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WIENER TYPE THEOREMS FOR FOURIER–VILENIKIN SERIES WITH NONNEGATIVE COEFFICIENTS AND SOLID SPACES

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Abstract. In the present paper we show that L^p-integrability near zero of a function on Vilenkin group G with nonnegative Fourier-Vilenkin coefficients implies L^p-integrability on G, if p is even integer. This is an analog of N. Wiener-S. Wainger result. A refinement of Hausdorff-Young-F. Riesz inequality is obtained and several examples concerning embeddings of solid function spaces on G are given.

1. Introduction

Let $\{\varphi_k\}_{k=0}^{\infty}$ be an arbitrary orthonormal system on $[a,b]$ such that $|\varphi_n(x)| \leqslant M$ for all $n \in \mathbb{Z}_+$ and $x \in [a,b]$. Since $L^p[a,b] \subset L^1[a,b]$ for $p \geqslant 1$, we define the n-th Fourier coefficient of a function $f \in L^p[a,b]$, $1 \leqslant p \leqslant \infty$, by

$$(f,\varphi_k) = (b-a)^{-1}\int_a^b f(t)\overline{\varphi_k(t)}\,dt.$$

The following theorems are well known (see [3, Chap. II, §3] and [19, Chap. XII, (2.3) and (3.19)]).

THEOREM A. (F. Hausdorff – W. H. Young – F. Riesz) *Let* $1 < p < \infty$ *and* $1/p + 1/p' = 1$.

(i) *If* $1 < p \leqslant 2$ *and* $f \in L^p[a,b]$, *then*

$$\left(\sum_{n=0}^{\infty} |(f,\varphi_n)|^{p'}\right)^{1/p'} \leqslant M^{2/p-1}\|f\|_{L^p[a,b]}.$$

(ii) *If* $2 \leqslant p < \infty$ *and* $\{c_n\}_{n=0}^{\infty}$ *is a sequence in* $\mathbb{C}$ *such that* $\sum_{n=0}^{\infty} |c_n|^{p'} < \infty$, *then there exists* $f \in L^p[a,b]$ *such that* $(f,\varphi_k) = c_k$, $k \in \mathbb{Z}_+$, *and*

$$\|f\|_{L^p[a,b]} \leqslant M^{2/p-1}\left(\sum_{n=0}^{\infty} |(f,\varphi_n)|^{p'}\right)^{1/p'}.$$

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THEOREM B. (G. Hardy – J. E. Littlewood – R. Paley) (i) *If* $1 < p \leqslant 2$ *and* $f \in L^p[a,b]$, *then*

$$\left(\sum_{n=0}^{\infty} |(f,\varphi_n)|^p (n+1)^{p-2}\right)^{1/p} \leqslant C\|f\|_{L^p[a,b]}.$$

(ii) *If* $2 \leqslant p < \infty$ *and* $\{c_n\}_{n=0}^{\infty}$ *is a sequence in* $\mathbb{C}$ *such that* $\sum\limits_{n=0}^{\infty} |c_n|^p (n+1)^{p-2} < \infty$, *then there exists* $f \in L^p[a,b]$ *such that* $(f,\varphi_k) = c_k$, $k \in \mathbb{Z}_+$, *and*

$$\|f\|_{L^p[a,b]} \leqslant C\left(\sum_{n=0}^{\infty} |(f,\varphi_n)|^p (n+1)^{p-2}\right)^{1/p}.$$

Let $\mathbb{T}$ be the interval $[-\pi,\pi]$. In this section we denote the complex trigonometric Fourier coefficients of $f \in L^1(\mathbb{T})$ by $\hat{f}(n)$. For $p > 1$ let

$$L^p_{loc+}(\mathbb{T}) = \left\{f \in L^1(\mathbb{T}) : \hat{f}(n) \geqslant 0,\ n \in \mathbb{Z}, \text{ and } \int_{-\delta}^{\delta} |f(x)|^p\,dx < \infty \quad \text{for some} \quad \delta > 0\right\}.$$

N. Wiener proved a theorem concerning power series with positive coefficients (see [5] or [17]) which is equivalent to the following proposition.

THEOREM C. *If* $f \in L^2_{loc+}(\mathbb{T})$, *then* $f \in L^2(\mathbb{T})$.

S. Wainger [17] noted that Theorem C is also valid for $p = 2n$, $n \in \mathbb{N}$, but it is false for all $1 < p < 2$. H. S. Shapiro [13] established a general result providing that for $p > 2$, $p \neq 2n$, $n \in \mathbb{N}$, the property $f \in L^p_{loc+}(\mathbb{T})$ does not imply $f \in L^p(\mathbb{T})$. J. M. Ash, M. Rains and S. Vagi [1] obtained an extension of Theorem A (i) for the case of trigonometric system.

THEOREM D. *If* $1 < p \leqslant 2$ *and* $f \in L^p_{loc+}(\mathbb{T})$, *then* $\sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)|^{p'} < \infty$.

Recently, J. M. Ash, S. Tikhonov and J. Tung [2] considered so called solid spaces $X(\mathbb{T})$ with the following property: if $f \in X(\mathbb{T})$ have Fourier series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ and $|d_n| \leqslant |c_n|$ for all $n \in \mathbb{Z}_+$, then $g(x)$ with Fourier series $\sum_{n\in\mathbb{Z}} d_n e^{inx}$ belongs to $X(\mathbb{T})$. Earlier, a similar property with condition $|d_n| \leqslant c_n$ was called the upper majorant property (see [4] or [8]). G.Bachelis [4] gave many conditions equivalent to the upper majorant property. The main positive result in [2] is

THEOREM E. *If* $L^p(\mathbb{T}) \subset X(\mathbb{T})$, $1 < p < \infty$, *and* $X(\mathbb{T})$ *is a solid space, then* $L^p_{loc+}(\mathbb{T}) \subset X(\mathbb{T})$.

Theorem E is a generalization of Theorem D. Also, [2] contains several counterexamples concerning embeddings between $L^p_{loc+}(\mathbb{T})$, $L^p(\mathbb{T})$, $l^{p'}(\mathbb{T}) := \{f \in L^1(\mathbb{T}) : \sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)|^{p'} < \infty\}$ and $HL^p(\mathbb{T}) := \{f \in L^1(\mathbb{T}) : \sum\limits_{n\in\mathbb{Z}} |\hat{f}(n)|^p (|n|+1)^{p-2} < \infty\}$. The aim of the present paper is to obtain results similar to Theorems C-E and to refine Theorems A and B for the character system of Vilenkin group of bounded type.

2. Definitions

Let $\mathbf{P} = \{p_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ and $2 \leqslant p_n \leqslant N$ for all $n \in \mathbb{N}$, $m_0 := 1$, $m_n = p_1 \dots p_n$ for $n \in \mathbb{N}$. For all $k \in \mathbb{N}$ we denote by $\mathbb{Z}(p_k)$ the discrete cyclic group of order p_k with elements $0,1,\dots,p_k-1$. The complete direct product of $\mathbb{Z}(p_k)$ will be denoted by $G = G(\mathbf{P})$, where G is a compact abelian group with addition $+$ and with Haar measure dx ($|G|$, the measure of G, is equal to 1.) Elements of G has the form $x = (x_1,x_2,\dots)$, where $x_i \in \mathbb{Z}(p_i)$, $i \in \mathbb{N}$. If $G_n = \{x \in G : x_1 = x_2 = \dots = x_n = 0\}$, then G_n is a subgroup of $G = G_0$ whose Haar measure is equal to $1/m_n$. It is well known that $\{G_n\}_{n=0}^{\infty}$ is a base of topology in G and that characters of G form a complete orthonormal system $\widehat{G} = \{\chi_n(x)\}_{n\in\mathbb{Z}_+}$ in $L^1(G)$. Moreover, if $r_k(x) := \exp(2\pi i x_k/p_k)$ for $x \in G$ and

$$n = \sum_{k=1}^{\infty} n_k m_{k-1}, \quad n_k \in \mathbb{Z}(p_k), \quad k \in \mathbb{N}, \tag{1}$$

then $\chi_n(x) = \prod_{k=1}^{\infty} r_k^{n_k}(x)$. Let us note that $\chi_0(x) \equiv 1$ on G. If $f \in L^1(\mathbb{G})$, then its Fourier coefficients and partial Fourier sums are

$$\hat{f}(k) = \int_G f(x)\overline{\chi_k(x)}\,dx, \quad k \in \mathbb{Z}_+, \quad S_n(f)(x) = \sum_{k=0}^{n-1} \hat{f}(k)\chi_k(x), \quad n \in \mathbb{N}. \tag{2}$$

Further we use notation $\hat{f}(k)$ in the sense of (2). Let $D_n := \sum_{k=0}^{n-1} \chi_k$, $n \in \mathbb{N}$, and $\Delta_k(f) := \sum_{i=m_{k-1}}^{m_k-1} \hat{f}(i)\chi_i$, $k \in \mathbb{N}$, $\Delta_0(f) := \hat{f}(0)$. It is well known that $D_{m_n} = m_n X_{G_n}$, where X_E is the indicator of E (see [7, §1.5]). If $n, r \in \mathbb{Z}_+$ are represented in the form (1), then by definition

$$n \oplus r = s = \sum_{k=1}^{\infty} s_k m_{k-1}, \quad s_k \in \mathbb{Z}(p_k), \quad s_k = n_k + r_k \pmod{p_k}.$$

The set $\mathbb{Z}_+$ with operation $\oplus$ is a group. The inverse operation $\ominus$ is defined similarly. It is easy to see that $\chi_n\chi_r = \chi_{n\oplus r}$, $\chi_n\overline{\chi_r} = \chi_{n\ominus r}$ for all $n, r \in \mathbb{Z}_+$. Since χ_n are characters of G, we obtain $\chi_n(x+y) = \chi_n(x)\chi_n(y)$ and $\chi_n(x-y) = \chi_n(x)\overline{\chi_n(y)}$ for all $n \in \mathbb{Z}_+$, $x, y \in G$. If $f \in L^1(G)$ and $g = \sum_{i=0}^{n-1} a_i\chi_i$ is a polynomial with respect to the system $\{\chi_i\}_{i=0}^{\infty}$, then

$$\widehat{fg}(k) = \sum_{i=0}^{n-1} a_i \int_G f(x)\overline{\chi_k(x)}\chi_i(x)\,dx = \sum_{i=0}^{n-1} a_i\hat{f}(k \ominus i) = \sum_{i\oplus j=k} a_i\hat{f}(j), \tag{3}$$

where $a_i = 0$ for $i \geqslant n$. Main properties of $\{\chi_i\}_{i=0}^{\infty}$ may be found in [14] and [7, §1.5].

A pseudomeasure on G is a linear continuous functional on $A(G) = \{f \in L^1(G) : \sum_{i=0}^{\infty} |\hat{f}(i)| < \infty\}$. If F is a pseudomeasure ($F \in PM(G)$), then it may be considered as a formal series $\sum\limits_{i=0}^{\infty} a_i\chi_i$, such that $\sup\limits_{i\in\mathbb{Z}_+} |a_i| < \infty$ and for $g \in A(G)$ we have $F(g) = \sum\limits_{i=0}^{\infty} a_i\overline{\hat{g}(i)}$. In particular, $\hat{F}(n) = F(\overline{\chi_n}) = a_n$, $n \in \mathbb{Z}_+$.

The space $L^p(G)$ consists of measurable complex-valued functions f with finite norm $\|f\|_p = (\int_G |f(x)|^p\,dx)^{1/p}$ for $1 \leqslant p < \infty$ and $\|f\|_\infty = ess\sup\{|f(x)| : x \in G\}$. If $f \in L^p(G_n) \cap L^1(G)$ for some $n \in \mathbb{N}$ and $\hat{f}(k) \geqslant 0$ for all $k \in \mathbb{Z}_+$, then we write $f \in L^p_{loc+}(G)$. Let $\mathscr{P}_n = \{f \in L^1(G) : \hat{f}(k) = 0, k \geqslant n\}$ and $E_n(f)_p = \inf\{\|f - t_n\|_p : t_n \in \mathscr{P}_n\}$, $n \in \mathbb{N}$. If $1 \leqslant p, r \leqslant \infty$, $\alpha > 0$, then the Besov space $B^\alpha_{p,r}$ consists of all $f \in L^p[0,1)$ with the property

$$\|f\|_{B^\alpha_{pr}} = \|f\|_p + \left(\sum_{k=0}^{\infty} m_k^{\alpha r} E^r_{m_k}(f)_p\right)^{1/r} < \infty$$

for $1 \leqslant r < \infty$ and $\|f\|_{B^\alpha_{p,\infty}} = \|f\|_p + \sup\limits_{k\in\mathbb{Z}_+} m_k^\alpha E_{m_k}(f)_p < \infty$ for $r = \infty$. We say $f \in (B^\alpha_{p,r})_{loc}$, if $fD_{m_n} \in B^\alpha_{p,r}$ for some $n \in \mathbb{N}$.

If the values A and B depend on a parameter, then the expression $A \asymp B$ means that $C_1 A \leqslant B \leqslant C_2 A$ for some constants $C_2 \geqslant C_1 > 0$ which are not depended on the parameter.

3. Auxiliary propositions

The following counterpart of famous Littlewood-Paley theorem [19, Chap. XV, (2.1)] is valid only for bounded $\{p_i\}_{i=1}^\infty$ and is due to C.Watari [18].

LEMMA 1. (i) *Let* $f \in L^p(G)$, $1 < p < \infty$, *and* $Q(f) = \left(\sum\limits_{k=0}^{\infty} |\Delta_k(f)|^2\right)^{1/2}$. *Then* $\|f\|_p \asymp \|Q(f)\|_p$.

(ii) *If for* $p \in (1,\infty)$ *and the series* $\sum\limits_{i=0}^{\infty} a_i\chi_i$ *the inequality*

$$I_p := \left\| \left(\sum_{k=1}^{\infty} \left| \sum_{i=m_{k-1}}^{m_k - 1} a_i\chi_i \right|^2 + |a_0|^2 \right)^{1/2} \right\|_p < \infty$$

holds, then there exists $f \in L^p(G)$ *with* $\hat{f}(k) = a_k$, $k \in \mathbb{Z}_+$, *and* $\|f\|_p \leqslant CI_p$.

Lemma 2 is a variant of Minkowski inequality (see [6]).

LEMMA 2. *Let* $\mathbf{g} = \{g_k\}_{k=1}^\infty$, *where* $g_k \in L^p(G)$, $k \in \mathbb{N}$, $1 \leqslant p, q < \infty$ *and*

$$\|\mathbf{g}\|_{L^p(l^q)} = \left\| \left(\sum_{k=1}^{\infty} |g_k|^q\right)^{1/q} \right\|_p, \quad \|\mathbf{g}\|_{l^q(L^p)} = \left(\sum_{k=1}^{\infty} \|g_k\|_p^q\right)^{1/q}.$$

Then the inequality $\|\mathbf{g}\|_{L^p(l^2)} \geqslant \|\mathbf{g}\|_{l^2(L^p)}$ *is valid for* $1 < p \leqslant 2$. *If* $p \geqslant 2$, *then we have* $\|\mathbf{g}\|_{L^p(l^2)} \leqslant \|\mathbf{g}\|_{l^2(L^p)}$.

Lemma 3 is the particular case of Theorem 9 in [15].

LEMMA 3. *Let* $1 < p < \infty$, $\{a_n\}_{n=0}^{\infty} \downarrow 0$. *Then* $S(x) = \sum_{n=0}^{\infty} a_n \chi_n(x)$ *belongs to* $L^p(G)$ *if and only if* $\sum_{n=0}^{\infty} a_n^p (n+1)^{p-2} < \infty$.

Lemma 4 follows from lemma 1.

LEMMA 4. *Let* $1 < p < \infty$, $\{a_i\}_{i=0}^{\infty} \subset \mathbb{C}$. *Then* $S(x) = \sum_{i=0}^{\infty} a_i \chi_{m_i}(x)$ *belongs to* $L^p(G)$ *if and only if* $\sum_{i=0}^{\infty} |a_i|^2 < \infty$.

Proof. If $S \in L^p(G)$, then $|\Delta_k(S)| = |a_{k-1}|$, $k \in \mathbb{N}$, and $\Delta_0(S) = 0$. By Lemma 1 (i) we obtain $\sum_{i=0}^{\infty} |a_i|^2 < \infty$. The converse assertion easily follows from Lemma 1 (ii). □

Lemma 5 is known (see, for example, Lemma 11 in [16]).

LEMMA 5. *Let* $1 \leqslant p, r \leqslant \infty$, $\alpha > 0$. *The norms* $\|f\|_{B_{pr}^{\alpha}}$ *and*

$$\|f\|_{B_{pr}^{\alpha}}^{(1)} = \|f\|_p + \left(\sum_{k=0}^{\infty} m_k^{\alpha r} \|S_{m_{k+1}}(f) - S_{m_k}(f)\|_p \right)^{1/r} \quad \textit{for} \quad 1 \leqslant r < \infty$$

(or $\|f\|_{B_{p,\infty}^{\alpha}}^{(1)} = \|f\|_p + \sup_{k \in \mathbb{Z}_+} m_k^{\alpha} \|S_{m_{k+1}}(f) - S_{m_k}(f)\|_p$ *for* $r = \infty$*) are equivalent.*

4. Main results

Theorem 1 is an analogue of C. N. Kellogg's result [10], who used in its proof H^p-multiplier results of J. H. Hedlund [9].

THEOREM 1. (i) *If* $1 < p \leqslant 2$, $f \in L^p(G)$, $1/p + 1/p' = 1$, *then*

$$\left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k - 1} |\hat{f}(i)|^{p'} \right)^{2/p'} \right)^{1/2} \leqslant C_1(p) \|f\|_p.$$

(ii) *If* $p \geqslant 2$, $\mathbf{a} = \{a_i\}_{i=0}^{\infty} \subset \mathbb{C}$ *and*

$$\|\mathbf{a}\|_{p',2} := \left(|a_0|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k - 1} |a_i|^{p'} \right)^{2/p'} \right)^{1/2} < \infty,$$

then there exists $f \in L^p(G)$ *with* $\hat{f}(k) = a_k$, $k \in \mathbb{Z}_+$, *and* $\|f\|_p \leqslant C_2(p) \|\mathbf{a}\|_{p',2}$.

Both inequalities of Theorem 1 are sharpenings of corresponding inequalities from Theorem A for the case of Fourier-Vilenkin systems of bounded type.

Proof. (i) Using Lemmas 1 (i), 2 and Theorem A(i) for $\{\chi_i\}_{i=0}^{\infty}$, we obtain

$$\|f\|_p \geqslant C_1 \left\| \left(\sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_p \geqslant C_1 \left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \|\Delta_k(f)\|_p^2 \right)^{1/2}$$
$$\geqslant C_1 \left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k-1} |\hat{f}(i)|^{p'} \right)^{2/p'} \right)^{1/2}.$$

Applying inequality $p' \geqslant 2$ and Jensen inequality, we see that

$$\left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k-1} |\hat{f}(i)|^{p'} \right)^{2/p'} \right)^{1/2} \geqslant \left(|\hat{f}(0)|^{p'} + \sum_{k=1}^{\infty} \sum_{i=m_{k-1}}^{m_k-1} |\hat{f}(i)|^{p'} \right)^{1/p'}$$
$$= \left(\sum_{k=0}^{\infty} |\hat{f}(i)|^{p'} \right)^{1/p'}$$

and the statement of Theorem 1 (i) is a sharpening of Theorem A (i) for the case of Fourier-Vilenkin systems of bounded type.

(ii) Due to Lemmas 1 (ii), 2 and Theorem A (ii) we have

$$\|f\|_p \leqslant C_2 \left\| \left(\sum_{k=0}^{\infty} |\Delta_k(f)|^2 \right)^{1/2} \right\|_p \leqslant C_2 \left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \|\Delta_k(f)\|_p^2 \right)^{1/2}$$
$$\leqslant C_2 \left[|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k-1} |\hat{f}(i)|^{p'} \right)^{2/p'} \right]^{1/2}.$$

Using inequality $p' \leqslant 2$ and Jensen inequality we conclude that

$$\left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty} \left(\sum_{i=m_{k-1}}^{m_k-1} |\hat{f}(i)|^{p'} \right)^{2/p'} \right)^{1/2} \leqslant \left(|\hat{f}(0)|^{p'} + \sum_{i=1}^{\infty} |\hat{f}(i)|^{p'} \right)^{1/p'}$$

and the statement of Theorem 1 (ii) is a sharpening of Theorem A (ii) for the case of Fourier-Vilenkin systems of bounded type. Theorem is proved. □

Using Theorem B for the system $\{\chi_i\}_{i=0}^{\infty}$ we can similarly establish the following theorem.

THEOREM 2. (i) *If* $1 < p \leqslant 2,\ f \in L^p(G)$, *then*

$$\left(|\hat{f}(0)|^2 + \sum_{k=1}^{\infty}\left(\sum_{i=m_{k-1}}^{m_k-1}|\hat{f}(i)|^p i^{p-2}\right)^{2/p}\right)^{1/2} \leqslant C_1(p)\|f\|_p.$$

(ii) *If* $p \geqslant 2,\ \mathbf{a} = \{a_i\}_{i=0}^{\infty} \subset \mathbb{C}$ *and*

$$\|\mathbf{a}\|_{hl^{p,2}} := \left(|a_0|^2 + \sum_{k=1}^{\infty}\left(\sum_{i=m_{k-1}}^{m_k-1}|a_i|^p i^{p-2}\right)^{2/p}\right)^{1/2} < \infty,$$

then there exists $f \in L^p(G)$ *with* $\hat{f}(k) = a_k,\ k \in \mathbb{Z}_+$, *and* $\|f\|_p \leqslant C_2(p)\|\mathbf{a}\|_{hl^{p,2}}$.

Theorem 3 is an analogue of Theorem E for Vilenkin groups. The notion of solid space $X(G)$ is introduced similarly to that of $X(\mathbb{T})$.

THEOREM 3. *If* $1 < p < \infty,\ L^p(G) \subset X(G)$ *and* $X(G)$ *is a solid space, then* $L^p_{loc+}(G) \subset X(G)$.

Proof. Let $f \in L^p_{loc+}(G)$, then $fD_{m_n} = m_n f X_{G_n} \in L^p(G)$ for some $n \in \mathbb{N}$. Since $D_{m_n} = \sum_{i=0}^{m_n-1}\chi_i$ and $\widehat{D_{m_n}}(0) = 1$, by (3) we have

$$\widehat{fD_{m_n}}(k) = \sum_{j=0}^{m_n-1}\hat{f}(k \ominus j)\hat{D}_{m_n}(j) \geqslant \hat{f}(k) \geqslant 0, \quad k \in \mathbb{Z}_+. \tag{4}$$

Thus, $fD_{m_n} \in X(G)$ and $f \in X(G)$, since $X(G)$ is a solid space. Theorem is proved. □

DEFINITION 1. If $f \in L^1(G)$, $1 < p < \infty$, and for $\hat{\mathbf{f}} = \{\hat{f}(i)\}_{i=0}^{\infty}$ we have $\|\hat{\mathbf{f}}\|_{p,2} < \infty$ ($\|\hat{\mathbf{f}}\|_{hl^{p,2}} < \infty$), then it is said that f belongs to $l^{p,2}(G)$ ($HL^{p,2}(G)$). Similarly, if $\|\hat{\mathbf{f}}\|_p := \left(\sum_{i=0}^{\infty}|\hat{f}(i)|^p\right)^{1/p} < \infty$ ($\|\hat{\mathbf{f}}\|_{hl^p} := \left(\sum_{i=0}^{\infty}|\hat{f}(i)|^p(i+1)^{p-2}\right)^{1/p} < \infty$), then it is said that f belongs to $l^p(G)$ ($HL^p(G)$). If it is necessary, the condition $f \in L^1(G)$ may be substituted by $f \in PM(G)$.

COROLLARY 1. *If* $1 < p \leqslant 2$, *then* $L^p_{loc+}(G) \subset l^{p',2}(G) \subset l^{p'}(G)$ *and* $L^p_{loc+}(G) \subset HL^{p,2}(G) \subset HL^p(G)$.

Proof. It is clear that $l^{p',2}(G)$, $HL^{p,2}(G)$, $l^{p'}(G)$, $HL^p(G)$ are solid spaces. By Theorems 1 and 2 we see that $L^p(G)$, $1 < p \leqslant 2$, is contained in all these spaces. Right embeddings are also established in the proof of Theorem 1 and Theorem 2. In virtue of Theorem 3 we obtain the assertion of Corollary 1.

REMARK 1. A dual proposition to Theorem 3 (if $1 < p < \infty$, $X(G) \subset L^p(G)$, then $X_+(G) = \{f \in X(G) : \hat{f}(i) \geqslant 0, i \in \mathbb{Z}_+\} \subset L^p_{loc+}(G)$) is trivial.

Theorem 4 extends Theorem C and Wainger's positive statement from [17] onto Vilenkin groups.

THEOREM 4. *If $p=2m$, $m\in\mathbb{N}$, then $L^p_{loc+}(G)\subset L^p(G)$.*

Proof. Let $p=2m$, $m\in\mathbb{N}$, and $f\in L^2_{loc+}(G)$. From (4) and Parseval identity it follows that

$$m_n^2\int_{G_n}|f(x)|^2\,dx=\|fD_{m_n}\|_2^2=\sum_{k=0}^{\infty}|\widehat{fD_{m_n}}(k)|^2\geqslant\sum_{k=0}^{\infty}|\hat f(k)|^2=\|f\|_2^2,$$

if $f\in L^2(G_n)$. Thus, we obtain inequality $\|f\|_2\leqslant|G_n|^{-1}\left(\int_{G_n}|f(x)|^2\,dx\right)^{1/2}$, where $|G_n|$ is the Haar measure of G_n. If $p=2m$, $m\in\mathbb{N}$, then $f\in L^p_{loc+}(G)$ implies $f^m\in L^2_{loc+}(G)$. Indeed, the condition $f\in L^{2m}(G_n)$, $n\in\mathbb{Z}_+$, means that $|f^m|^2\in L^1(G_n)$. For arbitrary $g,h\in L^2(G)$ and $S_j(g)=\sum\limits_{i=0}^{j-1}\hat g(i)\chi_i$ we obtain by Cauchy-Bunyakovsky-Schwarz inequality and (3) that $gh\in L^1(G)$, $\lim\limits_{j\to\infty}\|h(g-S_j(g))\|_1=0$ and

$$\widehat{hg}(k)=\lim_{j\to\infty}\widehat{S_j(g)h}(k)=\lim_{j\to\infty}\sum_{i=0}^{j-1}\hat g(i)\hat h(k\ominus i)=\sum_{i=0}^{\infty}\hat g(i)\hat h(k\ominus i).$$

Hence, if $\hat f(k)\geqslant 0$, $k\in\mathbb{Z}_+$, then f^i also have non-negative Fourier-Vilenkin coefficients for all $i\in\mathbb{N}$ and $f^m\in L^2_{loc+}(G)$. By just proved assertion we find that $f^m\in L^2(G)$ and $f\in L^{2m}(G)=L^p(G)$. Theorem is proved. □

Theorems 5 and 6 are close to Theorems 4 and 7 from [2], where the spaces $HL^p(\mathbb{T})$ and $l^{p'}(\mathbb{T})$ are considered.

THEOREM 5. (i) *There exist $f_0\in HL^{p,2}(G)\setminus l^{p',2}(G)$ for $1<p<2$ and $g_0\in l^{p',2}(G)\setminus HL^{p,2}(G)$ for $2<p<\infty$.*
(ii) *For $2<p<\infty$ there exists $f\in L^p_{loc+}(G)\setminus(HL^{p,2}(G)\cup l^{p'}(G))$.*

Proof. (i) Let us consider $f_\alpha=\sum\limits_{i=1}^{\infty}i^{-\alpha}\chi_{m_i}$ with $\alpha>1/2$. Then by Lemma 4 we have $f_\alpha\in L^p(G)$ for all $1<p<\infty$. Since

$$\left(\sum_{i=m_{k-1}}^{m_k-1}|\hat f_\alpha(i)|^{p'}\right)^{2/p'}=(k-1)^{-2\alpha},\quad k\geqslant 2,$$

and

$$\left(\sum_{i=m_{k-1}}^{m_k-1}|\hat f_\alpha(i)|^p i^{p-2}\right)^{2/p}=(k-1)^{-2\alpha}m_{k-1}^{2-4/p},\quad k\geqslant 2,$$

we find that $f_\alpha\in l^{p',2}(G)\setminus HL^{p,2}(G)$ for $p>2$ and $\alpha>1/2$. If $0<\alpha<1/2$, then we can consider f_α as pseudomeasure (see Definition 1) and for $1<p<2$ we obtain that $f_\alpha\in HL^{p,2}(G)\setminus l^{p',2}(G)$.

(ii) Using f_α from (i) with $\alpha \in (1/2, 1/p')$ again, we see that $f_\alpha \in L^p_{loc+}(G)$ by Lemma 4, $f_\alpha \notin HL^{p,2}(G)$ by (i) and

$$\sum_{i=m_1}^{\infty} |\hat{f}_\alpha(i)|^{p'} = \sum_{k=1}^{\infty} k^{-\alpha p'} = \infty.$$

Theorem is proved. □

THEOREM 6. *For $1 < p < 2$ there exists $f_0 \in l^{p',2}(G) \setminus L^p(G)$. Also, we have $f_0 \in l^{p',2}(G) \setminus HL^p(G)$.*

Proof. Let us consider $f_\alpha = \sum_{i=2}^{\infty} i^{-1/p'}(\ln i)^{-\alpha}\chi_i$, $\alpha > 0$. Since $i^{-1/p'}\ln^{-\alpha} i \downarrow 0$, then by Lemma 3 we have $f_\alpha \in L^p(G)$ if and only if $\sum_{i=2}^{\infty} i^{1-p}(\ln i)^{-\alpha p} i^{p-2} < \infty$. Hence, for $\alpha = 1/p$ we have $f_{1/p} \notin L^p(G)$ and $f_{1/p} \notin HL^p(G)$, while $f_{1/p} \in L^r(G)$, $r \in (1,p)$. Since for $k \geqslant 2$

$$\begin{aligned}\left(\sum_{i=m_{k-1}}^{m_k-1} |\hat{f}_{1/p}(i)|^{p'}\right)^{2/p'} &\leqslant \left(\sum_{i=m_{k-1}}^{m_k-1} i^{-1}(\ln i)^{-p'/p}\right)^{2/p'} \\ &\leqslant C_1\left(k^{-p'/p}\sum_{i=m_{k-1}}^{m_k-1} i^{-1}\right)^{2/p'} \leqslant C_2 k^{-2/p}, \qquad (5)\end{aligned}$$

we see that $f_{1/p} \in l^{p',2}(G)$. In the last inequality of (5) we use the boundedness of $\{p_i\}_{i=1}^{\infty}$. Theorem is proved. □

REMARK 2. The question of validity for Vilenkin groups of S. Wainger and H. S. Shapiro results noted after Theorem C in Introduction remains open.

Now we give an analogue of Paley theorem [12], concerning the case $p = \infty$.

THEOREM 7. *If $f \in L^\infty_{loc+}$, then its Fourier series $\sum_{k=0}^{\infty} \hat{f}(k)\chi_k$ converges absolutely and uniformly on G.*

Proof. Under condition of Theorem 7 we have $fD_{m_n} \in L^\infty(G)$ for some $n \in \mathbb{N}$. Using (4), we obtain for arbitrary $j \in \mathbb{N}$

$$\sum_{k=0}^{m_j-1} \hat{f}(k) \leqslant \sum_{k=0}^{m_j-1} \widehat{fD_{m_n}}(k) = S_{m_j}(fD_{m_n})(0) = \|S_{m_j}(fD_{m_n})\|_\infty \leqslant \|fD_{m_n}\|_\infty < \infty.$$

Since n is independent of j and $\hat{f}(k) \geqslant 0$, we conclude that $\sum_{k=0}^{\infty} |\hat{f}(k)| < \infty$. Here we use the following properties: 1) for $a_k \geqslant 0$ and $x \in G$

$$\left|\sum_{k=0}^{n-1} a_k\chi_k(x)\right| \leqslant \sum_{k=0}^{n-1} a_k\chi_k(0) = \sum_{k=0}^{n-1} a_k \quad \text{and} \quad \left\|\sum_{k=0}^{n-1} a_k\chi_k(x)\right\|_\infty = \sum_{k=0}^{n-1} a_k;$$

and 2) $S_{m_j}(f)(x) = m_j \int_{x+G_j} f(t)\,dt$ for $x \in G$ (see [14]). The property 2) gives inequality $\|S_{m_n}(f)\|_\infty \leqslant \|f\|_\infty$. Theorem is proved. $\square$

The last theorem is an analogue of Theorem 3.6 from [11], where the case $p = \infty$ is studied.

THEOREM 8. *Let $1 \leqslant r < \infty$, $\alpha > 0$, $p = 2$ or $p = \infty$ (correspondingly, $p' = 2$ or $p' = 1$) and $f \in L^p(G)$ such that $\hat{f}(k) \geqslant 0$, $k \in \mathbb{Z}_+$. The following three statements are equivalent.*

(a) $f \in (B^\alpha_{pr})_{loc}$;

(b) $f \in B^\alpha_{pr}$;

(c) $\left(\sum_{j=0}^{\infty} m_j^{\alpha r} \left(\sum_{k=m_j}^{m_{j+1}-1} (\hat{f}(k))^{p'} \right)^{r/p'} \right)^{1/r} < \infty.$

Proof. Since $\hat{f}(k) \geqslant 0$, we obtain $\|S_{m_{j+1}}(f) - S_{m_j}(f)\|_\infty = \sum_{k=m_j}^{m_{j+1}-1} \hat{f}(k)$ and $\|S_{m_{j+1}}(f) - S_{m_j}(f)\|_2 = \left(\sum_{k=m_j}^{m_{j+1}-1} (\hat{f}(k))^2 \right)^{1/2}$. Therefore, equivalence of (b) and (c) follows from Lemma 5. It is clear that (b) implies (a). Let (a) holds for $p = \infty$ and $fD_{m_n} \in B^\alpha_{\infty,r}$ for some $n \in \mathbb{N}$. In virtue of (4) we have

$$E_{m_j}(f)_\infty = \sum_{k=m_j}^{\infty} \hat{f}(k) \leqslant \sum_{k=m_j}^{\infty} \widehat{fD_{m_n}}(k) = E_{m_j}(fD_{m_n})_\infty$$

and $\|f\|_{B^\alpha_{\infty,r}} \leqslant \|fD_{m_n}\|_{B^\alpha_{\infty,r}}$. For $p = 2$ similarly we obtain

$$E_{m_j}(f)_2 = \left(\sum_{k=m_j}^{\infty} (\hat{f}(k))^2 \right)^{1/2} \leqslant \left(\sum_{k=m_j}^{\infty} (\widehat{fD_{m_n}}(k))^2 \right)^{1/2} = E_{m_j}(fD_{m_n})_2$$

and $\|f\|_{B^\alpha_{2,r}} \leqslant \|fD_{m_n}\|_{B^\alpha_{2,r}}$. Thus, (a) implies (b). Theorem is proved. $\square$

REFERENCES

[1] J. M. ASH, M. RAINS AND S. VAGI, *Fourier series with positive coefficients*, Proc. Amer. Math. Soc., **101**, 2 (1987), 392–393.

[2] J. M. ASH, S. TIKHONOV AND S. J. TUNG, *Wiener's positive Fourier coefficients in variants of L^p spaces*, Michigan Math. J., **59**, 1 (2010), 143–151.

[3] N. K. BARI, *Trigonometric series*, Fizmatgiz, Moskva, 1961 (in Russian). English transl.: N. Bary, *A treatise on trigonometric series. V 1,2.*, Pergamon Press, New York, 1964.

[4] G. F. BACHELIS, *On the upper and lower majorant properties in $L^p(G)$*, Quart. J. Math. Oxford ser. (2), **24**, 1 (1973), 119–128.

[5] R. P. BOAS, *Entire functions*, Academic Press, New York, 1964.

[6] S. FRIDLI, *On the rate of convergence of Cesaro means of Walsh-Fourier series*, J. Approxim. Theory, **76**, 1 (1994), 31–53.

[7] B. I. GOLUBOV, A. V. EFIMOV AND V. A. SKVORTSOV, *Walsh series and transforms*, Kluwer, Dordrecht, 1991.
[8] G. H. HARDY, J. E. LITTLEWOOD, *Notes on the theory of series (XIX): A problem concerning majorant of Fourier series*, Quart. J. Math. Oxford ser., **6** (1935), 304–315.
[9] J. H. HEDLUND, *Multipliers of H^p spaces*, J. Math. Mech., **18**, 11 (1969), 1067–1074.
[10] C. N. KELLOGG, *An extension of the Hausdorff-Young theorem*, Michigan Math. J., **18**, 2 (1971), 121–128.
[11] H. N. MAZHAR AND S. TIKHONOV, *Wiener-type theorems for Jacobi series with nonnegative coefficients*, Proc. Amer. Math. Soc., **140**, 3 (2012), 977–986.
[12] R. E. A. C. PALEY, *On Fourier series with positive coefficients*, J. London Math. Soc., **7**, 3 (1932), 205–208.
[13] H. S. SHAPIRO, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford ser. (2), **26**, 1 (1975), 9–18.
[14] N. YA. VILENKIN, *A class of complete orthonormal series* (in Russian), Izv. Akad. nauk SSSR. Ser. Mat., **11**, 4 (1947), 363–400.
[15] S. S. VOLOSIVETS, *On certain conditions in the theory of series with resprect to multiplicative systems* (in Russian), Analysis Math., **33**, 3 (2007), 227–246.
[16] S. S. VOLOSIVETS, *Modified Hardy and Hardy-Littlewood operators and their behaviour in various spaces*, Izvestiya: Math., **75**, 1 (2011), 29–51.
[17] S. WAINGER, *A problem of Wiener and the failure of a principle for Fourier series with positive coefficients*, Proc. Amer. Math. Soc., **20**, 1 (1969), 16–18.
[18] C. WATARI, *On generalized Walsh-Fourier series*, Tohoku Math. J., **10**, 3 (1958), 211–241.
[19] A. ZYGMUND, *Trigonometric series*, 2nd ed., Vol. 2., Cambridge Univ. Press, London, 1958.

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