\mathscr{E}(r)-r'^2\mathscr{K}(r)]/\pi$ and $J_2(r)=1-2r'+r'^2$. Then $J(r)=J_1(r)/J_2(r)$, $J_1(0)=J_2(0)=0$ and

$$\frac{J_1'(r)}{J_2'(r)}=\frac{1-4r'[\mathscr{E}(r)-r'^2\mathscr{K}(r)]/(\pi r^2)}{1-r'}. \tag{3.4}$$

Let $J_3(r)=1-4r'[\mathscr{E}(r)-r'^2\mathscr{K}(r)]/(\pi r^2)$ and $J_4(r)=1-r'$. Then $J_1'(r)/J_2'(r)=J_3(r)/J_4(r)$, $J_3(0)=J_4(0)=0$ and

$$\frac{J_3'(r)}{J_4'(r)}=\frac{4}{\pi}\frac{[\mathscr{E}(r)-r'^2\mathscr{K}(r)]-r'^2[\mathscr{K}(r)-\mathscr{E}(r)]}{r^4}. \tag{3.5}$$

It follows from (3.5) and Lemma 2(2) that $J_3'(r)/J_4'(r)$ is strictly increasing from $(0,1)$ onto $(3/4,4/\pi)$. Then equations (3.4) and (3.5), and Lemma 2.1 lead to the conclusion that $J(r)$ is strictly increasing in $(0,1)$. Moreover, making use of l'Hôptial's rule we get

$$\lim_{r\to 0^+}J(r)=\frac{3}{4}, \tag{3.6}$$

$$\lim_{r\to 1^-}J(r)=6-\frac{16}{\pi}. \tag{3.7}$$

Therefore, inequality (3.1) follows from (3.2), (3.3), (3.6) and (3.7) together with the monotonicity of $J(r)$.

Next, we prove that $p = 16/\pi - 5$ and $q = 1/4$ are the best possible parameters such that inequality (3.1) holds for all $a, b > 0$ with $a \neq b$.

For $\alpha \in (0,1)$ and $t \in (0,1)$. Let $r = (1-t)/(1+t)$. Then

$$\begin{aligned}
&\alpha\left(\frac{3}{2}A(1,t) - \frac{1}{2}G(1,t)\right) + (1-\alpha)\left(\frac{5}{4}A(1,t) - \frac{1}{4}H(1,t)\right) - T(1,t) \\
&= \left(\frac{3}{2}A(1,t) - \frac{1}{2}G(1,t) - T(1,t)\right) - (1-\alpha)\left(\frac{1}{4}A(1,t) - \frac{1}{2}G(1,t) + \frac{1}{4}H(1,t)\right) \\
&= \left(\frac{1}{4}A(1,t) - \frac{1}{2}G(1,t) + \frac{1}{4}H(1,t)\right)[J(r) + \alpha - 1] \\
&= \frac{(1-r')^2}{4(1+r)}[J(r) + \alpha - 1], \qquad (3.8)
\end{aligned}$$

where $J(r)$ is defined as in (3.3).

We divide the proof into two cases.

Case 1. $\alpha > 16/\pi - 5$. Then from (3.7) we know that

$$\lim_{r \to 1^-}[J(r) + \alpha - 1] = \alpha + 5 - \frac{16}{\pi} > 0. \qquad (3.9)$$

Inequality (3.9) implies that for any $\alpha > 16/\pi - 5$ there exists $0 < \delta_1 = \delta_1(r) < 1$, such that $J(r) + \alpha - 1 > 0$ for $r \in (\delta_1, 1)$. Then from (3.8) we conclude that $\alpha[3A(1,t)/2 - G(1,t)/2] + (1-\alpha)[5A(1,t)/4 - H(1,t)/4] > T(1,t)$ for $t \in (0, (1-\delta_1)/(1+\delta_1))$.

Case 2. $\alpha < 1/4$. Then from (3.6) we clearly see that

$$\lim_{r \to 0^+}[J(r) + \alpha - 1] = \alpha - \frac{1}{4} < 0. \qquad (3.10)$$

Inequality (3.10) implies that for any $\alpha < 1/4$ there exists $0 < \delta_2 = \delta_2(r) < 1$, such that $J(r) + \alpha - 1 < 0$ for $r \in (0, \delta_2)$. Then equation (3.8) leads to the conclusion that $\alpha[3A(1,t)/2 - G(1,t)/2] + (1-\alpha)[5A(1,t)/4 - H(1,t)/4] < T(1,t)$ for $t \in ((1-\delta_2)/(1+\delta_2), 1)$. □

The following Theorem 3.2 can be derived directly from (1.3) and Theorem 3.1 with $p = 16/\pi - 5$ and $q = 1/4$.

THEOREM 3.2. *The double inequality*

$$\begin{aligned}
&8A(a,b) - (16 - 5\pi)G(a,b) - (3\pi - 8)H(a,b) < L(a,b) \\
&< \frac{\pi[21A(a,b) - 2G(a,b) - 3H(a,b)]}{8} \qquad (3.11)
\end{aligned}$$

holds for all $a > b > 0$.

THEOREM 3.3. *Let* $x \in (0,1]$,

$$g(x) = F(-1/2, 1/2; 1; x) = \sum_{n=0}^{\infty} A_n x^n \tag{3.12}$$

and

$$H(x) = \frac{\left[21A(1,\sqrt{1-x}) - 2G(1,\sqrt{1-x}) - 3H(1,\sqrt{1-x})\right]}{16} = \sum_{n=0}^{\infty} B_n x^n. \tag{3.13}$$

Then

$$A_k \leqslant B_k \text{ for all } k = 0, 1, 2, \cdots, n, \cdots. \tag{3.14}$$

In particular, the function $F(x) = [H(x) - g(x)]/x^6$ *is convex and strictly increasing from* $(0,1]$ *onto* $(\alpha_1, \beta_1]$, *where* $\alpha_1 = 2^{-20} = 0.00000095\cdots$ *and* $\beta_1 = 21/32 - 2/\pi = 0.01963022\cdots$.

Proof. Making use of series expansions, equations (3.12) and (3.13) give

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1/2)_n (1/2)_n}{(n!)^2} x^n = 1 - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (1/2)_{n+1}}{[(n+1)!]^2} x^{n+1} = \sum_{n=0}^{\infty} A_n x^n \tag{3.15}$$

and

$$\begin{aligned}
H(x) &= \frac{21}{32}(1+\sqrt{1-x}) - \frac{1}{8}(1-x)^{1/4} - \frac{3}{8}\left(1 - \frac{1}{1+\sqrt{1-x}}\right) \\
&= \frac{21}{32}\left(1 + \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} x^n\right) - \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1/4)_n}{n!} x^n - \frac{3}{8}\left(1 - \frac{1-\sqrt{1-x}}{x}\right) \\
&= \frac{21}{32}\left(1 + \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} x^n\right) - \frac{3}{8}\left(1 - \frac{1 - \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} x^n}{x}\right) - \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1/4)_n}{n!} x^n \\
&= 1 + \sum_{n=0}^{\infty} \frac{[2(3/4)_n (n+2) - 9(1/2)_n (n+4)]}{64(n+2)!} x^{n+1} = \sum_{n=0}^{\infty} B_n x^n.
\end{aligned} \tag{3.16}$$

Equations (3.15) and (3.16) lead to

$$F(x) = \sum_{n=0}^{\infty} B_n x^n - \sum_{n=0}^{\infty} A_n x^n = \sum_{n=0}^{\infty} \frac{C_n}{64(n+1)!(n+2)!} x^{n+1}, \tag{3.17}$$

where $C_n = 2(n+2)!(3/4)_n + 32(n+2)(1/2)_n(1/2)_{n+1} - 9(n+4)(n+1)!(1/2)_n$.

From equation (3.17) we know that to prove inequality (3.14) it is sufficient to prove that $C_k \geqslant 0$ for all $k = 0, 1, 2, \cdots, n, \cdots$. Note that $C_0 = C_1 = C_2 = C_3 = C_4 = 0$, $C_5 = 14175/64$ and $C_6 = 1091475/32$. Next, we use mathematical induction to prove that $C_k > 0$ for $k \geqslant 5$ $(k \in \mathbf{N}^*)$. If we assume that $C_k > 0$ for $k = 5,6,7,\cdots,n$ $(n \geqslant 5)$ hold, then

$$\begin{aligned}
C_{n+1} =& 2(n+3)!(3/4)_{n+1} + 32(n+3)(1/2)_{n+1}(1/2)_{n+2} \\
& -9(n+5)(n+2)!(1/2)_{n+1} \\
=& 2(n+3/4)(n+3)(n+2)!(3/4)_n + 32(n+2)(1/2)_n(1/2)_{n+1} \\
& \times \frac{(n+1/2)(n+3/2)(n+3)}{n+2} - 9(n+5)(n+2)!(1/2)_{n+1} \\
=& (n+3/4)(n+3)\Big\{2(n+2)!(3/4)_n + 32(n+2)(1/2)_n(1/2)_{n+1} \\
& \times \left[1 - \frac{3(n+1)}{(4n+3)(n+2)}\right]\Big\} - 9(n+5)(n+2)!(1/2)_{n+1} \\
>& (n+3/4)(n+3)\,[9(n+4)(n+1)!(1/2)_n] - 24(n+1)(n+3) \\
& \times (1/2)_n(1/2)_{n+1} - 9(n+5)(n+2)!(1/2)_{n+1} \\
=& 9(n+1)!(n^2/4 + 15n/4 + 4)(1/2)_n - 24(n+1)(n+3)(1/2)_n(1/2)_{n+1} \\
=& \frac{12(n+1)!(n+3)(2n+1)(1/2)_n}{\sqrt{\pi}}\left[\frac{3\sqrt{\pi}(n^2+15n+16)}{16(n+3)(2n+1)} - \frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right].
\end{aligned} \tag{3.18}$$

The well known Wallis' inequality [13] reveals

$$\frac{\Gamma(n+1/2)}{\Gamma(n+1)} < \frac{1}{\sqrt{n+1/4}} \tag{3.19}$$

for all $n \geqslant 1$.

Therefore, $C_{n+1} > 0$ follows from (3.18) and (3.19) together with Lemma 2.3.

Finally, the convexity and monotonicity of $F(x)$ are clear. By l'Hôptial's rule, $F(0^+) = B_6 - A_6 = 2^{-20} = 1/1048576$, while the value of $F(1)$ is clear. □

The following Theorem 3.4 can be derived directly from (1.1) and Theorem 3.3 with $x = e^2 = (a^2 - b^2)/a^2$.

THEOREM 3.4. *The double inequality*

$$\begin{aligned}
&\frac{\pi\,[21A(a,b) - 2G(a,b) - 3H(a,b)]}{8} - 2\beta_1 \pi a \left(1 - \frac{b^2}{a^2}\right)^6 < L(a,b) \\
&< \frac{\pi\,[21A(a,b) - 2G(a,b) - 3H(a,b)]}{8} - 2\alpha_1 \pi a \left(1 - \frac{b^2}{a^2}\right)^6
\end{aligned} \tag{3.20}$$

holds for all $a > b > 0$, *where* α_1 *and* β_1 *are defined as in Theorem 3.3.*

4. Comparison with some well-known results

As we mentioned in the introduction, the perimeter of an ellipse has been studied intensively by many mathematicians, and some well-known bounds for it were presented. For example, Barnard, Pearce and Richards [6, 7] established that

$$2\pi M_{3/2}(a,b) < L(a,b) < 2\pi S(a,b) \tag{4.1}$$

for all $a > b > 0$. In 2004, Alzer and Qiu [3] found an upper power mean bound for $L(a,b)$ as follows:

$$L(a,b) < 2\pi M_{\log 2/\log(\pi/2)}(a,b) \tag{4.2}$$

for all $a > b > 0$.

Recently, Chu and Wang [14] proved that

$$L(a,b) < 2\pi L_{1/4}(a,b) \tag{4.3}$$

for all $a > b > 0$, where $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$ is p-th Lehmer mean.

In this section, we will compare our bounds with those in (4.1)–(4.3).

LEMMA 4.1. *Inequality*

$$8A(a,b) - (16-5\pi)G(a,b) - (3\pi-8)H(a,b) > 2\pi M_{3/2}(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$.

Proof. Clearly $A(a,b)$, $G(a,b)$, $H(a,b)$ and $M_{3/2}(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a > b = 1$. Let $t = \sqrt{a} > 1$. Then

$$\begin{aligned}
&[8A(a,b) - (16-5\pi)G(a,b) - (3\pi-8)H(a,b)]^3 - \left[2\pi M_{3/2}(a,b)\right]^3 \\
=& \left[4(1+t^2) - (16-5\pi)t - \frac{2(3\pi-8)t^2}{1+t^2}\right]^3 - 2\pi^3(1+t^3)^2 \\
=& \frac{(t-1)^4}{(1+t^2)^3} g(t),
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
g(t) =& (64-2\pi^3)t^8 + (-512+240\pi-8\pi^3)t^7 + (1792-1248\pi+300\pi^2-26\pi^3)t^6 \\
&+ (-3584+2832\pi-720\pi^2+57\pi^3)t^5 + (4480-3648\pi+1032\pi^2-102\pi^3)t^4 \\
&+ (-3584+2832\pi-720\pi^2+57\pi^3)t^3 + (1792-1248\pi+300\pi^2-26\pi^3)t^2 \\
&+ (-512+240\pi-8\pi^3)t + 64 - 2\pi^3.
\end{aligned} \tag{4.5}$$

Note that

$$64 - 2\pi^3 = 1.987\cdots,$$

$$-512 + 240\pi - 8\pi^3 = -6.067\cdots,$$

$$1792-1248\pi+300\pi^2-26\pi^3=26.010\cdots,$$
$$-3584+2832\pi-720\pi^2+57\pi^3=-25.767\cdots,$$
$$4480-3648\pi+1032\pi^2-102\pi^3=42.261\cdots.$$

It follows from (4.5) that

$$\begin{aligned} g(t) &> \frac{7}{5}t^8-7t^7+21t^6-28t^5+42t^4-28t^3+21t^2-7t+\frac{7}{5} \\ &= \frac{7}{5}\left(t^8-5t^7+15t^6-20t^5+30t^4-20t^3+15t^2-5t+1\right) \\ &= \frac{7}{5}\left[t^6\left(t-\frac{5}{2}\right)^2+\frac{35}{4}t^4\left(t-\frac{8}{7}\right)^2+\frac{130}{7}t^2\left(t-\frac{7}{13}\right)^2\right. \\ &\quad \left.+\left(\frac{5}{2}t-1\right)^2+\frac{175}{52}t^2\right]>0. \end{aligned} \tag{4.6}$$

Therefore, Lemma 4.1 follows from (4.4) and (4.6). □

REMARK 4.2. From Lemma 4.1 we know that the lower bound in (3.11) is better than that in (4.1).

LEMMA 4.3. *Inequalities*

$$\frac{21A(a,b)-2G(a,b)-3H(a,b)}{16}<L_{1/4}(a,b)$$

and

$$\frac{21A(a,b)-2G(a,b)-3H(a,b)}{16}<S(a,b)$$

hold for all $a,b>0$ *with* $a\neq b$.

Proof. Clearly $A(a,b)$, $G(a,b)$, $H(a,b)$, $S(a,b)$ and $L_{1/4}(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b=1$. Let $t=\sqrt[4]{a}>1$. Then

$$\begin{aligned} &\frac{21A(a,b)-2G(a,b)-3H(a,b)}{16}-L_{1/4}(a,b) \\ &= \frac{21(1+t^4)/2-2t^2-6t^4/(1+t^4)}{16}-\frac{1+t^5}{1+t} \\ &= -\frac{(t-1)^4(11t^4+12t^3+18t^2+12t+11)}{32(1+t^4)}<0 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &[16S(a,b)]^2-[21A(a,b)-2G(a,b)-3H(a,b)]^2 \\ &=128(1+t^4)-\left[\frac{21(1+t^4)^2-4t^2(1+t^4)-12t^4}{2(1+t^4)}\right]^2 \\ &=\frac{(t^2-1)^2}{4(1+t^4)^2}\left[71t^{12}+310t^{10}+297t^8+692t^6+297t^4+310t^2+71\right]>0. \end{aligned} \tag{4.8}$$

Therefore, Lemma 4.3 follows from inequalities (4.7) and (4.8). □

REMARK 4.4. From Lemma 4.3 we conclude that both the upper bounds in (3.11) and (3.20) are better than those in (4.1) and (4.3).

REMARK 4.5. If a and b are the semiaxes of an ellipse with eccentricity $e = \sqrt{a^2-b^2}/a$. Without loss of generality we can take one of the semiaxes, say a, to be 1. Then computational and numerical experiments show that the upper bound in (3.11) is better than that in (4.2) for $0 < e \leqslant 0.995$. However, the upper bound in (4.2) is better than that in (3.11) when $e \to 1$. In fact, if we let

$$I_1(e) = \frac{\pi\left[21A(1,\sqrt{1-e^2}) - 2G(1,\sqrt{1-e^2}) - 3H(1,\sqrt{1-e^2})\right]}{8},$$

and

$$I_2(e) = 2\pi M_{\log 2/\log(\pi/2)}(1,\sqrt{1-e^2}),$$

then

$$\lim_{e\to 1} I_1(e) = \frac{21\pi}{16} > 4 = \lim_{e\to 1} I_2(e).$$

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