Mathematical
Inequalities
& Applications
Volume 17, Number 1 (2014), 121–135 doi:10.7153/mia-17-09

REVERSE ORDER LAW FOR WEIGHTED MOORE–PENROSE INVERSES OF MULTIPLE MATRIX PRODUCTS

ZHIPING XIONG AND YINGYING QIN

(*Communicated by G. P. H. Styan*)

Abstract. In this paper by using some matrix rank theories, we derive equivalent conditions for reverse order law of weighted Moore-Penrose inverses of multiple matrix products. In addition, we also give a variety of necessary and sufficient conditions for the reverse product $(A_n)^{\dagger}_{M_n,M_{n+1}}(A_{n-1})^{\dagger}_{M_{n-1},M_n}\cdots(A_1)^{\dagger}_{M_1,M_2}$ to be a $\{1\}$-, $\{1,2\}$-, $\{1,3M_1\}$-, $\{1,4M_{n+1}\}$-, $\{1,2,3M_1\}$- or $\{1,2,4M_{n+1}\}$-inverse of matrix product $A_1A_2\cdots A_n$.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ denotes the set of $m\times n$ matrices with complex entries and $\mathbb{C}^m$ denotes the set of m-dimensional vectors. I_k denotes the identity matrix of order k and $O_{m\times n}$ denotes the $m\times n$ matrix of all zero entries (if no confusion occurs, we will omit the subscript). The symbols A^*, $r(A)$, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote the conjugate transpose, the rank, the range space and the null space of the matrix $A\in\mathbb{C}^{m\times n}$, respectively.

The weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ of $A\in\mathbb{C}^{m\times n}$ with respect to the positive definite Hermitian matrices $M\in\mathbb{C}^{m\times m}$ and $N\in\mathbb{C}^{n\times n}$ is defined as the solution of the following four matrix equations see [1, 8, 12]:

$$(1)\ AXA=A,\ \ (2)\ XAX=X,\ \ (3M)\ (MAX)^*=MAX,\ \ (4N)\ (NXA)^*=NXA.$$

For a subset $\{i,j,\cdots,k\}$ of the set $\{1,2,3M,4N\}$, the set of $n\times m$ matrices satisfying the equations (i), (j), $\cdots$, (k) from among equations $(1)-(4N)$ is denoted by $A\{i,j,\cdots,k\}$. A matrix in $A\{i,j,\cdots,k\}$ is called an $\{i,j,\cdots,k\}$-inverse of A and is denoted by $A^{(i,j,\cdots,k)}$. It is well known that the $\{1,2,3M,4N\}$-inverses of A is unique, which is also called the the weighted Moore-Penrose inverse of A and is denoted by $A^{\dagger}_{M,N}$. In particular, when $M=I_m$ and $N=I_n$, the weighted Moore-Penrose inverse

Mathematics subject classification (2010): 15A03, 15A09, 15A24.
Keywords and phrases: Reverse order law, generalized inverse, weighted generalized inverse, elementary block matrix operations, matrix rank theory.
The work of first author was supported by the NSFC Mathematics TianYuan Youth Fund (11226126) and the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (2012LYM-0126) and the Student Innovation Training Program of Guangdong, China (1134912033). The work of second author was supported by the Wuyi University Youth Fund, Jiangmen 529020, Guangdong, China (201210041650504).

Paper MIA-17-09

reduces to the Moore-Penrose inverse of A and is denoted by $A^{\dagger}_{I_m,I_n} = A^{\dagger}$, see [1, 9]. In fact, the weighted Moore-Penrose inverse $A^{\dagger}_{M,N}$ can be obtained by the expression

$$A^{\dagger}_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^{\dagger}M^{1/2},$$

where $M^{1/2}$ and $N^{1/2}$ are the positive definite square roots of M and N, respectively. We refer the reader to [1, 9, 12] for basic results on the weighted generalized inverses.

Let A_1, A_2, $\cdots$, A_n be n matrices such that the product $A_1A_2\cdots A_n$ exists. If each of the n matrices is nonsingular, then the product $A_1A_2\cdots A_n$ is nonsingular too and the inverse of $A_1A_2\cdots A_n$ satisfies the reverse order law $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1}\cdots A_1^{-1}$. This law, however, cannot trivially be extended to the weighted Moore-Penrose inverses of $A_1A_2\cdots A_n$ when the product is a singular matrix. In other words, the reverse order law

$$(A_1A_2\cdots A_n)^{\dagger}_{M_1,M_{n+1}} = (A_n)^{\dagger}_{M_n,M_{n+1}}(A_{n-1})^{\dagger}_{M_{n-1},M_n}\cdots(A_1)^{\dagger}_{M_1,M_2} \tag{1.1}$$

does not automatically hold, where $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1,2,\cdots,n$ and $M_i \in \mathbb{C}^{l_i\times l_i}$, $i = 1,2,\cdots,n+1$ are $n+1$ positive definite Hermitian matrices.

One of the fundamental research problems in the theory of generalized inverses of matrices is to give necessary and sufficient conditions for various reverse order laws for generalized inverses of matrix products. In 1966, Greville [4] first gave a necessary and sufficient condition for the reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. Since then, the problem of the reverse order law for generalized inverses was widely studied, see [2, 3, 4, 6, 10, 11, 13, 14, 15, 16, 17, 19].

In this paper, we will discuss the reverse order law (1.1) and present necessary and sufficient conditions for (1.1) to hold. In addition, we will also study the relationship between the reverse product $(A_n)^{\dagger}_{M_n,M_{n+1}}(A_{n-1})^{\dagger}_{M_{n-1},M_n}\cdots(A_1)^{\dagger}_{M_1,M_2}$ and the weighted generalized inverse of $A_1A_2\cdots A_n$, providing a variety of necessary and sufficient conditions for the reverse product $(A_n)^{\dagger}_{M_n,M_{n+1}}(A_{n-1})^{\dagger}_{M_{n-1},M_n}\cdots(A_1)^{\dagger}_{M_1,M_2}$ to be a $\{1\}$-, $\{1,2\}$-, $\{1,3M_1\}$-, $\{1,4M_{n+1}\}$-, $\{1,2,3M_1\}$- or $\{1,2,4M_{n+1}\}$-inverse of matrix product $A_1A_2\cdots A_n$.

We first mention the following lemmas, which will be used in this paper.

LEMMA 1.1. [1, 11, 19] *For any matrices A and B, such that the product AB is well defined, the Moore-Penrose inverses of matrix products satisfy the following simple properties:*

(1). $A^*(AA^*)^{\dagger} = (A^*A)^{\dagger}A^* = A^*(A^*AA^*)^{\dagger}A^* = A^{\dagger}$;

(2). $B^{\dagger}B(AB)^{\dagger}AA^{\dagger} = B^{\dagger}B(AB)^{\dagger} = (AB)^{\dagger}AA^{\dagger} = (AB)^{\dagger}$.

LEMMA 1.2. [9, 11, 18] *Let $A \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{n\times m}$; $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ be two positive definite Hermitian matrices. Then*

(1). $G \in A\{1,2\} \Leftrightarrow AGA = A$ and $r(G) = r(A)$;

(2). $G \in A\{1,3M\} \Leftrightarrow A^*MAG = A^*M$;

(3). $G \in A\{1,4N\} \Leftrightarrow GAN^{-1}A^* = N^{-1}A^*$;

(4). $G \in A\{1,2,3M\} \Leftrightarrow A^*MAG = A^*M$ and $r(G) = r(A)$;

(5). $G \in A\{1,2,4N\} \Leftrightarrow GAN^{-1}A^* = N^{-1}A^*$ and $r(G) = r(A)$;
(6). $G = A^{\dagger}_{M,N} \Leftrightarrow A^*MAGM^{-1} = NGAN^{-1}A^* = A^*$ and $r(G) = r(A)$.

LEMMA 1.3. [1, 9, 12] *Let $A \in \mathbb{C}^{m\times n}$; L, M be two complementary subspaces of $\mathbb{C}^n$ and $P_{L,M}$ denotes a projector on L along M. Then for any matrix A, we have the following:*
(1). $P_{L,M}A^* = A^* \Leftrightarrow \mathscr{R}(A^*) \subseteq L$;
(2). $AP_{L,M} = A \Leftrightarrow \mathscr{N}(A) \supseteq M$.

LEMMA 1.4. [1, 9, 12] *Let $A \in \mathbb{C}^{m\times n}$ and $M \in \mathbb{C}^{m\times m}$, $N \in \mathbb{C}^{n\times n}$ be two positive definite Hermitian matrices. Then*
(1). $\mathscr{R}(AA^{\dagger}_{M,N}) = \mathscr{R}(A)$, $\mathscr{N}(AA^{\dagger}_{M,N}) = M^{-1}\mathscr{N}(A^*)$;
(2). $\mathscr{R}(A^{\dagger}_{M,N}A) = N^{-1}\mathscr{R}(A^*)$, $\mathscr{N}(A^{\dagger}_{M,N}A) = \mathscr{N}(A)$;
(3). $A^{\dagger}_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^{\dagger}M^{1/2} = A^{(1,2)}_{N^{-1}\mathscr{R}(A^*),\, M^{-1}\mathscr{N}(A^*)}$.

In addition, the following two lemmas are widely used in the context to simplify various operations on ranks and ranges of matrices.

LEMMA 1.5. [7] *Suppose A, B, C and D satisfy the following conditions:*

$$\mathscr{R}(B) \subseteq \mathscr{R}(A) \ \ \textit{and} \ \ \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$$

or

$$\mathscr{R}(C) \subseteq \mathscr{R}(D) \ \ \textit{and} \ \ \mathscr{R}(B^*) \subseteq \mathscr{R}(D^*).$$

Then

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(A) + r(D - CA^{\dagger}B) \tag{1.2}$$

or

$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r(D) + r(A - BD^{\dagger}C). \tag{1.3}$$

LEMMA 1.6. [5] *Suppose B, C and D satisfy*

$$\mathscr{R}(D) \subseteq \mathscr{R}(C) \ \ \textit{and} \ \ \mathscr{R}(D^*) \subseteq \mathscr{R}(B^*).$$

Then the Moore-Penrose inverse of the block matrix

$$J = \begin{pmatrix} O & B \\ C & D \end{pmatrix},$$

may be expressed as

$$J^{\dagger} = \begin{pmatrix} O & B \\ C & D \end{pmatrix}^{\dagger} = \begin{pmatrix} -C^{\dagger}DB^{\dagger} & C^{\dagger} \\ B^{\dagger} & O \end{pmatrix}. \tag{1.4}$$

As the main tools in our discussion, we now state the weighted Moore-Penrose inverse of a special $n\times n$ block matrix.

THEOREM 1.1. *Let* $A_i\in\mathbb{C}^{l_i\times l_{i+1}}$, $i=1,2,\cdots,n$ *and* $M_i\in\mathbb{C}^{l_i\times l_i}$, $i=1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. If* $\widetilde{A_i}=M_i^{1/2}A_iM_{i+1}^{-1/2}\in\mathbb{C}^{l_i\times l_{i+1}}$, $i=1,2,\cdots,n$ *and* $\widetilde{B_i}\in\mathbb{C}^{l_{i+1}\times l_{i+1}}$, $i=1,2,\cdots,n-1$ *satisfy*

$$\widetilde{B_i}=\widetilde{A_{i+1}}X_{i+1}\widetilde{A_i},\quad i=1,2,\cdots,n-1,\quad for\ some\ X_{i+1}\in\mathbb{C}^{l_{i+2}\times l_i}. \tag{1.5}$$

Then the Moore-Penrose inverse of the $n\times n$ *block matrix*

$$T_n=\begin{pmatrix} & & & & \widetilde{A_1}\\ & & & \widetilde{A_2} & \widetilde{B_1}\\ & & \diagup & \diagup & \\ & \diagup & \diagup & & \\ & \widetilde{A_{n-1}} & \widetilde{B_{n-2}} & & \\ \widetilde{A_n} & \widetilde{B_{n-1}} & & & \end{pmatrix}, \tag{1.6}$$

may be expressed as

$$(T_n)^\dagger=\begin{pmatrix} E(1,n) & E(2,n) & \cdots & E(n-1,n) & E(n,n)\\ E(1,n-1) & E(2,n-1) & \cdots & E(n-1,n-1) & \\ \vdots & \vdots & \diagup & & \\ E(1,2) & E(2,2) & & & \\ E(1,1) & & & & \end{pmatrix}, \tag{1.7}$$

with

$$E(i,i)=(\widetilde{A_i})^\dagger,\quad i=1,2,\cdots,n \tag{1.8}$$

and

$$E(i,j)=(-1)^{j-i}(\widetilde{A_j})^\dagger\widetilde{B_{j-1}}(\widetilde{A_{j-1}})^\dagger\widetilde{B_{j-2}}\cdots(\widetilde{A_{i+1}})^\dagger\widetilde{B_i}(\widetilde{A_i})^\dagger,\ 1\leqslant i<j\leqslant n. \tag{1.9}$$

Proof. We shall use induction on n. For $n=2$

$$T_2=\begin{pmatrix} O & \widetilde{A_1}\\ \widetilde{A_2} & \widetilde{B_1}\end{pmatrix}. \tag{1.10}$$

According to (1.5), we have

$$\mathscr{R}(\widetilde{B_1})\subseteq\mathscr{R}(\widetilde{A_2})\ \text{ and }\ \mathscr{R}(\widetilde{B_1}^*)\subseteq\mathscr{R}(\widetilde{A_1}^*).$$

Then, by the formula (1.4) in Lemma 1.6, we get

$$T_2^\dagger=\begin{pmatrix} -(\widetilde{A_2})^\dagger\widetilde{B_1}(\widetilde{A_1})^\dagger & (\widetilde{A_2})^\dagger\\ (\widetilde{A_1})^\dagger & O\end{pmatrix}=\begin{pmatrix} E(1,2) & E(2,2)\\ E(1,1) & O\end{pmatrix}. \tag{1.11}$$

Suppose the hypothesis is also true for $n-1$, that is, under the conditions in (1.5) and the equality (1.7), the Moore-Penrose of T_{n-1} in (1.6) is given by

$$(T_{n-1})^{\dagger} = \begin{pmatrix} E(1,n-1) & E(2,n-1) & \cdots & E(n-2,n-1) & E(n-1,n-1) \\ E(1,n-2) & E(2,n-2) & \cdots & E(n-2,n-2) & \\ \vdots & \vdots & \diagup & & \\ E(1,2) & E(2,2) & & & \\ E(1,1) & & & & \end{pmatrix}. \quad (1.12)$$

Next consider the Moore-Penrose inverse of T_n in (1.6). First partition T_n in (1.6) into the form

$$T_n = \begin{pmatrix} O & T_{n-1} \\ \widetilde{A_n} & H \end{pmatrix},$$

where $H = \left(\widetilde{B_{n-1}},\, O,\, \cdots,\, O\right)$. It is easy to see from the conditions (1.5) that

$$\mathscr{R}(H) \subseteq \mathscr{R}(\widetilde{B_{n-1}}) \subseteq \mathscr{R}(\widetilde{A_n})$$

and

$$\mathscr{R}(H^*) = \mathscr{R}\begin{pmatrix} \widetilde{B_{n-1}}^* \\ O \\ \vdots \\ O \end{pmatrix} = \mathscr{R}\begin{pmatrix} \widetilde{A_{n-1}}^* X_n \widetilde{A_n}^* \\ O \\ \vdots \\ O \end{pmatrix} \subseteq \mathscr{R}\begin{pmatrix} \widetilde{A_{n-1}}^* \\ O \\ \vdots \\ O \end{pmatrix} \subseteq \mathscr{R}(T_{n-1}^*).$$

Hence, by Lemma 1.6, the Moore-Penrose inverse of T_n can be written as

$$(T_n)^{\dagger} = \begin{pmatrix} -(\widetilde{A_n})^{\dagger} H (T_{n-1})^{\dagger} & (\widetilde{A_n})^{\dagger} \\ (T_{n-1})^{\dagger} & O \end{pmatrix}. \quad (1.13)$$

According to the hypothesis of the induction for $T_{n-1}^{\dagger}$ in (1.12) and the structure of H, $E(i,i)$ and $E(i,j)$ in $T_{n-1}^{\dagger}$, we have

$$\begin{aligned} -(\widetilde{A_n})^{\dagger} H (T_{n-1})^{\dagger} &= \left(-(\widetilde{A_n})^{\dagger}\widetilde{B_{n-1}} E(1,n-1),\, \cdots,\, -(\widetilde{A_n})^{\dagger}\widetilde{B_{n-1}} E(n-1,n-1)\right) \\ &= \left(E(1,n),\, E(2,n),\, \cdots,\, E(n-1,n)\right). \end{aligned} \quad (1.14)$$

Substituting $T_{n-1}^{\dagger}$ from (1.12) and $-(\widetilde{A_n})^{\dagger} H (T_{n-1})^{\dagger}$ from (1.14) into (1.13) directly produces (1.7). This fact shows that the conclusion of this theorem is true. □

According to Theorem 1.1 and Lemma 1.1 (1), we can immediately get the following two corollaries:

COROLLARY 1.1. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$ *and* $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* $\widetilde{A_i} = M_i^{1/2} A_i M_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$

and build an $n\times n$ block matrix by using these n matrices as follow

$$\widetilde{T}=\begin{pmatrix} & & & & \widetilde{A_1}\widetilde{A_1}^* \\ & & & \widetilde{A_2}^*\widetilde{A_2}\widetilde{A_2}^* & \widetilde{A_2}^*\widetilde{A_1}^* \\ & & \diagup & \diagup & \\ & \diagup & \diagup & & \\ & \widetilde{A_{n-1}}^*\widetilde{A_{n-1}}\widetilde{A_{n-1}}^* & \widetilde{A_{n-1}}^*\widetilde{A_{n-2}}^* & & \\ \widetilde{A_n}^*\widetilde{A_n} & \widetilde{A_n}^*\widetilde{A_{n-1}}^* & & & \end{pmatrix}$$
$$=\begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix}, \tag{1.15}$$

where $E_1=\big(O,\cdots,O,I_{l_1}\big)$ and $E_2=\big(O,\cdots,O,I_{l_{n+1}}\big)^*$. Then the $l_{n+1}\times l_1$ submatrix in the upper left corner of the Moore-Penrose inverse of $\widetilde{T}$ can be expressed as

$$\begin{aligned} P\widetilde{T}^\dagger Q &= (-1)^{n-1}(\widetilde{A_n}^*\widetilde{A_n})^\dagger\widetilde{A_n}^*\widetilde{A_{n-1}}^*(\widetilde{A_{n-1}}^*\widetilde{A_{n-1}}\widetilde{A_{n-1}}^*)^\dagger\widetilde{A_{n-1}}^*\widetilde{A_{n-2}}^*\cdots \\ &\quad\cdots\widetilde{A_3}^*\widetilde{A_2}^*(\widetilde{A_2}^*\widetilde{A_2}\widetilde{A_2}^*)^\dagger\widetilde{A_2}^*\widetilde{A_1}^*(\widetilde{A_1}\widetilde{A_1}^*)^\dagger \\ &= (-1)^{n-1}(\widetilde{A_n})^\dagger(\widetilde{A_{n-1}})^\dagger\cdots(\widetilde{A_1})^\dagger, \end{aligned} \tag{1.16}$$

where $P=\big(I_{l_{n+1}},O,\cdots,O\big)$ and $Q=\big(I_{l_1},O,\cdots,O\big)^*$. Furthermore, let

$$M=\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & \ddots \\ & & & & M_n \end{pmatrix} \text{ and } N=\begin{pmatrix} M_{n+1} & & & \\ & M_n & & \\ & & \ddots & \\ & & & \ddots \\ & & & & M_2 \end{pmatrix}, \tag{1.17}$$

be two $n\times n$ block positive definite Hermitian matrices. Then combining the structures of M, N in (1.17) and the results in (1.16) with the formulas (3) in Lemma 1.4, we have

$$\begin{aligned} PN^{-1/2}\widetilde{T}^\dagger M^{1/2}Q &= (-1)^{n-1}M_{n+1}^{-1/2}(\widetilde{A_n})^\dagger(\widetilde{A_{n-1}})^\dagger\cdots(\widetilde{A_1})^\dagger M_1^{1/2} \\ &= (-1)^{n-1}(A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\cdots(A_1)^\dagger_{M_1,M_2}. \end{aligned} \tag{1.18}$$

COROLLARY 1.2. *Let $\widetilde{T}$, P, Q, M, N be given as Corollary 1.1 and $A=A_1A_2\cdots A_n$. Then*

$$r(\widetilde{T})=r(A_1)+r(A_2)+\cdots+r(A_n). \tag{1.19}$$

$$\mathscr{R}(M^{1/2}QA_1)\subseteq\mathscr{R}(\widetilde{T}),\qquad \mathscr{R}(N^{-1/2}P^*A_n^*)\subseteq\mathscr{R}(\widetilde{T}^*). \tag{1.20}$$

$$\mathscr{R}(M^{1/2}QA)\subseteq\mathscr{R}(\widetilde{T}),\qquad \mathscr{R}(N^{-1/2}P^*A^*)\subseteq\mathscr{R}(\widetilde{T}^*). \tag{1.21}$$

2. The relationship between $(A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\dots(A_1)^\dagger_{M_1,M_2}$ and the weighted generalized inverses of $A_1A_2\dots A_n$

In this section, applying some matrix rank theories, we will show the relationship between the reverse order product $(A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\cdots(A_1)^\dagger_{M_1,M_2}$ and the common types of weighted generalized inverses of the product $A_1A_2\cdots A_n$. For the sake of the simplicity in the later discussion, we will adopt the following notations for the matrix products:

$$A = A_1A_2\cdots A_n, \quad \widetilde{A} = M_1^{1/2}AM_{n+1}^{-1/2} \tag{2.1}$$

and

$$X = (A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\cdots(A_1)^\dagger_{M_1,M_2}, \tag{2.2}$$

where $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1,2,\cdots,n$ and $M_i \in \mathbb{C}^{l_i\times l_i}$, $i = 1,2,\cdots,n+1$ are $n+1$ positive definite Hermitian matrices.

According to the equality (1.18) in Corollary 1.1, the matrix X in (2.2) can be expressed as

$$X = (-1)^{n-1}PN^{-1/2}\widetilde{T}^\dagger M^{1/2}Q, \tag{2.3}$$

where N, M, $\widetilde{T}$, P and Q are defined as in Corollary 1.1. Based on the expression for X in (2.3), we obtain the following result.

THEOREM 2.1. *Let $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1,2,\cdots,n$ and $M_i \in \mathbb{C}^{l_i\times l_i}$, $i = 1,2,\cdots,n+1$ be $n+1$ positive definite Hermitian matrices. Let A, $\widetilde{A}$ and X be given as* (2.1) and (2.2), $\widetilde{A_i} = M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$. Then $X \in A\{1\}$, if and only if

$$r\begin{pmatrix} (-1)^n\widetilde{A} & \widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix} = r(A_1)+r(A_2)+\cdots+r(A_n)-r(A), \tag{2.4}$$

where E_1, E_2 and S are as in Corollary 1.1.

Proof. According to the definition of $\{1\}$-inverse and (2.3), we know that $X \in A\{1\}$ if and only if

$$\begin{aligned} r(A-AXA) &= r(A-(-1)^{n-1}APN^{-1/2}\widetilde{T}^\dagger M^{1/2}QA) \\ &= r((-1)^{n-1}A-APN^{-1/2}\widetilde{T}^\dagger M^{1/2}QA) \\ &= 0. \end{aligned} \tag{2.5}$$

On the other hand, from Corollary 1.2, we have

$$\mathscr{R}(M^{1/2}QA) \subseteq \mathscr{R}(\widetilde{T}) \quad \text{and} \quad \mathscr{R}(N^{-1/2}P^*A^*) \subseteq \mathscr{R}(\widetilde{T}^*). \tag{2.6}$$

Then by Lemma 1.5, we obtain

$$
\begin{aligned}
& r((-1)^{n-1}A - APN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA) \\
&= r\begin{pmatrix} (-1)^{n-1}A & APN^{-1/2} \\ M^{1/2}QA & \widetilde{T} \end{pmatrix} - r(\widetilde{T}) \\
&= r\left(\begin{pmatrix} I & O \\ (-1)^n M^{1/2}Q & I \end{pmatrix}\begin{pmatrix} (-1)^{n-1}A & APN^{-1/2} \\ M^{1/2}QA & \widetilde{T} \end{pmatrix}\begin{pmatrix} I & (-1)^n PN^{-1/2} \\ O & I \end{pmatrix}\right) - r(\widetilde{T}) \\
&= r\begin{pmatrix} (-1)^{n-1}A & O \\ O & \widetilde{T} + (-1)^n M^{1/2}QAPN^{-1/2} \end{pmatrix} - r(\widetilde{T}) \\
&= r(\widetilde{T} + (-1)^n M^{1/2}QAPN^{-1/2}) + r(A) - r(\widetilde{T}). \qquad (2.7)
\end{aligned}
$$

By the structure of $\widetilde{T}$, M, Q, P and N as given in Corollary 1.1, we further have

$$
\begin{aligned}
& r(\widetilde{T} + (-1)^n M^{1/2}QAPN^{-1/2}) \\
&= r\left(\begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix} + \begin{pmatrix} (-1)^n\widetilde{A} & O \\ O & O \end{pmatrix}\right) \\
&= r\begin{pmatrix} (-1)^n\widetilde{A} & \widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix}. \qquad (2.8)
\end{aligned}
$$

Substituting (2.8), (1.19) into (2.7) and combining the result with (2.5), we arrive at (2.4). □

By a similar approach with Theorem 2.1, we can also prove the following two theorems.

THEOREM 2.2. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$ *and* $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2), $\widetilde{A_i} = M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$. Then $X \in A\{1,2\}$, if and only if A_1, A_2, $\cdots$, A_n and A satisfy (2.4) and the following rank equality:

$$r(S) = r(A) + r(A_2) + r(A_3) + \cdots + r(A_{n-1}), \qquad (2.9)$$

where S is given as in Corollary 1.1.

Proof. According to Lemma 1.2 (1), $X \in A\{1,2\}$ holds if and only if

$$r(A - AXA) = 0 \quad \text{and} \quad r(X) = r(A). \qquad (2.10)$$

In Theorem 2.1 we proved that $r(A - AXA) = 0$ is equivalent to (2.4). Next, we will claim that $r(X) = r(A)$ is equivalent to (2.9). In fact, from (2.2) we have

$$
\begin{aligned}
& (A_n)^{\dagger}_{M_n,M_{n+1}} A_n X A_1 (A_1)^{\dagger}_{M_1,M_2} \\
&= (A_n)^{\dagger}_{M_n,M_{n+1}} A_n (A_n)^{\dagger}_{M_n,M_{n+1}} (A_{n-1})^{\dagger}_{M_{n-1},M_n} \cdots (A_1)^{\dagger}_{M_1,M_2} A_1 (A_1)^{\dagger}_{M_1,M_2} \\
&= (A_n)^{\dagger}_{M_n,M_{n+1}} (A_{n-1})^{\dagger}_{M_{n-1},M_n} \cdots (A_1)^{\dagger}_{M_1,M_2} \\
&= X. \qquad (2.11)
\end{aligned}
$$

Combining (2.11) with (2.3), we have

$$r(X) = r((A_n)^{\dagger}_{M_n,M_{n+1}} A_n X A_1 (A_1)^{\dagger}_{M_1,M_2}) \leqslant r(A_n X A_1) \leqslant r(X) \tag{2.12}$$

and

$$r(X) = r(A_n X A_1) = r(A_n P N^{-1/2} \widetilde{T}^{\dagger} M^{1/2} Q A_1). \tag{2.13}$$

On the other hand, by Corollary 1.2, we get

$$\mathscr{R}(M^{-1/2} Q A_1) \subseteq \mathscr{R}(\widetilde{T}) \text{ and } \mathscr{R}(N^{-1/2} P A_n^*) \subseteq \mathscr{R}(\widetilde{T}^*).$$

Then applying Lemma 1.5, we have

$$\begin{aligned}
& r(A_n P N^{-1/2} \widetilde{T}^{\dagger} M^{1/2} Q A_1) \\
&= r\begin{pmatrix} \widetilde{T} & M^{1/2} Q A_1 \\ A_n P N^{-1/2} & O \end{pmatrix} - r(\widetilde{T}) \\
&= r\begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^* E_1 & M_1^{1/2} A_1 \\ E_2 \widetilde{A_n}^* \widetilde{A_n} & S & O \\ A_n M_{n+1}^{-1/2} & O & O \end{pmatrix} - r(\widetilde{T}) \\
&= r\left(\begin{pmatrix} I & O & O \\ O & I & -E_2 \widetilde{A_n}^* M_n^{1/2} \\ O & O & I \end{pmatrix} \begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^* E_1 & M_1^{1/2} A_1 \\ E_2 \widetilde{A_n}^* \widetilde{A_n} & S & O \\ A_n M_{n+1}^{-1/2} & O & O \end{pmatrix} \begin{pmatrix} I & O & O \\ O & I & O \\ O & -M_2^{-1/2} \widetilde{A_1}^* E_1 & I \end{pmatrix} \right) \\
&\quad - r(\widetilde{T}) \\
&= r\begin{pmatrix} O & O & M_1^{1/2} A_1 \\ O & S & O \\ A_n M_{n+1}^{-1/2} & O & O \end{pmatrix} - r(\widetilde{T}) \\
&= r(S) + r(A_1) + r(A_n) - r(\widetilde{T}).
\end{aligned} \tag{2.14}$$

Finally, combining the results in (1.19), (2.10), (2.13) with (2.14), we have (2.9). □

THEOREM 2.3. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \cdots, n$ and $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1, 2, \cdots, n+1$ be $n+1$ positive definite Hermitian matrices. Let A, $\widetilde{A}$ and X be given as* (2.1) and (2.2), $\widetilde{A_i} = M_i^{1/2} A_i M_{i+1}^{-1/2}$, $i = 1, 2, \cdots, n$. Then $X \in A\{1, 3M_1\}$, if and only if

$$r\begin{pmatrix} (-1)^n (\widetilde{A})^* \widetilde{A} & (\widetilde{A})^* \widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2 \widetilde{A_n}^* \widetilde{A_n} & S \end{pmatrix} = r(A_2) + r(A_3) + \cdots + r(A_n), \tag{2.15}$$

where E_1, E_2 and S are given as in Corollary 1.1.

Proof. From Lemma 1.2 (2), $X \in A\{1, 3M_1\}$ holds if and only if

$$r(A^* M_1 - A^* M_1 A X) = 0.$$

Note that

$$A^*M_1A_1(A_1)^\dagger_{M_1,M_2} = A_n^*A_{n-1}^*\cdots A_1^*(A_1^*)^\dagger_{M_1,M_2}A_1^*M_1 = A^*M_1$$

and

$$\begin{aligned} A^*M_1AXA_1(A_1)^\dagger_{M_1,M_2} &= A^*M_1A(A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\cdots(A_1)^\dagger_{M_1,M_2}A_1(A_1)^\dagger_{M_1,M_2} \\ &= A^*M_1AX \end{aligned}$$

and

$$\begin{aligned} r(A^*M_1 - A^*M_1AX) &= r((A^*M_1A_1 - A^*M_1AXA_1)(A_1)^\dagger_{M_1,M_2})) \\ &\leqslant r(A^*M_1A_1 - A^*M_1AXA_1) \\ &\leqslant r(A^*M_1 - A^*M_1AX) \end{aligned}$$

Thus, by the expression for X as in (2.3), we have

$$\begin{aligned} & r(A^*M_1 - A^*M_1AX) \\ &= r(A^*M_1A_1 - A^*M_1AXA_1) \\ &= r(A^*M_1A_1 - (-1)^{n-1}A^*M_1APN^{-1/2}\widetilde{T}^\dagger M^{1/2}QA_1) \\ &= r((-1)^{n-1}A^*M_1A_1 - A^*M_1APN^{-1/2}\widetilde{T}^\dagger M^{1/2}QA_1). \end{aligned} \tag{2.16}$$

On the other hand, from Corollary 1.2, we obtain $R(M^{1/2}QA_1) \subseteq R(\widetilde{T})$ and $R(N^{-1/2}P^*A^*M_1A) \subseteq R(N^{-1/2}PA_n^*) \subseteq R(\widetilde{T})$. Then applying Lemma 1.5, we have

$$\begin{aligned} & r((-1)^{n-1}A^*M_1A_1 - A^*M_1APN^{-1/2}\widetilde{T}^\dagger M^{1/2}QA_1) \\ &= r\begin{pmatrix} \widetilde{T} & M^{1/2}QA_1 \\ A^*M_1APN^{-1/2} & (-1)^{n-1}A^*M_1A_1 \end{pmatrix} - r(\widetilde{T}) \\ &= r\begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^*E_1 & M_1^{1/2}A_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S & O \\ A^*M_1AM_{n+1}^{-1/2} & O & (-1)^{n-1}A^*M_1A_1 \end{pmatrix} - r(\widetilde{T}) \\ &= r\begin{pmatrix} O & O & M_1^{1/2}A_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S & O \\ A^*M_1AM_{n+1}^{-1/2} & (-1)^nA^*M_1A_1M_2^{-1/2}\widetilde{A_1}^*E_1 & O \end{pmatrix} - r(\widetilde{T}) \\ &= r\begin{pmatrix} E_2\widetilde{A_n}^*\widetilde{A_n} & S \\ A^*M_1AM_{n+1}^{-1/2} & (-1)^nA^*M_1^{1/2}\widetilde{A_1}\widetilde{A_1}^*E_1 \end{pmatrix} + r(A_1) - r(\widetilde{T}). \end{aligned} \tag{2.17}$$

Further,

$$
\begin{aligned}
& r\begin{pmatrix} E_2\widetilde{A_n}^*\widetilde{A_n} & S \\ A^*M_1AM_{n+1}^{-1/2} & (-1)^nA^*M_1^{1/2}\widetilde{A_1}\widetilde{A_1}^*E_1 \end{pmatrix} \\
&= r\begin{pmatrix} (-1)^nA^*M_1AM_{n+1}^{-1/2} & A^*M_1^{1/2}\widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix} \\
&= r\left(\begin{pmatrix} M_{n+1}^{-1/2} & O \\ O & I \end{pmatrix} \times \begin{pmatrix} (-1)^nA^*M_1AM_{n+1}^{-1/2} & A^*M_1^{1/2}\widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix}\right) \\
&= r\begin{pmatrix} (-1)^n(\widetilde{A})^*\widetilde{A} & (\widetilde{A})^*\widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix}. \qquad (2.18)
\end{aligned}
$$

Therefore from (1.19), (2.16), (2.17) and (2.18), we know that $X \in A\{1,3M_1\}$ holds if and only if

$$
r\begin{pmatrix} (-1)^n(\widetilde{A})^*\widetilde{A} & (\widetilde{A})^*\widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S \end{pmatrix} = r(A_2)+r(A_3)+\cdots+r(A_n). \quad \square
$$

Notice that $GAN^{-1}A^* = N^{-1}A^*$ is equivalent to the equation $AN^{-1}A^*G^* = AN^{-1}$. This implies that, by Lemma 1.2 (2) and (3), $G \in A\{1,4N\}$ if and only if $G^* \in A^*\{1,3N^{-1}\}$. So from the results obtained in Theorem 2.3, we have

THEOREM 2.4. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$ *and* $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2), $\widetilde{A_i} = M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$. Then $X \in A\{1,4M_{n+1}\}$, if and only if

$$
r\begin{pmatrix} (-1)^n\widetilde{A}(\widetilde{A})^* & \widetilde{A_1}\widetilde{A_1}^*E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n}(\widetilde{A})^* & S \end{pmatrix} = r(A_1)+r(A_2)+\cdots+r(A_{n-1}), \qquad (2.19)
$$

where E_1, E_2 and S are given as Corollary 1.1.

Furthermore, according to Lemma 1.2 (4) and (5) as well as the results in Theorem 2.2, Theorem 2.3 and Theorem 2.4, we can immediately draw the following two conclusions:

COROLLARY 2.1. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$ *and* $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2), $\widetilde{A_i} = M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$. Then $X \in A\{1,2,3M_1\}$, if and only if A_1, A_2, $\cdots$, A_n and A satisfy the rank equalities (2.9) and (2.15).

COROLLARY 2.2. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1,2,\cdots,n$ *and* $M_i \in \mathbb{C}^{l_i \times l_i}$, $i = 1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2) , $\widetilde{A_i} = M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i = 1,2,\cdots,n$. Then $X \in A\{1,2,4M_{n+1}\}$, if and only if A_1, A_2, $\cdots$, A_n and A satisfy the rank equalities (2.9) and (2.19).

3. The necessary and sufficient conditions for the reverse order law (1.1) to hold

From Lemma 1.2 (6), we know that, the reverse order law (1.1):

$$(A_1A_2\cdots A_n)^\dagger_{M_1,M_{n+1}} = (A_n)^\dagger_{M_n,M_{n+1}}(A_{n-1})^\dagger_{M_{n-1},M_n}\cdots(A_1)^\dagger_{M_1,M_2}$$

holds if and only if A and X in (2.1)and (2.2) satisfy the following three rank equalities:

$$r(X)=r(A),\ \ r(A^*M_1-A^*M_1AX)=0 \ \text{ and } \ r(M_{n+1}^{-1}A^*-XAM_{n+1}^\dagger A^*)=0.$$

Thus, from Theorem 2.2, Theorem 2.3 and Theorem 2.4, we immediately obtain the following key result in this section.

THEOREM 3.1. *Let* $A_i\in\mathbb{C}^{l_i\times l_{i+1}}$, $i=1,2,\cdots,n$ *and* $M_i\in\mathbb{C}^{l_i\times l_i}$, $i=1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2), $\widetilde{A_i}=M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i=1,2,\cdots,n$. Then the reverse order law (1.1) holds if and only if A_1, A_2, $\cdots$, A_n and A satisfy the rank equalities (2.9), (2.15) and (2.19).

In addition to the result in Theorem 3.1, we deduce another necessary and sufficient condition for the reverse order law in (1.1) to hold.

THEOREM 3.2. *Let* $A_i\in\mathbb{C}^{l_i\times l_{i+1}}$, $i=1,2,\cdots,n$ *and* $M_i\in\mathbb{C}^{l_i\times l_i}$, $i=1,2,\cdots,n+1$ *be* $n+1$ *positive definite Hermitian matrices. Let* A, $\widetilde{A}$ *and* X *be given as* (2.1) and (2.2), $\widetilde{A_i}=M_i^{1/2}A_iM_{i+1}^{-1/2}$, $i=1,2,\cdots,n$. Then the reverse order law (1.1) holds if and only if A_1, A_2, $\cdots$, A_n and A satisfy the following rank equality

$$r\begin{pmatrix}(-1)^n(\widetilde{A})^*\widetilde{A}(\widetilde{A})^* & (\widetilde{A})^*\widetilde{A_1}\widetilde{A_1}^*E_1\\ E_2\widetilde{A_n}^*\widetilde{A_n}(\widetilde{A})^* & S\end{pmatrix}=r(A)+r(A_2)+r(A_3)+\cdots+r(A_{n-1}), \quad (3.1)$$

where E_1, E_2 and S are given as in Corollary 1.1.

Proof. It is obvious that $A^\dagger_{M_1,M_{n+1}}=X$ holds if and only if

$$r(A^\dagger_{M_1,M_{n+1}}-X)=0.$$

By Lemma 1.3 and Lemma 1.4 and the structure of A in (2.1), we have

$$\begin{aligned}&(A_n)^\dagger_{M_n,M_{n+1}}A_nA^\dagger_{M_1,M_{n+1}}A_1(A_1)^\dagger_{M_1,M_2}\\ &=P_{N^{-1}R(A^*),N(A)}A^\dagger_{M_1,M_{n+1}}P_{R(A),M^{-1}N(A^*)}\\ &=P_{N^{-1}R(A^*),N(A)}A^{(1,2)}_{N^{-1}R(A^*),M^{-1}N(A^*)}P_{R(A),M^{-1}N(A^*)}\\ &=A^{(1,2)}_{N^{-1}R(A^*),M^{-1}N(A^*)}\\ &=A^\dagger_{M_1,M_{n+1}}.\end{aligned} \quad (3.2)$$

On the other hand, from the structure of X as given in (2.2) and the equality (2.11), we have

$$(A_n)^\dagger_{M_n,M_{n+1}}A_nXA_1(A_1)^\dagger_{M_1,M_2}=X. \quad (3.3)$$

Combining (3.2) and (3.3), we have

$$\begin{aligned} & r(A^{\dagger}_{M_1,M_{n+1}} - X) \\ & = r((A_n)^{\dagger}_{M_n,M_{n+1}}(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 - A_nXA_1)(A_1)^{\dagger}_{M_1,M_2}) \\ & \leqslant r(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 - A_nXA_1) \\ & = r(A_n(A^{\dagger}_{M_1,M_{n+1}} - X)A_1) \\ & \leqslant r(A^{\dagger}_{M_1,M_{n+1}} - X). \end{aligned} \tag{3.4}$$

Thus from (2.3) and (3.4), we have

$$\begin{aligned} & r(A^{\dagger}_{M_1,M_{n+1}} - X) \\ & = r(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 - A_nXA_1) \\ & = r(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 + (-1)^n A_nPN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA_1). \end{aligned} \tag{3.5}$$

So from (3.5), we know that the rank equality $r(A^{\dagger}_{M_1,M_{n+1}} - X) = 0$ holds if and only if

$$r(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 + (-1)^n A_nPN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA_1) = 0. \tag{3.6}$$

Using the matrices in (3.6), we construct a 3×3 block matrix:

$$G = \begin{pmatrix} \widetilde{T} & O & M^{1/2}QA_1 \\ O & (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* & \widetilde{A}^*M_1^{1/2}A_1 \\ A_nPN^{-1/2} & A_nM_{n+1}^{-1/2}\widetilde{A}^* & O \end{pmatrix}. \tag{3.7}$$

According to (1.20) and (1.21) in Corollary 1.2, we obtain

$$\mathscr{R}\begin{pmatrix} M^{1/2}QA_1 \\ \widetilde{A}^*M_1^{1/2}A_1 \end{pmatrix} \subseteq \mathscr{R}\begin{pmatrix} \widetilde{T} & O \\ O & (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* \end{pmatrix}$$

and

$$\mathscr{R}\begin{pmatrix} N^{-1/2}P^*A_n^* \\ \widetilde{A}M_{n+1}^{-1/2}A_n^* \end{pmatrix} \subseteq \mathscr{R}\begin{pmatrix} \widetilde{T}^* & O \\ O & \widetilde{A}\widetilde{A}^*\widetilde{A} \end{pmatrix}.$$

Hence by Lemma 1.1 (1) and Lemma 1.5, we have

$$\begin{aligned} r(G) &= r\begin{pmatrix} \widetilde{T}^* & O \\ O & (-1)^n\widetilde{A}\widetilde{A}^*\widetilde{A} \end{pmatrix} \\ &\quad + r\left(\left(A_nPN^{-1/2},\, A_nM_{n+1}^{-1/2}\widetilde{A}^*\right)\begin{pmatrix} \widetilde{T}^{\dagger} & O \\ O & (-1)^n(\widetilde{A}^*\widetilde{A}\widetilde{A}^*)^{\dagger} \end{pmatrix}\begin{pmatrix} M^{1/2}QA_1 \\ \widetilde{A}^*M_1^{1/2}A_1 \end{pmatrix}\right) \\ &= r(\widetilde{T}) + r(A) \\ &\quad + r((-1)^nA_nPN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA_1 + A_nM_{n+1}^{-1/2}\widetilde{A}^*(\widetilde{A}^*\widetilde{A}\widetilde{A}^*)^{\dagger}\widetilde{A}^*M_1^{1/2}A_1) \\ &= r(\widetilde{T}) + r(A) + r(A_nPN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA_1 + (-1)^nA_nA^{\dagger}_{M_1,M_{n+1}}A_1). \end{aligned} \tag{3.8}$$

On the other hand

$$r(G) = r\begin{pmatrix} O & \widetilde{A_1}\widetilde{A_1}^* E_1 & O & M_1^{1/2}A_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S & O & O \\ O & O & (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* & \widetilde{A}^*M_1^{1/2}A_1 \\ A_nM_{n+1}^{-1/2} & O & A_nM_{n+1}^{-1/2}\widetilde{A}^* & O \end{pmatrix}$$

$$= r\begin{pmatrix} O & O & O & M_1^{1/2}A_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n} & S & O & O \\ O & -\widetilde{A}^*\widetilde{A_1}\widetilde{A_1}^* E_1 & (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* & \widetilde{A}^*M_1^{1/2}A_1 \\ A_nM_{n+1}^{-1/2} & O & A_nM_{n+1}^{-1/2}\widetilde{A}^* & O \end{pmatrix}$$

$$= r\begin{pmatrix} O & O & O & M_1^{1/2}A_1 \\ O & S & -E_2\widetilde{A_n}^*\widetilde{A_n}\widetilde{A}^* & O \\ O & -\widetilde{A}^*\widetilde{A_1}\widetilde{A_1}^* E_1 & (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* & \widetilde{A}^*M_1^{1/2}A_1 \\ A_nM_{n+1}^{-1/2} & O & A_nM_{n+1}^{-1/2}\widetilde{A}^* & O \end{pmatrix}$$

$$= r\begin{pmatrix} (-1)^n\widetilde{A}^*\widetilde{A}\widetilde{A}^* & \widetilde{A}^*\widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n}\widetilde{A}^* & S \end{pmatrix} + r(A_1) + r(A_n) \tag{3.9}$$

Therefore, combining (3.8), (3.9) with (1.19), we have

$$r(A_nA^{\dagger}_{M_1,M_{n+1}}A_1 + (-1)^nA_nPN^{-1/2}\widetilde{T}^{\dagger}M^{1/2}QA_1)$$
$$= r\begin{pmatrix} (-1)^n(\widetilde{A})^*\widetilde{A}(\widetilde{A})^* & (\widetilde{A})^*\widetilde{A_1}\widetilde{A_1}^* E_1 \\ E_2\widetilde{A_n}^*\widetilde{A_n}(\widetilde{A})^* & S \end{pmatrix} - r(A) - r(A_2) - r(A_3) - \cdots - r(A_{n-1}).$$

Let the right-hand of above equality be zero and combining the result with (3.6), we complete the proof of Theorem 3.2. □

Acknowledgements. The authors would like to thank Professor Iva Franjic and the anonymous referees for their very detailed comments and constructive suggestions, which greatly improved the presentation of this paper.

REFERENCES

[1] A. BEN-ISRAEL AND T. N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, Wiley-Interscience, 1974, 2nd Edition, Springer-Verlag, New York, 2002.

[2] D. T. BARWICK AND J. D. GILBERT, *On generalization of the reverse order law with related results*, SIAM J. Appl. Math. **27** (1974), 326–330.

[3] D. S. DJORDJEVIĆ, *Furthuer results on the reverse order law for generalized inverses*, SIAM J. Matrix. Anal. Appl. **29** (2007), 1242–1246.

[4] T. N. E. GREVILLE, *Note on the generalized inverse of a matrix product*, SIAM Review **8** (1966), 518–521.

[5] R. E. HARTWIG, *Block generalized inverses*, Arch. Rational. Mech. Anal. **61** (1976), 197–251.

[6] Q. LIU AND M. WEI, *Reverse order law for least squares g-inverses of multiple matrix products*, Linear and Multilinear Algebra **56** (2008), 491–506.
[7] G. MARSAGLIA AND G. P. H. STYAN, *Equalities and inequalities for ranks of matrices*, Linear and Multilinear Algebra **2** (1974), 269–292.
[8] R. PENROSE, *A generalized inverse for matrices*, Proc. Cambridge. Philos. Soc. **51** (1955), 406–413.
[9] C. R. RAO AND S. K. MITRA, *Generalized Inverse of Matrices and Its Applications*, Wiley, New York, 1971.
[10] W. SUN AND Y. WEI, *Inverse order rule for weighted generalized inverse*, SIAM J. Matrix Anal. Appl. **19** (1998), 772–775.
[11] Y. TIAN, *Reverse order laws for generalized inverse of multiple matrix products*, Linear Algebra Appl. **211** (1994), 85–100.
[12] G. WANG, Y. WE AND S. QIAO, *Generalized Inverses: Theory and Computations*, Science Press, Beijing, 2004.
[13] G. WANG AND B. ZHENG, *The reverse order law for the generalized inverse* $A^{(2)}_{T,S}$, Appl. Math. Comput. **157** (2004), 295–305.
[14] M. WEI, *Reverse order laws for generalized inverse of multiple matrix products*, Linear Algebra Appl. **293** (1999), 273–288.
[15] M. WANG, M. WEI AND Z. JIA, *Mixed-type reverse order law of* $(AB)^{(1,3)}$, Linear Algebra Appl. **43** (2009), 1691–1699.
[16] Z. XIONG AND B. ZHENG, *The reverse order laws for* $\{1,2,3\}$*- and* $\{1,2,4\}$*-inverses of two matrix product*, Appl. Math. Let. **21** (2008), 649–655.
[17] B. ZHENG AND Z. XIONG, *An new equivalent condition of the reverse order law for g-inverses of multiple matrix products*, Electronic Journal of Linear Algebra **17** (2008), 1–8.
[18] B. ZHENG AND Z. XIONG, *On reverse order laws for the weighted generalized inverse*, Arab. J. Sci. Eng. **34** (2A) (2009), 195–203.
[19] S. ZLOBEC, *An explicit form of the Moore-Penrose inverse of an arbitrary complex matrix*, SIAM Review **12** (1970), 132–134.

(Received December 22, 2011)

Zhiping Xiong
School of Mathematics and Computational Science
Wuyi University
Jiangmen 529020, Guangdong
P. R. China
e-mail: xzpwhere@163.com

Yingying Qin
School of Mathematics and Computational Science
Wuyi University
Jiangmen 529020, Guangdong
P. R. China