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SOBOLEV EMBEDDINGS FOR RIESZ POTENTIALS OF FUNCTIONS IN MORREY SPACES $L^{1,\nu,\beta_1,\beta_2}(\mathbf{R}^n)$

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Abstract. Our aim in this paper is to deal with Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{1,\nu,\beta_1,\beta_2}(\mathbf{R}^n)$, as an extension of Trudinger [17], Serrin [14] and the authors [5] in the case of bounded open sets. We are mainly concerned with Trudinger's type exponential integrability.

1. Introduction

The space introduced by Morrey [9] in 1938 has become a useful tool of the study for the existence and regularity of solutions of partial differential equations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. In the present paper, we aim to establish Sobolev embeddings for Riesz potentials of functions in Morrey spaces $L^{1,\nu,\beta_1,\beta_2}(\mathbf{R}^n)$, as an extension of Trudinger [17], Serrin [14] and the authors [5] in the case of bounded open sets. We are mainly concerned with Trudinger's type exponential integrability.

Let $B(x,r)$ denote the open ball centered at x of radius r. In particular, we set $\mathbf{B} = B(0,1)$. For $0 < \alpha < n$, we define the generalized Riesz potential of order α for a locally integrable function f on $\mathbf{R}^n$ by

$$I_{\alpha,0}f(x) = \int_{\mathbf{R}^n} \{|x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)\} f(y)\,dy,$$

where χ_E denotes the characteristic functions of a measurable set $E \subset \mathbf{R}^n$; the integral converges almost everywhere when

$$\int_{\mathbf{R}^n} (1+|y|)^{\alpha-n-1}|f(y)|\,dy < \infty. \tag{1}$$

We set

$$\tilde{I}_\alpha f(x) = \int_{\mathbf{R}^n} ||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)|\,|f(y)|\,dy. \tag{2}$$

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For an integrable function u on a measurable set $E \subset \mathbf{R}^n$ of positive measure, we define the integral mean over E by

$$\fint_E u(x)\,dx = \frac{1}{|E|}\int_E u(x)\,dx,$$

where $|E|$ denotes the Lebesgue measure of E.

In the present paper, f is assumed to satisfy the Morrey condition : if $0 \leqslant \nu \leqslant n$ and β_1 and β_2 are real numbers, then

$$\fint_{B(x,r)} |f(y)|dy \leqslant r^{-\nu}(\log(e+r^{-1}))^{-\beta_1}(\log(e+r))^{-\beta_2} \tag{3}$$

for all $x \in \mathbf{R}^n$ and $r > 0$. We denote by $L^{1,\nu,\beta_1,\beta_2}(\mathbf{R}^n)$ the family of all measurable functions f on $\mathbf{R}^n$ satisfying condition (3); for Morrey spaces, we refer to [9] and [13].

The famous Trudinger's inequality ([17]) insists that Sobolev functions in $W^{1,n}$ satisfy finite exponential integrability (see also [2], [4] and [18]). For another proof, see Serrin [14]. The authors [5] gave a result on Sobolev embeddings for Riesz potentials of functions satisfying (3) with $\beta_2 = 0$ in the case of bounded open sets.

Our first aim in this paper is to give a Morrey version of Trudinger's type exponential integrability for Riesz potentials of functions satisfying (3), as an extension of [14], [17] and [5, Theorem 1.1].

THEOREM 1. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$ and real numbers β_1 and β_2. If $0 < \varepsilon < \alpha/2$, then there exist constants $c_j = C(n,\alpha,\beta_1,\beta_2,\varepsilon) > 0$ $(j = 1,2)$ such that*

(1) *in case $1 > \beta_1 \geqslant \beta_2$,*

$$\fint_{B(z,r)} (e+|x|)^{-\varepsilon}\exp\left(\frac{1}{c_1}\left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}}\right)^{1/(1-\beta_1)}\right)dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$;

(2) *in case $1 = \beta_1 > \beta_2$,*

$$\fint_{B(z,r)} \exp\left(\exp\left(\frac{\tilde{I}_\alpha f(x)}{c_1(\log(e+|x|))^{1-\beta_2}}\right)\right)dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$;

(3) *in case $\beta_1 > 1$,*

$$|I_{\alpha,0}f(x) - I_{\alpha,0}f(z)| \leqslant c_2(\log|x-z|^{-1})^{-\beta_1+1}$$

for all $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$.

The sharpness of Theorem 1 will be discussed in Section 3 (see Remark 3 below).

REMARK 1. Suppose

$$\fint_{B(z,r)} (e+|x|)^{-\varepsilon_0} \exp\left(\frac{1}{c_1} \left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}} \right)^{1/(1-\beta_1)} \right) dx \leqslant c_2(1+r^{-\varepsilon_0})$$

for $z \in \mathbf{R}^n$ and $r>0$. Then Jensen's inequality gives

$$\begin{aligned}
&\fint_{B(z,r)} (e+|x|)^{-\varepsilon} \exp\left(\frac{\varepsilon}{c_1\varepsilon_0} \left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}} \right)^{1/(1-\beta_1)} \right) dx \\
&\leqslant \left(\fint_{B(z,r)} (e+|x|)^{-\varepsilon_0} \exp\left(\frac{1}{c_1} \left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}} \right)^{1/(1-\beta_1)} \right) dx \right)^{\varepsilon/\varepsilon_0} \\
&\leqslant \{c_2(1+r^{-\varepsilon_0})\}^{\varepsilon/\varepsilon_0} \\
&\leqslant (2c_2)^{\varepsilon/\varepsilon_0}(1+r^{-\varepsilon})
\end{aligned}$$

when $0<\varepsilon<\varepsilon_0$.

REMARK 2. Theorem 1 (1) gives the exponential inequality by letting $r \to \infty$:

$$\int_{\mathbf{R}^n} (e+|x|)^{-n-\varepsilon} \exp\left(\frac{1}{c_1} \left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}} \right)^{1/(1-\beta_1)} \right) dx \leqslant c_3$$

with $c_3 = C(n,\alpha,\beta_1,\beta_2,\varepsilon) > 0$.

We next give the following Morrey version of Sobolev's type inequality for Riesz potentials of functions satisfying (3), as an extension of [14] and [5, Theorem 1.2].

THEOREM 2. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\alpha<\nu\leqslant n$ and real numbers β_1 and β_2. If $p=\nu/(\nu-\alpha)$ and $\gamma>1$, then there exists a constant $C=C(n,\alpha,\nu,\beta_1,\beta_2,\gamma)>0$ such that*

(1) *in case $\beta_2<1$,*

$$\begin{aligned}
&\left(\fint_{B(z,r)} (\tilde{I}_\alpha f(x))^p (\log(e+\tilde{I}_\alpha f(x)))^{-\gamma+\alpha\beta_1 p/\nu} (\log(e+(\tilde{I}_\alpha f(x))^{-1}))^{-\gamma+\alpha\beta_2 p/\nu} dx \right)^{1/p} \\
&\leqslant Cr^{\alpha-\nu}(\log(e+r^{-1}))^{(1-\gamma-\beta_1)/p}(\log(e+r))^{-\beta_2/p} \\
&\quad +C(\log(e+r+|z|))^{-\beta_2+1}(\log(e+(\log(e+r+|z|))^{-\beta_2+1}))^{-\gamma/p+\alpha\beta_1/\nu} \qquad (4)
\end{aligned}$$

for all $z \in \mathbf{R}^n$ and $r>0$;

(2) *in case $\beta_2=1$,*

$$\begin{aligned}
&\left(\fint_{B(z,r)} (\tilde{I}_\alpha f(x))^p (\log(e+\tilde{I}_\alpha f(x)))^{-\gamma+\alpha\beta_1 p/\nu} (\log(e+(\tilde{I}_\alpha f(x))^{-1}))^{-\gamma+\alpha p/\nu} dx \right)^{1/p} \\
&\leqslant Cr^{\alpha-\nu}(\log(e+r^{-1}))^{(1-\gamma-\beta_1)/p}(\log(e+r))^{-1/p}
\end{aligned}$$

for all $z \in \mathbf{R}^n$ *and* $r > 0$;

(3) *in case* $\beta_2 > 1$,

$$\left(\fint_{B(z,r)} (\tilde{I}_\alpha f(x))^p (\log(e + \tilde{I}_\alpha f(x)))^{-\gamma + \alpha\beta_1 p/\nu} (\log(e + (\tilde{I}_\alpha f(x))^{-1}))^{-\gamma + \alpha\beta_2 p/\nu} dx \right)^{1/p}$$
$$\leqslant C r^{\alpha - \nu} (\log(e + r^{-1}))^{(1-\gamma-\beta_1)/p} (\log(e + r))^{-\beta_2/p}$$
$$+ C(\log(e + r + |z|))^{-\beta_2+1} (\log(e + (\log(e + r + |z|))^{\beta_2 - 1}))^{-\gamma/p + \alpha\beta_2/\nu}$$

for all $z \in \mathbf{R}^n$ *and* $r > 0$.

We will show that the condition $\gamma > 1$ is sharp in Theorem 2 (see Remark 5 below).

For related results, we also refer to Adams [1], Chiarenza-Frasca [3] and the authors [6, 7, 8, 11, 12].

2. Preliminary lemmas

Throughout this paper, let C denote various positive constants independent of the variables in question and $C(a,b,\cdots)$ be a constant which may depend on $a,b,\cdots$.

First we note the following lemma.

LEMMA 1. ([5, Lemma 2.2]) *Let* f *be a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (3) *with* $\nu = \alpha$.

(1) *If* $0 < \varepsilon < \alpha/2$, *then*

$$\int_{B(x,1)\setminus B(x,\delta)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \leqslant C\delta^{-\varepsilon} (\log(e+\delta^{-1}))^{-\beta_1};$$

(2) *if* $\beta_1 < 1$, *then*

$$\int_{B(x,1)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C(\log(e+\delta^{-1}))^{-\beta_1+1};$$

(3) *if* $\beta_1 = 1$, *then*

$$\int_{B(x,1)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C\log(\log(e^e+\delta^{-1}))$$

for $x \in \mathbf{R}^n$ *and* $0 < \delta < 1$, *where* $C = C(n,\beta_1,\varepsilon)$.

LEMMA 2. *Let* f *be a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (3) *with* $\nu = \alpha$. *If* $0 < \varepsilon < \alpha/2$, *then*

$$\int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \leqslant C\delta^{-\varepsilon} (\log(e+\delta^{-1}))^{-\beta_1} + C$$

for $x \in \mathbf{R}^n$ *and* $\delta > 0$, *where* $C = C(n,\alpha,\beta_1,\beta_2,\varepsilon)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. If $0 < \varepsilon < \alpha/2$, then we have by Lemma 1 (1)

$$\int_{B(x,1)\setminus B(x,\delta)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \leqslant C\delta^{-\varepsilon}(\log(e+\delta^{-1}))^{-\beta_1}.$$

Next we will estimate $\int_{\mathbf{R}^n\setminus B(x,1)} |x-y|^{\alpha-\varepsilon-n} f(y)dy$. Integrating by parts and changing to polar coordinates, we have by (3)

$$\begin{aligned}
&\int_{\mathbf{R}^n\setminus B(x,1)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \\
&= \left[r^{\alpha-\varepsilon-n}\int_{B(x,r)} f(y)dy\right]_1^\infty + \int_1^\infty \left(\int_{B(x,r)} f(y)dy\right) d(-r^{\alpha-\varepsilon-n}) \\
&\leqslant C\int_1^\infty r^{-\varepsilon}(\log(e+r))^{-\beta_2}\frac{dr}{r}.
\end{aligned}$$

When $\beta_2 \geqslant 0$, we have

$$\int_1^\infty r^{-\varepsilon}(\log(e+r))^{-\beta_2}\frac{dr}{r} \leqslant C\int_1^\infty r^{-\varepsilon}\frac{dr}{r} = C/\varepsilon.$$

Next consider the case $\beta_2 < 0$. Note that $s^{\varepsilon/2}(\log(e+1/s))^{-\beta_2}$ attains a maximum value of $e^{\beta_2}(-2\beta_2/\varepsilon)^{-\beta_2}$ at $s = e^{2\beta_2/\varepsilon}$. Therefore

$$\begin{aligned}
\int_1^\infty r^{-\varepsilon}(\log(e+r))^{-\beta_2}\frac{dr}{r} &= \int_0^1 s^{\varepsilon}(\log(e+1/s))^{-\beta_2}\frac{ds}{s} \\
&\leqslant e^{\beta_2}(-2\beta_2/\varepsilon)^{-\beta_2}\int_0^1 s^{\varepsilon/2}\frac{ds}{s} < \infty.
\end{aligned}$$

Hence

$$\int_{\mathbf{R}^n\setminus B(x,1)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \leqslant C.$$

Thus it follows that

$$\int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-\varepsilon-n} f(y)dy \leqslant C\delta^{-\varepsilon}(\log(e+\delta^{-1}))^{-\beta_1} + C,$$

where C is a positive constant depending on $n, \alpha, \beta_1, \beta_2, \varepsilon$. □

LEMMA 3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$.*

(1) *If $\beta_1 < 1$ and $\beta_2 < 1$, then*

$$\int_{B(0,1+2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C(\log(e+\delta^{-1}))^{-\beta_1+1} + C(\log(e+|x|))^{-\beta_2+1};$$

(2) *if* $1 = \beta_1 > \beta_2$, *then*

$$\int_{B(0,1+2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C\log(\log(e^e+\delta^{-1})) + C(\log(e+|x|))^{-\beta_2+1}$$

for $x \in \mathbf{R}^n$ *and* $0 < \delta < 1$, *where* $C = C(n,\alpha,\beta_1,\beta_2)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. If $\beta_1 < 1$, then we have by Lemma 1 (2)

$$\int_{B(x,1)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C(\log(e+\delta^{-1}))^{-\beta_1+1}.$$

Next we will estimate $\int_{B(0,1+2|x|)\setminus B(x,1)} |x-y|^{\alpha-n} f(y)dy$. Integrating by parts and changing to polar coordinates, we have by (3)

$$\begin{aligned}
&\int_{B(0,1+2|x|)\setminus B(x,1)} |x-y|^{\alpha-n} f(y)dy \\
&\leqslant \int_{B(x,1+3|x|)\setminus B(x,1)} |x-y|^{\alpha-n} f(y)dy \\
&= \left[r^{\alpha-n} \int_{B(x,r)} f(y)dy \right]_1^{1+3|x|} + \int_1^{1+3|x|} \left(\int_{B(x,r)} f(y)dy \right) d(-r^{\alpha-n}) \\
&\leqslant C(\log(e+|x|))^{-\beta_2} + C\int_1^{1+3|x|} (\log(e+r))^{-\beta_2} \frac{dr}{r} \\
&\leqslant C(\log(e+|x|))^{-\beta_2} + C(\log(e+|x|))^{-\beta_2+1} \\
&\leqslant C(\log(e+|x|))^{-\beta_2+1}.
\end{aligned}$$

Thus it follows that

$$\int_{B(0,1+2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C(\log(e+\delta^{-1}))^{-\beta_1+1} + C(\log(e+|x|))^{-\beta_2+1},$$

where C is a positive constant depending on $n, \alpha, \beta_1, \beta_2$.

The remaining case can be proved similarly. □

As in the proof of Lemma 3, we can prove the following lemma.

LEMMA 4. (cf. [5, Lemma 2.3]) *Let* $\alpha < \nu \leqslant n$. *Let* f *be a nonnegative measurable function on* $\mathbf{R}^n$ *satisfying* (3). *Then*

$$\int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \leqslant C\delta^{\alpha-\nu}(\log(e+\delta^{-1}))^{-\beta_1}(\log(e+\delta))^{-\beta_2}$$

for $x \in \mathbf{R}^n$ *and* $\delta > 0$, *where* $C = C(n,\alpha,\nu,\beta_1,\beta_2)$.

As in the proof of [5, Lemma 2.4], we can prove the following lemma in view of Lemma 2.

LEMMA 5. *Let $0 < \varepsilon < \alpha/2$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. Then*

$$\fint_{B(z,\delta)} I_{\alpha-\varepsilon} f(x)dx \leqslant C\delta^{-\varepsilon}(\log(e+\delta^{-1}))^{-\beta_1}(\log(e+\delta))^{-\beta_2} + C$$

for $z \in \mathbf{R}^n$ and $\delta > 0$, where $C = C(n,\alpha,\beta_1,\beta_2,\varepsilon)$.

LEMMA 6. *Let $\beta_2 < 1$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. Then*

$$G_1(x) \equiv \int_{B(0,|x|)\setminus \mathbf{B}} |y|^{\alpha-n} f(y)dy \leqslant C(\log(e+|x|))^{-\beta_2+1}$$

for $x \in \mathbf{R}^n$, where $C = C(n,\alpha,\beta_1,\beta_2)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. We have

$$\begin{aligned} G_1(x) &= \left[r^{\alpha-n} \int_{B(0,r)} f(y)dy \right]_1^{|x|} + \int_1^{|x|} \left(\int_{B(0,r)} f(y)dy \right) d(-r^{\alpha-n}) \\ &\leqslant C(\log(e+|x|))^{-\beta_2} + C\int_1^{|x|} (\log(e+r))^{-\beta_2} \frac{dr}{r} \\ &\leqslant C(\log(e+|x|))^{-\beta_2+1}, \end{aligned}$$

as required. □

LEMMA 7. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. Then*

$$G_2(x) \equiv |x| \int_{\mathbf{R}^n \setminus B(0,2|x|)} |y|^{\alpha-n-1} f(y)dy \leqslant C(\log(e+|x|))^{-\beta_2}$$

for $x \in \mathbf{R}^n \setminus \mathbf{B}$, where $C = C(n,\alpha,\beta_1,\beta_2)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. We have

$$\begin{aligned} G_2(x) &= |x| \left\{ \left[r^{\alpha-n-1} \int_{B(0,r)} f(y)dy \right]_{2|x|}^{\infty} + \int_{2|x|}^{\infty} \left(\int_{B(0,r)} f(y)dy \right) d(-r^{\alpha-n-1}) \right\} \\ &\leqslant C|x| \int_{2|x|}^{\infty} (\log(e+r))^{-\beta_2} r^{-2} dr \\ &\leqslant C(\log(e+|x|))^{-\beta_2}, \end{aligned}$$

as required. □

3. Proof of Theorem 1

To complete the proof of Theorem 1, we prepare the following results.

THEOREM 3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$ and real numbers β_1 and β_2 with $1 > \beta_1 \geqslant \beta_2$. If $\varepsilon > 0$, then there exist constants $c_j = C(n,\alpha,\beta_1,\beta_2,\varepsilon) > 0$ $(j = 1,2)$ such that*

$$\fint_{B(z,r)} (e+|x|)^{-\varepsilon} \exp\left(\frac{1}{c_1}\left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}}\right)^{1/(1-\beta_1)}\right) dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$. Note that

$$||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)| \leqslant C|x||y|^{\alpha-n-1}$$

whenever $|y| > 2|x|$, which is derived by the mean value theorem; see e.g. [15, Section 3]. For $0 < \varepsilon < \alpha/2$, by Lemmas 3, 6 and 7, we have

$$\begin{aligned}
\tilde{I}_\alpha f(x) &= \int_{B(0,1+2|x|)} ||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)|\, f(y)\, dy \\
&\quad + \int_{\mathbf{R}^n\setminus B(0,1+2|x|)} ||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)|\, f(y)\, dy \\
&\leqslant \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{B(0,1+2|x|)\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
&\quad + \int_{B(0,1+2|x|)\setminus\mathbf{B}} |y|^{\alpha-n} f(y)dy + C|x| \int_{\mathbf{R}^n\setminus B(0,1+2|x|)} |y|^{\alpha-n-1} f(y)dy \\
&\leqslant \delta^\varepsilon \int_{B(x,\delta)} |x-y|^{\alpha-\varepsilon-n} f(y)dy + C(\log(e+\delta^{-1}))^{-\beta_1+1} + C(\log(e+|x|))^{-\beta_2+1} \\
&\leqslant \delta^\varepsilon I_{\alpha-\varepsilon} f(x) + C(\log(e+\delta^{-1}))^{-\beta_1+1} + C(\log(e+|x|))^{-\beta_2+1}
\end{aligned}$$

for $0 < \delta < 1$. Considering

$$\delta = \min\{1/2, (I_{\alpha-\varepsilon} f(x))^{-1/\varepsilon} (\log(e + I_{\alpha-\varepsilon} f(x)))^{(1-\beta_1)/\varepsilon}\},$$

we see that

$$\tilde{I}_\alpha f(x) \leqslant C(\log(e + I_{\alpha-\varepsilon} f(x)))^{-\beta_1+1} + C(\log(e+|x|))^{-\beta_2+1},$$

so that

$$\begin{aligned}
&\fint_{B(z,r)} (e+|x|)^{-\varepsilon} \exp\left(\frac{1}{c_1}\left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}}\right)^{1/(1-\beta_1)}\right) dx \\
&\leqslant 1 + \fint_{B(z,r)} I_{\alpha-\varepsilon} f(x) dx
\end{aligned}$$

for $z \in \mathbf{R}^n$ and $r > 0$. Hence Lemma 5 gives

$$\begin{aligned}
&\fint_{B(z,r)} (e+|x|)^{-\varepsilon} \exp\left(\frac{1}{c_1}\left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{\beta_1-\beta_2}}\right)^{1/(1-\beta_1)}\right) dx \\
&\leqslant 1 + Cr^{-\varepsilon}(\log(e+r^{-1}))^{-\beta_1}(\log(e+r))^{-\beta_2} + C \\
&\leqslant C\{1 + r^{-\varepsilon}(\log(e+r^{-1}))^{-\beta_1}(\log(e+r))^{-\beta_2}\} \\
&\leqslant c_2(1+r^{-2\varepsilon})
\end{aligned}$$

for such z and r, which proves the present theorem. □

THEOREM 4. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$ and real numbers $1 = \beta_1 > \beta_2$. If $\varepsilon > 0$, then there exist constants $c_j = C(n,\alpha,\beta_2,\varepsilon) > 0$ $(j = 1,2)$ such that*

$$\fint_{B(z,r)} \exp\left(\exp\left(\frac{\tilde{I}_\alpha f(x)}{c_1(\log(e+|x|))^{1-\beta_2}}\right)\right) dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$.

Proof. Let $1 = \beta_1 > \beta_2$. For $0 < \varepsilon < \alpha/2$, by Lemmas 3, 6 and 7, we have as above

$$\tilde{I}_\alpha f(x) \leqslant \delta^\varepsilon I_{\alpha-\varepsilon} f(x) + C\log(\log(e^e+\delta^{-1})) + C(\log(e+|x|))^{-\beta_2+1}$$

for $0 < \delta < 1$. Considering

$$\delta = \min\{1/2, (I_{\alpha-\varepsilon} f(x))^{-1/\varepsilon}(\log(\log(e^e + I_{\alpha-\varepsilon} f(x))))^{1/\varepsilon}\},$$

we see that

$$(\log(e+|x|))^{\beta_2-1}\tilde{I}_\alpha f(x) \leqslant C\log\left(\log(e^e + I_{\alpha-\varepsilon} f(x))\right),$$

so that

$$\fint_{B(z,r)} \exp\left(\exp\left(\frac{\tilde{I}_\alpha f(x)}{c_1(\log(e+|x|))^{-\beta_2+1}}\right)\right) dx \leqslant \fint_{B(z,r)} \{e^e + I_{\alpha-\varepsilon} f(x)\} dx$$

for $z \in \mathbf{R}^n$ and $r > 0$. Hence Lemma 5 gives

$$\begin{aligned}
&\fint_{B(z,r)} \exp\left(\exp\left(\frac{\tilde{I}_\alpha f(x)}{c_1(\log(e+|x|))^{-\beta_2+1}}\right)\right) dx \\
&\leqslant Cr^{-\varepsilon}(\log(e+r^{-1}))^{-1}(\log(e+r))^{-\beta_2} + C \\
&\leqslant c_2(1+r^{-2\varepsilon})
\end{aligned}$$

for such z and r, which proves the present theorem. □

THEOREM 5. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\nu = \alpha$ and real numbers $\beta_1 > 1$ and β_2. Then*

$$|I_{\alpha,0}f(x) - I_{\alpha,0}f(z)| \leqslant C(\log|x-z|^{-1})^{-\beta_1+1}$$

for all $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$.

Proof. Let $\beta_1 > 1$. Write

$$\begin{aligned} I_{\alpha,0}f(x) - I_{\alpha,0}f(z) = &\int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy - \int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy \\ &+ \int_{\mathbf{R}^n \setminus B(x,2|x-z|)} (|x-y|^{\alpha-n} - |z-y|^{\alpha-n}) f(y)dy. \end{aligned}$$

As in the proof of Lemma 1, we have

$$\int_{B(x,2|x-z|)} |x-y|^{\alpha-n} f(y)dy \leqslant C(\log|x-z|^{-1})^{-\beta_1+1}$$

and

$$\begin{aligned} \int_{B(x,2|x-z|)} |z-y|^{\alpha-n} f(y)dy &\leqslant \int_{B(z,3|x-z|)} |z-y|^{\alpha-n} f(y)dy \\ &\leqslant C(\log|x-z|^{-1})^{-\beta_1+1} \end{aligned}$$

for $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$. On the other hand, by the mean value theorem for analysis, we have by Lemma 4

$$\begin{aligned} &\int_{\mathbf{R}^n \setminus B(x,2|x-z|)} ||x-y|^{\alpha-n} - |z-y|^{\alpha-n}| f(y)dy \\ &\leqslant C|x-z| \int_{\mathbf{R}^n \setminus B(x,2|x-z|)} |x-y|^{\alpha-n-1} f(y)dy \\ &\leqslant C(\log|x-z|^{-1})^{-\beta_1} \end{aligned}$$

for all $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$. As a consequence we obtain

$$|I_{\alpha,0}f(x) - I_{\alpha,0}f(z)| \leqslant C(\log|x-z|^{-1})^{-\beta_1+1}$$

for $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$, which proves the present theorem. □

REMARK 3. Let $f(y) = (e+|y|)^{-\alpha}$. Then $f \in L^{1,\alpha,0,0}(\mathbf{R}^n)$. In fact, if $r < |x|/2$, then

$$⨍_{B(x,r)} f(y)dy \leqslant C|x|^{-\alpha} \leqslant Cr^{-\alpha}$$

and if $r \geqslant |x|/2$, then the inequality is easily obtained.

We see that

$$\begin{aligned}\tilde{I}_\alpha f(x) &\geqslant C\int_{B(0,|x|/3)}(e+|y|)^{\alpha-n}f(y)dy\\ &= C\int_{B(0,|x|/3)}(e+|y|)^{-n}dy\\ &\geqslant C\log(e+|x|),\end{aligned}$$

so that for $C_1>0$,

$$\begin{aligned}\fint_{B(0,r)}\exp\left(\frac{\tilde{I}_\alpha f(x)}{C_1}\right)dx &\geqslant \fint_{B(0,r)}(e+|x|)^{C/C_1}dx\\ &\geqslant C(e+r)^{C/C_1},\end{aligned}$$

which implies that one can not take $\varepsilon=0$ in Theorem 1 (1).

Further we find

$$\begin{aligned}&\fint_{B(0,r)}(e+|x|)^{-\varepsilon}\exp\left(\left(\frac{\tilde{I}_\alpha f(x)}{C_1}\right)^{1+\delta}\right)dx\\ &\geqslant C(e+r)^{-\varepsilon}\{\exp((C/C_1)^{1+\delta}(\log(e+r))^{1+\delta})\},\end{aligned}$$

which assures that the exponent 1 is sharp in Theorem 1 (1).

REMARK 4. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying

$$\left(\fint_{B(x,r)}f(y)^p(\log(e+f(y)))^{pq}dy\right)^{1/p}\leqslant Cr^{-\nu}(\log(e+r^{-1}))^{-\beta_1}(\log(e+r))^{-\beta_2} \quad (5)$$

for all $x\in\mathbf{R}^n$ and $r>0$, with $p>1$, $\nu>0$, $q\geqslant 0$ and real numbers β_1,β_2. Then $f\in L^{1,\nu,\beta_1+q,\beta_2}(\mathbf{R}^n)$, that is,

$$\fint_{B(x,r)}f(y)dy\leqslant Cr^{-\nu}(\log(e+r^{-1}))^{-\beta_1-q}(\log(e+r))^{-\beta_2}. \quad (6)$$

In fact, taking $a>\nu$ when $r>1$ ($0<a<\nu$ when $r\leqslant 1$), we have by (5)

$$\begin{aligned}\int_{B(x,r)}f(y)^pdy &\leqslant \int_{B(x,r)}r^{-ap}dy+\int_{B(x,r)}f(y)^p\left(\frac{\log(e+f(y))}{\log(e+r^{-a})}\right)^{pq}dy\\ &\leqslant Cr^{n-ap}+C\left(\log(e+r^{-1})\right)^{-pq}\int_{B(x,r)}f(y)^p\left(\log(e+f(y))\right)^{pq}dy\\ &\leqslant Cr^{n-ap}+Cr^{n-\nu p}\left(\log(e+r^{-1})\right)^{(-\beta_1-q)p}(\log(e+r))^{-\beta_2 p},\end{aligned}$$

so that Jensen's inequality yields (6).

Hence Theorem 1 yields the following result.

COROLLARY 1. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying*

$$\left(\fint_{B(x,r)} f(y)^{n/\alpha}(\log(e+f(y)))^{nq/\alpha}dy\right)^{\alpha/n} \leqslant Cr^{-\nu}(\log(e+r^{-1}))^{-\beta}$$

for all $x \in \mathbf{R}^n$ and $r > 0$, where $q \geqslant 0$. If $\varepsilon > 0$, then there exist constant $c_j = C(n,\alpha,q,\beta,\varepsilon) > 0$ $(j = 1,2)$ such that

(1) *in case $0 \leqslant q+\beta < 1$,*

$$\fint_{B(z,r)} (e+|x|)^{-\varepsilon}\exp\left(\frac{1}{c_1}\left(\frac{\tilde{I}_\alpha f(x)}{(\log(e+|x|))^{q+\beta}}\right)^{1/(1-q-\beta)}\right)dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$;

(2) *in case $q+\beta = 1$,*

$$\fint_{B(z,r)} \exp\left(\exp\left(\frac{\tilde{I}_\alpha f(x)}{c_1\log(e+|x|)}\right)\right)dx \leqslant c_2(1+r^{-\varepsilon})$$

for all $z \in \mathbf{R}^n$ and $r > 0$;

(3) *in case $q+\beta > 1$,*

$$|I_{\alpha,0}f(x) - I_{\alpha,0}f(z)| \leqslant C(\log|x-z|^{-1})^{1-q-\beta}$$

for all $x,z \in \mathbf{R}^n$ with $|x-z| < 1/e$.

4. Proof of Theorem 2

For $\gamma > 0$, let

$$\rho_\gamma(r) = r^{-n}(\log(e+r^{-1}))^{-\gamma}(\log(e+r))^{-\gamma}.$$

For a proof of Theorem 2, we prove the following lemma.

LEMMA 8. *Let $\alpha < \nu \leqslant n$ and $\gamma > 1$. If f is a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3), then*

$$\int_{B(z,r)}\left(\int_{\mathbf{R}^n}\rho_\gamma(|x-y|)f(y)dy\right)dx \leqslant Cr^{n-\nu}(\log(e+r^{-1}))^{-\gamma-\beta_1+1}(\log(e+r))^{-\beta_2}$$

for all $z \in \mathbf{R}^n$ and $r > 0$, where $C = C(n,\nu,\beta_1,\beta_2,\gamma)$.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3). Write

$$\begin{aligned}\int_{\mathbf{R}^n}\rho_\gamma(|x-y|)f(y)dy &= \int_{B(z,2r)}\rho_\gamma(|x-y|)f(y)dy + \int_{\mathbf{R}^n\setminus B(z,2r)}\rho_\gamma(|x-y|)f(y)dy \\ &= H_1(x) + H_2(x).\end{aligned}$$

By Fubini's theorem, we have by (3)

$$\begin{aligned}
&\int_{B(z,r)} H_1(x)dx \\
&\leqslant \int_{B(z,2r)} \left(\int_{B(z,r)} \rho_\gamma(|x-y|)dx \right) f(y)dy \\
&\leqslant \int_{B(z,2r)} \left(\int_{B(y,3r)} |x-y|^{-n}(\log(e+|x-y|^{-1}))^{-\gamma}(\log(e+|x-y|))^{-\gamma}dx \right) f(y)dy \\
&\leqslant C(\log(e+r^{-1}))^{-\gamma+1} \int_{B(z,2r)} f(y)dy \\
&\leqslant Cr^{n-\nu}(\log(e+r^{-1}))^{-\gamma-\beta_1+1}(\log(e+r))^{-\beta_2}.
\end{aligned}$$

For H_2, we have by (3)

$$\begin{aligned}
\int_{B(z,r)} H_2(x)dx &\leqslant \int_{B(z,r)} \left(\int_{\mathbf{R}^n \setminus B(x,r)} \rho_\gamma(|x-y|)f(y)dy \right) dx \\
&\leqslant Cr^n \left\{ \left[t^{-n}(\log(e+t^{-1}))^{-\gamma}(\log(e+t))^{-\gamma} \int_{B(x,t)} f(y)dy \right]_r^\infty \right. \\
&\qquad \left. + \int_r^\infty \left(\int_{B(x,t)} f(y)dy \right) d(-\rho_\gamma(t)) \right\} \\
&\leqslant Cr^n \int_r^\infty t^{-\nu}(\log(e+t^{-1}))^{-\beta_1-\gamma}(\log(e+t))^{-\beta_2-\gamma}\frac{dt}{t} \\
&\leqslant Cr^{n-\nu}(\log(e+r^{-1}))^{-\beta_1-\gamma}(\log(e+r))^{-\beta_2-\gamma}.
\end{aligned}$$

Thus this lemma is proved. □

LEMMA 9. *Let ω be a nonnegative continuous function on $(0,\infty)$ such that $r^a\omega(r)$ is almost increasing for some $0<a<n$, that is,*

$$r_1^a\omega(r_1) \leqslant Cr_2^a\omega(r_2) \qquad \text{whenever } 0<r_1<r_2.$$

Then

$$⨍_{B(z,r)} \omega(|x|)\, dx \leqslant C\omega(|z|+r)$$

for all $z \in \mathbf{R}^n$ and $r>0$.

Proof. If $x \in B(z,r)$, then $|x| \leqslant |z|+|z-x| \leqslant |z|+r$, so that

$$|x|^a\omega(|x|) \leqslant C(|z|+r)^a\omega(|z|+r).$$

Hence we obtain

$$\begin{aligned}
⨍_{B(z,r)} \omega(|x|)\, dx &\leqslant C(|z|+r)^a\omega(|z|+r) ⨍_{B(z,r)} |x|^{-a}\, dx \\
&\leqslant C\omega(|z|+r)
\end{aligned}$$

by considering two cases (i) $r < |z|/2$ and (ii) $r > |z|/2$. □

Proof of Theorem 2. Let f be a nonnegative measurable function on $\mathbf{R}^n$ satisfying (3) with $\alpha < \nu \leqslant n$. Let

$$J_\gamma(x) = \int_{\mathbf{R}^n} \rho_\gamma(|x-y|) f(y) dy$$

and

$$p = \frac{\nu}{\nu - \alpha}.$$

As in the proof of Theorem 1 (1), we find by Lemmas 4, 6 and 7,

$$\begin{aligned}
\tilde{I}_\alpha f(x) &= \int_{B(0,1+2|x|)} ||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)|\, f(y)\, dy \\
&\quad + \int_{\mathbf{R}^n\setminus B(0,1+2|x|)} ||x-y|^{\alpha-n} - |y|^{\alpha-n}\chi_{\mathbf{R}^n\setminus\mathbf{B}}(y)|\, f(y)\, dy \\
&\leqslant \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y) dy + \int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \\
&\quad + \int_{B(0,1+2|x|)\setminus\mathbf{B}} |y|^{\alpha-n} f(y) dy + C|x| \int_{\mathbf{R}^n\setminus B(0,1+2|x|)} |y|^{\alpha-n-1} f(y) dy \\
&\leqslant C\delta^\alpha (\log(e+\delta^{-1}))^\gamma (\log(e+\delta))^\gamma J_\gamma(x) \\
&\quad + C\delta^{\alpha-\nu} (\log(e+\delta^{-1}))^{-\beta_1} (\log(e+\delta))^{-\beta_2} + C(\log(e+|x|))^{-\beta_2+1}
\end{aligned}$$

for $\delta > 0$. Considering

$$\delta = J_\gamma(x)^{-1/\nu} (\log(e+J_\gamma(x)))^{-(\gamma+\beta_1)/\nu} (\log(e+J_\gamma(x)^{-1}))^{-(\gamma+\beta_2)/\nu},$$

we see that

$$\begin{aligned}
&\tilde{I}_\alpha f(x) \\
&\leqslant CJ_\gamma(x)^{(\nu-\alpha)/\nu} (\log(e+J_\gamma(x)))^{\gamma(\nu-\alpha)/\nu - \alpha\beta_1/\nu} (\log(e+J_\gamma(x)^{-1}))^{\gamma(\nu-\alpha)/\nu - \alpha\beta_2/\nu} \\
&\quad + C(\log(e+|x|))^{-\beta_2+1} \\
&= CJ_\gamma(x)^{1/p} (\log(e+J_\gamma(x)))^{\gamma/p - \alpha\beta_1/\nu} (\log(e+J_\gamma(x)^{-1}))^{\gamma/p - \alpha\beta_2/\nu} \\
&\quad + C(\log(e+|x|))^{-\beta_2+1},
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{B(z,r)} \{\tilde{I}_\alpha f(x) (\log(e+\tilde{I}_\alpha f(x)))^{-\gamma/p+\alpha\beta_1/\nu} (\log(e+(\tilde{I}_\alpha f(x))^{-1}))^{-\gamma/p+\alpha\beta_2/\nu}\}^p dx \\
&\leqslant C \int_{B(z,r)} J_\gamma(x) dx \\
&\quad + C \int_{B(z,r)} \{(\log(e+|x|))^{-\beta_2+1} (\log(e+(\log(e+|x|))^{-\beta_2+1}))^{-\gamma/p+\alpha\beta_1/\nu} \\
&\quad (\log(e+(\log(e+|x|)^{\beta_2-1}))^{-\gamma/p+\alpha\beta_2/\nu}\}^p dx
\end{aligned}$$

for all $z \in \mathbf{R}^n$ and $r > 0$. Hence Lemmas 8 and 9 give

$$\begin{aligned}
&\fint_{B(z,r)} \{\tilde{I}_\alpha f(x)(\log(e+\tilde{I}_\alpha f(x)))^{-\gamma/p+\alpha\beta_1/\nu}(\log(e+(\tilde{I}_\alpha f(x))^{-1}))^{-\gamma/p+\alpha\beta_2/\nu}\}^p dx \\
&\leqslant Cr^{-\nu}(\log(e+r^{-1}))^{-\gamma-\beta_1+1}(\log(e+r))^{-\beta_2} \\
&\quad + C\{(\log(e+r+|z|))^{-\beta_2+1}(\log(e+(\log(e+r+|z|))^{-\beta_2+1}))^{-\gamma/p+\alpha\beta_1/\nu} \\
&\quad (\log(e+(\log(e+r+|z|))^{\beta_2-1}))^{-\gamma/p+\alpha\beta_2/\nu}\}^p
\end{aligned}$$

for all $z \in \mathbf{R}^n$ and $r > 0$, which completes the proof of Theorem 2. □

REMARK 5. In general, (4) does not hold when $\gamma = 1$.

To show this when $n = 2$, we consider

$$f(y) = f(y_1,y_2) = |y_2|^{-1}(\log(e+|y_2|^{-1}))^{-1}(\log(e+\log(e+|y_2|^{-1})))^{-\delta}$$

with $1 < \delta < 2-\alpha$. Then

$$\begin{aligned}
&\fint_{B(x,r)} |f(y)|dy \\
&\leqslant \frac{C}{r}\int_0^r |y_2|^{-1}(\log(e+|y_2|^{-1}))^{-1}(\log(e+\log(e+|y_2|^{-1})))^{-\delta}dy_2 \leqslant Cr^{-1}
\end{aligned}$$

for $x \in \mathbf{B}$. For $0 < \alpha < 1$, consider the potential

$$\tilde{I}_\alpha f(x) = I_\alpha f(x) = \int_{\mathbf{B}} |x-y|^{\alpha-2} f(y)dy.$$

Here we may assume that $x_2 \neq 0$. Setting

$$Q(x) = \{y = (y_1,y_2) \in \mathbf{B} : |x_1 - y_1| < |x_2|, |y_2| < |x_2|\},$$

we note that

$$\begin{aligned}
\tilde{I}_\alpha f(x) &\geqslant \int_{Q(x)} |x-y|^{\alpha-2} f(y)dy \\
&\geqslant C|x_2|^{\alpha-2}\int_{Q(x)} f(y)dy \\
&\geqslant C|x_2|^{\alpha-1}\int_0^{|x_2|} |y_2|^{-1}(\log(e+|y_2|^{-1}))^{-1}(\log(e+\log(e+|y_2|^{-1})))^{-\delta}dy_2 \\
&\geqslant C|x_2|^{\alpha-1}(\log(e+\log(e+|x_2|^{-1})))^{-\delta+1},
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{B(0,1)} (\tilde{I}_\alpha f(x))^p(\log(e+\tilde{I}_\alpha f(x))^{-1}(\log(e+(\tilde{I}_\alpha f(x))^{-1})^{-1}dx \\
&\geqslant C\int_{B(0,1)} |x_2|^{-1}(\log(e+\log(e+|x_2|^{-1})))^{(-\delta+1)p}(\log(e+|x_2|^{-1}))^{-1}dx \\
&= \infty
\end{aligned}$$

when $p = 1/(1-\alpha), \nu = 1$ and $1 < \delta < 2-\alpha$.

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