Mathematical
Inequalities
& Applications
Volume 17, Number 4 (2014), 1441–1452 doi:10.7153/mia-17-106

MONOTONE MAPS ON DIAGONALIZABLE MATRICES

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(*Communicated by S. Puntanen*)

Abstract. We characterize injective maps preserving $\overset{\sharp}{\leqslant}$-order and maps strongly preserving $\overset{\sharp}{<}$-order on the set of diagonalizable matrices.

1. Introduction

Let $M_n(\mathbb{F})$ denote the space of square matrices of order n with entries from a field $\mathbb{F}$. In this work we always assume that $\mathbb{F}$ is an arbitrary algebraically closed field. Let rk denote the rank of a matrix.

DEFINITION 1.1. A matrix $A \in M_n(\mathbb{F})$ has *index* l ($\operatorname{Ind} A = l$) if $\operatorname{rk} A^l = \operatorname{rk} A^{l+1}$ and l is the smallest nonnegative integer with this property.

Equivalently it is possible to introduce index via the images of powers of A, namely, $\operatorname{Ind} A = l$ iff l is minimal such that $\operatorname{Im} A^l \subseteq \operatorname{Im} A^{l+1}$. Note that the other inclusion holds automatically. Let $\mathrm{I}_n^1(\mathbb{F})$ denote the subset of matrices from $M_n(\mathbb{F})$, which has index 1, i.e.,

$$\mathrm{I}_n^1(\mathbb{F}) = \{A \in M_n(\mathbb{F}) | \operatorname{rk}(A) = \operatorname{rk}(A^2)\}.$$

We observe that by the definition the set of matrices of index 1 contains the set of idempotents, however, the set of index 1 matrices is much larger. In particular, it contains all diagonalizable matrices. Also it contains all Jordan blocks with a non-zero eigenvalue.

THEOREM 1.2. *Let* $A \in M_n(\mathbb{F})$. *Then the system*

$$AXA = A, \quad XAX = X, \quad AX = XA$$

has a solution X *if and only if* $\operatorname{Ind} A = 1$ *and the solution is unique (see* [10, 19, 20, 25]*). It is called the group inverse of A, and is denoted* $A^\sharp$.

Mathematics subject classification (2010): 15A86, 15A04, 15A09.
Keywords and phrases: Monotone matrix transformations, sharp partial order, group inverse.
This work is partially financially supported by the grants MD-962.2014.1 and RFBR 12-01-00140.

So, the set $I_n^1(\mathbb{F})$ is precisely the set of matrices having a group inverse. More details on the properties of group inverses can be found in [2, 11, 24]. The classical approach to introduce many partial orders on matrices is based on the notion of generalized inverses, see [21] for details on matrix partial orders and their applications. In particular by means of the group inverse it is possible to introduce the following order relation on matrices:

DEFINITION 1.3. [20] Let $A,B \in M_n(\mathbb{F})$. Then $A \overset{\sharp}{\leqslant} B$ if and only if $A = B$ or $\operatorname{Ind} A = 1$ and $AA^\sharp = BA^\sharp = A^\sharp B$. Moreover if $A \overset{\sharp}{\leqslant} B$ and $A \neq B$ then $A \overset{\sharp}{<} B$.

Another important relation on matrices is the orthogonality (see [27]).

DEFINITION 1.4. [27] Let $A,B \in M_n(\mathbb{F})$. The matrices A and B are called *pairwise orthogonal*, denoted by $A \perp B$, if $AB = BA = 0$.

DEFINITION 1.5. [9] The map $T\colon I_n^1(\mathbb{F}) \to I_n^1(\mathbb{F})$ is *0-additive*, if for any matrices $A,B \in I_n^1(\mathbb{F})$ with $A \perp B$ we have:
(i) $T(A) \perp T(B)$; (ii) $T(A+B) = T(A) + T(B)$.

The following decomposition of an arbitrary matrix $A \in M_n(\mathbb{F})$ exists and is unique (see [2, Chapter 4.8]):

DEFINITION 1.6. [2] A *core-nilpotent decomposition* of a matrix $A \in M_n(\mathbb{F})$ is the following sum $A = C_A + N_A$, such that $C_A \perp N_A$, $\operatorname{Ind} C_A = 1$, N_A is a nilpotent matrix.

Below we provide the definitions of the matrix partial orders which will be useful for our further considerations.

DEFINITION 1.7. [15, 22] We say that $A \overline{\leqslant} B$ for an arbitrary pair of matrices A and B if and only if $\operatorname{rk}(B - A) = \operatorname{rk} B - \operatorname{rk} A$.

DEFINITION 1.8. [16] Let $A,B \in M_n(\mathbb{F})$. Then $A \overset{\text{cn}}{\leqslant} B$ if and only if $C_A \overset{\sharp}{\leqslant} C_B$ and $N_A \overline{\leqslant} N_B$. If $A \overset{\text{cn}}{\leqslant} B$ and $A \neq B$ then $A \overset{\text{cn}}{<} B$.

LEMMA 1.9. [16] *Let $A,B \in M_n(\mathbb{F})$. The condition $A \overset{\sharp}{\leqslant} B$ implies that $A \overline{\leqslant} B$.*

The class of monotone matrix transformations is introduced via matrix partial orderings in a standard way. Let $\leqslant$ be a certain partial order relation on $M_n(\mathbb{F})$.

DEFINITION 1.10. Let $M \subseteq M_n(\mathbb{F})$. The map $T\colon M \to M$ is called *monotone* with respect to $\leqslant$-order, if for arbitrary two matrices $A,B \in M$ it follows that $A \leqslant B$ implies $T(A) \leqslant T(B)$.

DEFINITION 1.11. Let $M \subseteq M_n(\mathbb{F})$. The map $T: M \to M$ is called *strongly monotone* with respect to $\leqslant$-order, if for arbitrary two matrices $A, B \in M$ conditions $A \leqslant B$ and $T(A) \leqslant T(B)$ are equivalent.

Monotone transformations were under intensive investigations during the recent decades, see for example [1, 3, 4, 5, 6, 7, 8, 13, 14, 17, 18, 23, 27] and references therein. In particular, monotone transformations defined via group inverse were also previously studied. In the paper [3] the characterization of linear bijective maps for matrices over an arbitrary field which are monotone with respect to $\overset{\sharp}{\leqslant}$- and $\overset{\mathrm{cn}}{\leqslant}$-orders was obtained. In the paper [8] the approach to remove the bijectivity assumption was discovered. In [7] additive transformations which are monotone with respect to these orders are investigated. In [9] the authors introduced and studied spectral orthogonal matrix decompositions and showed that these decompositions can serve as an efficient tool to characterize monotone matrix transformations. The present work is devoted to the investigations of injective monotone with respect to $\overset{\sharp}{\leqslant}$-order matrix transformations on the set of diagonalizable matrices over algebraically closed fields.

Let $\mathscr{D}_n(\mathbb{F}) \subseteq M_n(\mathbb{F})$ be the set of diagonalizable matrices. If A is a matrix, and f is a field endomorphism, then $A^f = [a_{ij}]^f = [f(a_{ij})]$ is a matrix obtained from A by applying f entrywise, $A^t = [a_{ij}]^t = [a_{ji}]$ is the transpose of A. The main results of the paper can be formulated as follows:

THEOREM 1.12. *Let $\mathbb{F}$ be an arbitrary algebraically closed field. Assume $n \geqslant 3$ and consider an injective map $T: \mathscr{D}_n(\mathbb{F}) \to \mathscr{D}_n(\mathbb{F})$ which is monotone with respect to $\overset{\sharp}{\leqslant}$-order. Then there exist a matrix $P \in GL_n(\mathbb{F})$, a nonzero endomorphism $f: \mathbb{F} \to \mathbb{F}$, and an injective map $\sigma: \mathbb{F} \to \mathbb{F}$ satisfying the condition $\sigma(0) = 0$ such that*

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} (S_A^2(\lambda))^f P \text{ for all } A \in \mathscr{D}_n(\mathbb{F})$$

or

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} [(S_A^2(\lambda))^f]^t P \text{ for all } A \in \mathscr{D}_n(\mathbb{F}),$$

here spectral orthogonal matrix decomposition $S_A^i(\lambda) \in M_n(\mathbb{F})$, $i = 1,2,3$ is defined below, see Definition 2.3.

THEOREM 1.13. *Let $\mathbb{F}$ be an arbitrary algebraically closed field, let $n \geqslant 3$, and let the map $T: \mathscr{D}_n(\mathbb{F}) \to \mathscr{D}_n(\mathbb{F})$ be strongly monotone with respect to $\overset{\sharp}{<}$-order. Then T is injective and there is $P \in GL_n(\mathbb{F})$, a nonzero endomorphism $f: \mathbb{F} \to \mathbb{F}$, and an injective map $\sigma: \mathbb{F} \to \mathbb{F}$ satisfying $\sigma(0) = 0$ such that*

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} (S_A^2(\lambda))^f P \text{ for all } A \in \mathscr{D}_n(\mathbb{F})$$

or

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} [(S_A^2(\lambda))^f]^t P \text{ for all } A \in \mathscr{D}_n(\mathbb{F}).$$

REMARK 1.14. 1. We note that on the set of matrices of index one, in particular, on its subset of diagonalizable matrices, $\overset{\sharp}{\leqslant}$- and $\overset{\text{cn}}{\leqslant}$-orders are equivalent, so we do not need to consider $\overset{\text{cn}}{\leqslant}$-order separately in this work.

2. Observe that no linearity or additivity is assumed in Theorems 1.12 and 1.13.

2. Preliminaries

The notion of spectrally orthogonal matrix decompositions are introduced and investigated in our paper [9]. It will be essentially used in this paper, so we provide below its definition and basic properties. Firstly we need the following counting functions.

DEFINITION 2.1. A function $k_A : \mathbb{F} \times \mathbb{N} \to \mathbb{Z}_+$ is defined as follows: for $\lambda \in \mathbb{F}$ and $r \in \mathbb{N}$ the value $k_A(\lambda, r)$ equals to the number of Jordan blocks of A of the size r, corresponding to the eigenvalue λ. If there are no Jordan blocks of A with λ of the size r then $k_A(\lambda, r) = 0$.

DEFINITION 2.2. A function $K_A : \mathbb{F} \to \mathbb{Z}_+$ determines the total number of Jordan blocks of A, corresponding to the eigenvalue λ,

$$K_A(\lambda) = \sum_{r=1}^{\infty} k_A(\lambda, r).$$

Let $\operatorname{Spec}(A)$ denotes the spectrum, i.e., the set of eigenvalues, of a matrix A. Observe that $\operatorname{Spec}(A) = \{\lambda \in \mathbb{F} \mid K_A(\lambda) > 0\}$.

Now we are ready to define spectrally orthogonal matrix decompositions.

DEFINITION 2.3. Let $\mathbb{F}$ be a field, $A \in M_n(\mathbb{F})$, $A = C_A + N_A$ be the core-nilpotent decomposition of A. The maps $S^i : \mathbb{F} \times M_n(\mathbb{F}) \to M_n(\mathbb{F})$, $i = 1,2,3$ are called *spectrally orthogonal decompositions* of A if $S^1_A(0) = N_A$ and for any $\lambda \neq 0$ the matrix $S^1_A(\lambda) = X_\lambda$ is such that $X_\lambda \overset{\sharp}{\leqslant} A$, $K_{X_\lambda}(\lambda) = K_A(\lambda)$ and $K_{X_\lambda}(\mu) = 0$ for all $\mu \in \mathbb{F} \setminus \{0, \lambda\}$.

$$S^2_A(\lambda) = S^1_{A+I}(\lambda + 1) - S^1_A(\lambda) \text{ for all } \lambda \in \mathbb{F};$$

$$S^3_A(\lambda) = S^1_A(\lambda) - \lambda S^2_A(\lambda) \text{ for all } \lambda \in \mathbb{F}.$$

The correctness of this definition is proved in [9, Lemma 2.14]. For convenience we denote $S^i(\lambda, A) = S^i_A(\lambda)$, $i = 1,2,3$. If $A \in M_n(\mathbb{F})$ is fixed, it is possible to consider S^i_A as maps $\mathbb{F} \to M_n(\mathbb{F})$, $i = 1,2,3$. Below we list the most important properties of these maps:

THEOREM 2.4. [9, Theorems 2.17, 2.19–2.20] *Let* $A \in M_n(\mathbb{F})$.

1. *If* $\lambda \notin \operatorname{Spec}(A) \subseteq \mathbb{F}$ *then* $S^i_A(\lambda) = 0$ *for* $i = 1,2,3$.

2. $\operatorname{rk}(S^2_A(\lambda)) = \deg_{\chi_A}(z - \lambda)$ *is the multiplicity of* λ *in the characteristic polynomial* χ_A.

3. $S_A^i(\lambda) \perp S_A^j(\mu)$ *for all* $\lambda \neq \mu$, $i, j = 1,2,3$.
4. $S_A^i(\lambda) S_A^2(\lambda) = S_A^2(\lambda) S_A^i(\lambda) = S_A^i(\lambda)$ *for all* $\lambda \in \mathbb{F}$, $i = 1,2,3$.
5. *The matrix* $S_A^2(\lambda)$ *is idempotent for all* $\lambda \in \mathbb{F}$.
6. *The matrix* $S_A^3(\lambda)$ *is nilpotent for all* $\lambda \in \mathbb{F}$.
7. $A = \sum_{\lambda \in \mathbb{F}} S_A^1(\lambda) = \sum_{\lambda \in \mathbb{F}} (\lambda S_A^2(\lambda) + S_A^3(\lambda))$, $I = \sum_{\lambda \in \mathbb{F}} S_A^2(\lambda)$.
8. *For any polynomial* $f \in \mathbb{F}[t]$ *it holds that*

$$f(A) = \sum_{\lambda \in \mathbb{F}} (f(\lambda) S_A^2(\lambda) + \frac{f'(\lambda)}{1!} S_A^3(\lambda) + \ldots + \frac{f^{(n-1)}(\lambda)}{(n-1)!} (S_A^3(\lambda))^{n-1}).$$

9. $\mathbb{F}[A] = \{f(A)\}_{f \in \mathbb{F}[t]} = \langle \{S_A^2(\lambda), S_A^3(\lambda), \ldots, (S_A^3(\lambda))^{n-1}\}_{\lambda \in \mathbb{F}} \rangle$, *and nonzero matrices from the system* $\{S_A^2(\lambda), S_A^3(\lambda), \ldots, (S_A^3(\lambda))^{n-1}\}_{\lambda \in \mathbb{F}}$ *are linearly independent.*
10. *If* $\lambda \in \mathbb{F}$ *then* $S_A^i(\lambda) \in M_n(\mathbb{F})$, $i = 1,2,3$.
11. *If* A *commutes with some* $B \in M_n(\mathbb{F})$, *then* $S_A^i(\lambda)$ *commutes with* B *for all* $\lambda \in \mathbb{F}$ *and* $i = 1,2,3$.
12. *If* $\operatorname{Ind} A = 1$ *and* A *is orthogonal to some* $B \in M_n(\mathbb{F})$ *then*
    a) all matrices $S_A^i(\lambda)$ *are orthogonal to* B,
    b) $S_{A+B}^i(\lambda) = S_A^i(\lambda) + S_B^i(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1,2,3$.
    c) $S_A^i(\lambda) \perp S_B^j(\mu)$ *for all* $\lambda, \mu \in \mathbb{F} \setminus \{0\}$, $i, j = 1,2,3$.
13. *If* $A \overset{\sharp}{\leqslant} C$ *for some* $C \in M_n(\mathbb{F})$, *then for all* $\Lambda \subseteq \mathbb{F} \setminus \{0\}$ *we have* $\sum_{\lambda \in \Lambda} S_A^i(\lambda) \overset{\sharp}{\leqslant} \sum_{\lambda \in \Lambda} S_C^i(\lambda)$, $i = 1,2$. *In particular,* $S_A^i(\lambda) \overset{\sharp}{\leqslant} S_C^i(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1,2$.

3. Monotone transformations on diagonalizable matrices

DEFINITION 3.1. We say that a matrix $A \in M_n(\mathbb{F})$ is *diagonalizable* if there exists $P \in GL_n(\mathbb{F})$ such that $P^{-1}AP$ is diagonal.

Denote by E_{ij} the matrix with 1 at the (i,j)-position and 0 elsewhere, by O_k the $k \times k$ zero matrix. In this section we characterize injective monotone and strongly monotone maps on diagonalizable matrices.

Proof of Theorem 1.12. We divide the proof into several steps.

Step 1. Let us show that T *preserves rank.*

Assume $A \in \mathscr{D}_n(\mathbb{F})$ and $\operatorname{rk} A = k$. Then there are matrices $A_0, A_1, \ldots, A_n \in \mathscr{D}_n(\mathbb{F})$ such that $A_0 \overset{\sharp}{<} A_1 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_n$ and $A_k = A$. Indeed since A is diagonalizable then there exists $Q \in GL_n(\mathbb{F})$ such that $A = Q \operatorname{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) Q^{-1}$. Set $\lambda_s = 1$ if $k < s \leqslant n$. Then set $A_0 = 0$ and for each s, $1 \leqslant s \leqslant n$, consider the matrices $A_s = Q \operatorname{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0) Q^{-1} \in M_n(\mathbb{F})$. Then for $0 \leqslant s \leqslant n$ we have $A_0 \overset{\sharp}{<} A_1 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_n$ and $A_k = A$.

Thus for any $A \in \mathscr{D}_n(\mathbb{F})$ with $\operatorname{rk} A = k$ we have

$$A_0 \overset{\sharp}{<} A_1 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_n, \quad A_k = A.$$

Since T is injective and monotone we have

$$T(A_0) \overset{\sharp}{<} T(A_1) \overset{\sharp}{<} \cdots \overset{\sharp}{<} T(A_n).$$

Note that if A and B are such that $A \overset{\sharp}{<} B$, then $A \overline{<} B$, hence $0 < \operatorname{rk}(B-A) = \operatorname{rk}B - \operatorname{rk}A$, and thus, $1 \leqslant \operatorname{rk}B - \operatorname{rk}A$.

Denote $d_i = \operatorname{rk}T(A_i)$ for $i = 0, \ldots, n$. Then we have

$$1 \leqslant d_1 - d_0, \quad \ldots, \quad 1 \leqslant d_k - d_{k-1}, \quad 1 \leqslant d_{k+1} - d_k, \quad \ldots, \quad 1 \leqslant d_n - d_{n-1}.$$

Adding the k first inequalities we get $k \leqslant d_k - d_0$. Hence $k \leqslant k + d_0 \leqslant d_k$. Adding the $n-k$ last inequalities we get $n - k \leqslant d_n - d_k$. Hence $d_k \leqslant k - n + d_n \leqslant k$. Therefore $d_k = k$. Thus T preserves rank.

Step 2. For any $\lambda \in \mathbb{F}$ we show that there exists $\mu \in \mathbb{F}$ such that $T(\lambda I) = \mu I$.

1. If $\lambda = 0$ then $\mu = 0$ from the rank preserving condition.

2. Let us consider now $\lambda \neq 0$. Assume in the contrary that $T(\lambda I)$ is not a scalar matrix. Consider the set

$$\Gamma = \{X \in \mathscr{D}_n(\mathbb{F}) | X \overset{\sharp}{\leqslant} \lambda I \text{ and there is no } \nu \in \mathbb{F} \text{ such that } T(X) \overset{\sharp}{\leqslant} \nu I\}.$$

Since T preserves rank and $T(\lambda I)$ is not scalar we have $\lambda I \in \Gamma$.

3. Denote $m = \min\limits_{X \in \Gamma} \operatorname{rk}X$ and show that $m > 1$. Since $T(0) = 0 \overset{\sharp}{\leqslant} I$, then $0 \notin \Gamma$, and $m > 0$. Moreover for an arbitrary rank 1 matrix $X \in \mathscr{D}_n(\mathbb{F})$ we have $T(X) \in \mathscr{D}_n(\mathbb{F})$ and $\operatorname{rk}T(X) = 1$, i.e. there exist $\alpha \in \mathbb{F}$ and $Q_1 \in GL_n(\mathbb{F})$ such that $T(X) = \alpha Q_1^{-1} E_{11} Q_1 \overset{\sharp}{\leqslant} \alpha I$. Hence $X \notin \Gamma$ and $m > 1$.

4. Let us show that $m \neq 2$. Indeed if $X \overset{\sharp}{\leqslant} \lambda I$ and $\operatorname{rk}X = 2$ then there exists $Q_2 \in GL_n(\mathbb{F})$ such that $X = Q_2^{-1} \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \oplus O_{n-2} \right] Q_2$. Thus there exist at least 3 different matrices $Y_1,\ Y_2,\ Y_3 \in \mathscr{D}_n(\mathbb{F})$ which have rank 1 and satisfy the condition $Y_j \overset{\sharp}{\leqslant} X$ for $j = 1,2,3$. Indeed let $\alpha_1, \alpha_2, \alpha_3$ be some distinct elements of $\mathbb{F}$ (they exist since $\mathbb{F}$ is algebraically closed, hence, infinite). Set $Y_j = Q_2^{-1} \left[\begin{pmatrix} \lambda & \alpha_j \\ 0 & 0 \end{pmatrix} \oplus O_{n-2} \right] Q_2$ if $j = 1,2,3$.

Then $T(Y_j) \overset{\sharp}{\leqslant} T(X)$ if $j = 1,2,3$. Moreover by the injectivity of T the matrices $T(Y_1)$, $T(Y_2)$ and $T(Y_3)$ are distinct. If $T(X)$ have distinct nonzero eigenvalues then by [9, Lemma 2.11] there exist at most two matrices Y of rank 1 such that $Y \overset{\sharp}{\leqslant} T(X)$. Thus nonzero eigenvalues of $T(X)$ must be equal. This shows that there exists $\nu \in \mathbb{F}$ such that $T(X) \overset{\sharp}{\leqslant} \nu I$. Hence $X \notin \Gamma$. Thus $m \geqslant 3$.

5. Let $A \in \Gamma$ and $\operatorname{rk}A = m$. Since A is diagonalizable, there exists a matrix $B \in \mathscr{D}_n(\mathbb{F})$ such that $B \overset{\sharp}{\leqslant} A$ and $\operatorname{rk}B = m - 1$. By the definition of m we have $B \notin \Gamma$.

Thus there exists $\beta \neq 0$ such that $T(B) \overset{\sharp}{\leqslant} \beta I$. Since $T(B) \overset{\sharp}{\leqslant} T(A)$ and $\operatorname{rk} T(B) = \operatorname{rk} B = m-1$ then β is an eigenvalue of $T(A)$ of the multiplicity at least $m-1$. Moreover since $\operatorname{rk} T(A) = \operatorname{rk} A = m$ and there is no ν such that $T(A) \overset{\sharp}{\leqslant} \nu I$ then the multiplicity of β is $m-1$. Hence $\operatorname{rk} S^1_{T(A)}(\beta) = m-1$ and the multiplicity of the other nonzero eigenvalue is 1.

6. For an arbitrary matrix $X \in \mathscr{D}_n(\mathbb{F})$ satisfying $X \overset{\sharp}{\leqslant} A$ and $\operatorname{rk} X = m-1$ we have that $X \notin \Gamma$. Hence $T(X) \overset{\sharp}{\leqslant} \gamma I$ for a certain element $\gamma \in \mathbb{F}$. It follows that $T(X) = S^1_{T(X)}(\gamma)$ and $\operatorname{rk} S^1_{T(X)}(\gamma) = m-1 \geqslant 2$ since $m \geqslant 3$.

We have $T(X) \overset{\sharp}{\leqslant} T(A)$ and by [9, Theorem 2.20, item 3] it follows that

$$S^1_{T(X)}(\gamma) \overset{\sharp}{\leqslant} S^1_{T(A)}(\gamma).$$

Therefore, γ can not be an eigenvalue of multiplicity 1 of $T(A)$. Hence $\gamma = \beta$.

7. By items 5 and 6 it follows that $T(X) = S^1_{T(X)}(\beta) \overset{\sharp}{\leqslant} S^1_{T(A)}(\beta)$ and also $\operatorname{rk} T(X) = m-1 = \operatorname{rk} S^1_{T(A)}(\beta)$. Thus $T(X) = S^1_{T(A)}(\beta)$ for any diagonalizable X of rank $m-1$ such that $X \overset{\sharp}{\leqslant} A$. This contradicts the injectivity of T since there exist at least two diagonalizable matrices X_1 and X_2 such that $X_j \overset{\sharp}{\leqslant} A$ and $\operatorname{rk} X_j = m-1$, $j = 1,2$. In particular it is possible to take $X_1 = Q_3 \operatorname{diag}(\alpha_1, \dots, \alpha_{m-1}, 0, \dots, 0) Q_3^{-1}$, $X_2 = Q_3 \operatorname{diag}(0, \alpha_2, \dots, \alpha_m, 0, \dots, 0) Q_3^{-1}$, where Q_3 is such that $Q_3 A Q_3^{-1}$ is diagonal with $\alpha_1, \dots, \alpha_m$ on the diagonal.

The obtained contradiction shows that for any $\lambda \in \mathbb{F}$ there exists $\mu = \mu(\lambda) \in \mathbb{F}$ such that the equality $T(\lambda I) = \mu(\lambda) I$ holds.

Step 3. Let us show that the map T is 0-additive.

1. Define a map $\sigma \colon \mathbb{F} \to \mathbb{F}$ such that $T(\lambda I) = \sigma(\lambda) I$ for all $\lambda \in \mathbb{F}$. It follows by Step 2 that such σ exists, it is injective and $\sigma(0) = 0$.

2. Assume $A \in \mathscr{D}_n(\mathbb{F})$. Let us prove that $T(S^1_A(\lambda_1)) \perp T(S^1_A(\lambda_2))$ for $\lambda_1 \neq \lambda_2$ and $T(A) = \sum\limits_{\lambda \in \mathbb{F}} T(S^1_A(\lambda))$. Indeed $S^1_A(\lambda) \overset{\sharp}{\leqslant} A$ for all $\lambda \in \mathbb{F} \setminus \{0\}$ by the definition of $S^1_A(\lambda)$. Since A is diagonalizable it follows that $S^1_A(0) = 0$. Then $S^1_A(\lambda) \overset{\sharp}{\leqslant} A$ for all $\lambda \in \mathbb{F}$.

3. T is monotone with respect to $\overset{\sharp}{\leqslant}$-order. Therefore we have $T(S^1_A(\lambda)) \overset{\sharp}{\leqslant} T(A)$ for all $\lambda \in \mathbb{F}$. Moreover $S^1_A(\lambda) \overset{\sharp}{\leqslant} \lambda I$ and $T(S^1_A(\lambda)) \overset{\sharp}{\leqslant} T(\lambda I) = \sigma(\lambda) I$. Then

$$T(S^1_A(\lambda)) = S^1_{T(S^1_A(\lambda))}(\sigma(\lambda)) \overset{\sharp}{\leqslant} S^1_{T(A)}(\sigma(\lambda))$$

for all $\lambda \in \mathbb{F}$. Since the function σ is injective, by [9, Theorem 2.17, item 3] we have $S^1_{T(A)}(\sigma(\lambda_1)) \perp S^1_{T(A)}(\sigma(\lambda_2))$ if $\lambda_1 \neq \lambda_2$ thus $T(S^1_A(\lambda_1)) \perp T(S^1_A(\lambda_2))$ if $\lambda_1 \neq \lambda_2$ due to [9, Lemma 2.5]. Moreover by [9, Lemma 2.22] we have $\sum\limits_{\lambda\in\mathbb{F}} T(S_A^1(\lambda)) \overset{\sharp}{\leqslant} T(A)$ and $T(A) = \sum\limits_{\lambda\in\mathbb{F}} T(S_A^1(\lambda))$ since ranks are equal.

4. Let $X,Y \in \mathcal{D}_n(\mathbb{F})$, $X \perp Y$ and $X = S_X^1(\lambda)$, $Y = S_Y^1(\lambda)$ for a certain $\lambda \in \mathbb{F}\setminus\{0\}$. By the condition $\mathbb{F}$ is algebraically closed, hence, it is infinite, thus there exist $\alpha,\beta \in \mathbb{F}$ such that $0,\alpha,\beta,\lambda$ are pairwise distinct elements in $\mathbb{F}$. Denote $X' = \alpha S_X^2(\lambda)$, $Y' = \beta S_X^2(0)$. Then $X \perp Y'$, $X' \perp Y$, $X' \perp Y'$ and $\operatorname{rk} X' + \operatorname{rk} Y' = n$. From the above discussion we have $T(X) \perp T(Y')$, $T(X') \perp T(Y)$, $T(X') \perp T(Y')$ and $\operatorname{rk} T(X') + \operatorname{rk} T(Y') = n$. Similarly to the proof of [9, Lemma 6.1] we have the relations $T(X) \perp T(Y)$, $T(X) + T(Y) = T(X+Y)$.

5. Let $A,B \in \mathcal{D}_n(\mathbb{F})$, $A \perp B$ are fixed arbitrary matrices. Then

$$T(A) = \sum_{\lambda\in\mathbb{F}} T(S_A^1(\lambda)) \perp \sum_{\lambda\in\mathbb{F}} T(S_B^1(\lambda)) = T(B)$$

by the above and since $S_A^1(\lambda_1) \perp S_B^1(\lambda_2)$ for all $\lambda_1,\lambda_2 \in \mathbb{F}$. Moreover

$$\begin{aligned} T(A+B) &= \sum_{\lambda\in\mathbb{F}} T(S_{A+B}^1(\lambda)) = \sum_{\lambda\in\mathbb{F}} T(S_A^1(\lambda) + S_B^1(\lambda)) \\ &= \sum_{\lambda\in\mathbb{F}} T(S_A^1(\lambda)) + \sum_{\lambda\in\mathbb{F}} T(S_B^1(\lambda)) = T(A) + T(B), \end{aligned}$$

and the map T is 0-additive.

Step 4. The final form of T.

1. Consider $T_1 \colon \mathcal{D}_n(\mathbb{F}) \to \mathcal{D}_n(\mathbb{F})$ defined by $T_1(A) = (\sigma(1))^{-1} T(A)$. It is straightforward to check that T_1 is monotone with respect to $\overset{\sharp}{\leqslant}$-order, T_1 is 0-additive, and T_1 satisfies $T_1(\lambda I) = \sigma_1(\lambda) I$ for all $\lambda \in \mathbb{F}$ where $\sigma_1(\lambda) = (\sigma(1))^{-1}\sigma(\lambda)$ for all $\lambda \in \mathbb{F}$. Moreover $\sigma_1(0) = 0$ and $\sigma_1(1) = 1$.

2. Assume $A \in M_n(\mathbb{F})$ is an idempotent of rank 1. Then $A \overset{\sharp}{\leqslant} I$ and $T_1(A) \overset{\sharp}{\leqslant} T_1(I) = I$, $\operatorname{rk} T_1(A) = 1$. Therefore $T_1(A)$ is also idempotent of rank 1. Hence by Theorem 2.3 from [26] there are a matrix $P \in GL_n(\mathbb{F})$ and a nonzero endomorphism f of the field $\mathbb{F}$ such that either $T_1(A) = P^{-1}A^f P$ for any rank one idempotent A or $T_1(A) = P^{-1}(A^f)^t P$ for any rank one idempotent A. We remark that in [26] this result was obtained only for the case $\mathbb{F} = \mathbb{C}$. However the same proof is applicable over an arbitrary algebraically closed field. Indeed, the nonsurjective analog of the main theorem of projective geometry, which is the key tool in obtaining the above characterization, is proved in [12] even for an arbitrary skewfield the other arguments from the proof are directly applicable not only for the field $\mathbb{C}$ but for any field.

3. Thus either $T_1(A) = P^{-1}A^f P$ or $T_1(A) = P^{-1}(A^f)^t P$ on the set of idempotents of rank 1. After the composition of T_1 with similarity and transposition (if necessary) we obtain a 0-additive map $T_2 \colon \mathcal{D}_n(\mathbb{F}) \to \mathcal{D}_n(\mathbb{F})$ satisfying the conditions $T_2(A) = A^f$ for all idempotents of rank 1 and $T_2(\lambda I) = \sigma_1(\lambda) I$.

4. By 0-additivity of T_2 we have $T_2(A) = A^f$ for all idempotents since any idempotent is a sum of pairwise orthogonal rank 1 idempotents.

5. For any $B \in \mathscr{D}_n(\mathbb{F})$ such that $\operatorname{rk} B = 1$ it holds that $B = S_B^1(\lambda)$ for a certain $\lambda \neq 0$. Then $B \perp S_B^2(0)$ and

$$T_2(B) \perp T_2(S_B^2(0)) = (S_B^2(0))^f.$$

Moreover $\operatorname{rk} T_2(B) + \operatorname{rk}(S_B^2(0))^f = n$. However

$$\frac{1}{\sigma_1(\lambda)} T_2(B) + (S_B^2(0))^f = I = (S_B^2(0) + S_B^2(\lambda))^f,$$

thus $T_2(B) = \sigma_1(\lambda)(S_B^2(\lambda))^f$.

6. Hence $T_2(A) = \sum_{\lambda \in \mathbb{F}} \sigma_1(\lambda)(S_A^2(\lambda))^f$ for all $A \in \mathscr{D}_n(\mathbb{F})$. However $\sigma(\lambda) = \sigma(1)\sigma_1(\lambda)$ hence

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1}(S_A^2(\lambda))^f P$$

for all $A \in \mathscr{D}_n(\mathbb{F})$ or

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1}((S_A^2(\lambda))^f)^t P$$

for all $A \in \mathscr{D}_n(\mathbb{F})$, which is our statement. □

Proof of Theorem 1.13. Similarly to item 1 of Theorem 1.12 we show that map T preserves rank. Let $X, Y \in \mathscr{D}_n(\mathbb{F})$ are such that $T(X) = T(Y)$. Then $\operatorname{rk} X = \operatorname{rk} Y = r$. There are two possible cases:

1) $r > 1$. Let nonzero matrices X_1, X_2 be such that $X_1 \perp X_2$, $X = X_1 + X_2$. Then $X_i \overset{\sharp}{<} X$, $T(X_i) \overset{\sharp}{<} T(X) = T(Y)$, $X_i \overset{\sharp}{<} Y$ for $i = 1, 2$. Therefore $X = X_1 + X_2 \overset{\sharp}{\leqslant} Y$ by [9, Lemma 2.22] and $X = Y$.

2) $r = 1$. Denote by λ a nonzero eigenvalue of a matrix X. We have $X \overset{\sharp}{<} \lambda I$ therefore $Y \overset{\sharp}{<} \lambda I$, $\lambda \in \operatorname{Spec}(Y)$. Let $\mu \in \mathbb{F} \setminus \{0, \lambda\}$, $A = X + \mu S_X^2(0)$. Then $X \overset{\sharp}{<} A$ and $Y \overset{\sharp}{<} A$,

$$Y = S_Y^1(\lambda) \overset{\sharp}{\leqslant} S_A^1(\lambda) = X, \quad X = Y.$$

Thus the map T is injective and has the required form by Theorem 1.12. □

The following proposition provides the uniqueness conditions for the aforesaid form of T.

PROPOSITION 3.2. *Let* $T_j(A) = \sum_{\lambda \in \mathbb{F}} \sigma_j(\lambda) P_j^{-1}[(S_A^2(\lambda))^{f_j}]^{t_j} P_j$, $j = 1, 2$, *where each* t_j *is either identity map or the transposition,* $P_j \in GL_n(\mathbb{F})$, f_1, f_2 *are field endomorphisms of* $\mathbb{F}$ *not both equal to zero,* $\sigma_j \colon \mathbb{F} \to \mathbb{F}$ *are some maps. Assume that* $T_1(A) = T_2(A)$ *for all* $A \in \mathscr{D}_n(\mathbb{F})$. *Then* $\sigma_1 \equiv \sigma_2$. *Assume in addition that* $\sigma_1 \neq \text{const}$ *then* $f_1 \equiv f_2$, $t_1 = t_2$ *and there exists* $\alpha \neq 0$ *such that* $P_2 = \alpha P_1$.

Proof. For all $\lambda \in \mathbb{F}$ we have

$$T_1(\lambda I) = \sigma_1(\lambda)I = \sigma_2(\lambda)I = T_2(\lambda I),$$

thus $\sigma_1 \equiv \sigma_2$. Set $\sigma(\lambda) = \sigma_1(\lambda) - \sigma_1(0)$, $T(A) = T_1(A) - \sigma_1(0)I$. Thus $T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P_j^{-1}[(S_A^2(\lambda))^{f_j}]^{t_j} P_j$ for $j = 1,2$. If $\sigma_1 \neq \text{const}$ then $\sigma \neq \text{const}$ and there exists such $\mu \in \mathbb{F}$ that $\sigma(\mu) \neq \sigma(0) = 0$. For any idempotent matrix A we have

$$T(\mu A) = \sigma(\mu) P_1^{-1}(A^{f_1})^{t_1} P_1 = \sigma(\mu) P_2^{-1}(A^{f_2})^{t_2} P_2.$$

Denote $Q = P_2 P_1^{-1}$ we obtain $(A^{f_1})^{t_1} = Q^{-1}(A^{f_2})^{t_2} Q$. Set $A = E_{ii}$ thus $E_{ii} = Q^{-1} E_{ii} Q$. Therefore $Q = \text{diag}(\alpha_1, \ldots, \alpha_n)$. Let $A = E_{11} + \lambda E_{i1}$, $i > 1$. Since $(A^{f_1})^{t_1} = Q^{-1}(A^{f_2})^{t_2} Q$ and $E_{11} = Q^{-1} E_{11} Q$ then $f_1(\lambda)(E_{i1})^{t_1} = f_2(\lambda) Q^{-1}(E_{i1})^{t_2} Q$. Thus $t_1 = t_2$. If t_1 is an identity map then $f_1(\lambda) E_{i1} = f_2(\lambda) \alpha_i^{-1} \alpha_1 E_{i1}$. If t_1 denotes the transposition then $f_1(\lambda) E_{1i} = f_2(\lambda) \alpha_1^{-1} \alpha_i E_{1i}$. In both cases we set $\lambda = 1$ and obtain $\alpha_i = \alpha_1$ for all $i > 1$ therefore $f_1 \equiv f_2$. Thus Q is a scalar matrix and there exists $\alpha \neq 0$ such that $P_2 = \alpha P_1$. □

4. Examples

Below we provide several examples showing that the assumptions of Theorems 1.12 and 1.13 are indispensable.

Our first two examples show that despite of the fact that monotone transformations on $\mathcal{D}_n(\mathbb{F})$ have some standard form, on the whole matrix space $M_n(\mathbb{F})$ they are uncontrollable.

EXAMPLE 4.1. Let $T_1 \colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined by $T_1(A) = 0$ if $\text{Ind}(A) > 1$, and $T_1(A) = A$ if $\text{Ind}(A) = 1$. Then T_1 is monotone with respect to the $\overset{\sharp}{\leqslant}$-order. However, T_1 is not of the form described in Theorem 1.12. Indeed, all transformations of the form described in Theorem 1.12 are injective, but T_1 is not.

Even under the conditions that T is bijective and strongly monotone on the whole $M_n(\mathbb{F})$ it can be of the form which is different from the one described in Theorem 1.12, as the following example show.

EXAMPLE 4.2. Let $T_2 \colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be defined as

$$T_2(A) = \sum_{\lambda \in \mathbb{F}} (\lambda S_A^2(\lambda) - S_A^3(\lambda)).$$

Here in the spectral-orthogonal decomposition of A via S^2 and S^3 we changed plus to minus. Then

(1) T_2 is bijective,

(2) T_2 is strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order,

however it is straightforward to see that on the whole $M_n(\mathbb{F})$ the map T_2 is not of the form described in Theorem 1.12. Note that on $\mathscr{D}_n(\mathbb{F})$ the map T_2 is identity.

Let us check the conditions (1) and (2).

Indeed $T_2(T_2(A)) = A$ for all $A \in M_n(\mathbb{F})$. This implies bijectivity.

T is strongly monotone with respect to the $\overset{\sharp}{\leqslant}$-order, since $S^2_{T_2(A)}(\lambda) = S^2_A(\lambda)$ and $S^3_{T_2(A)}(\lambda) = -S^3_A(\lambda)$ for all $A \in M_n(\mathbb{F})$.

Also $T_2(E_{12}) = -E_{12}$, thus T_2 is not identity map on the whole $M_n(\mathbb{F})$.

The following example shows that even on the set of diagonalizable matrices there are “wild” maps which are monotone with respect to the $\overset{\sharp}{<}$-order. Certainly, such maps are neither injective nor strongly monotone.

EXAMPLE 4.3. Let $T_3 \colon \mathscr{D}_n(\mathbb{F}) \to \mathscr{D}_n(\mathbb{F})$ be such that for each $A \in \mathscr{D}_n(\mathbb{F})$, $\operatorname{rk} A = k$, we define $T_3(A) = E_{11} + \ldots + E_{kk}$. Then T_3 is monotone with respect to the $\overset{\sharp}{<}$-order, T_3 is not injective, T_3 is not strongly monotone with respect to the $\overset{\sharp}{<}$-order, and T_3 is not of the form described in Theorem 1.12.

In our next paper we further emphasize our results in the case of injective continuous maps over the field of complex numbers. It will be shown that such maps have to be either linear or conjugate-linear. Also an example will be given showing that without these conditions non-additive monotone maps really exist.

Acknowledgements. The authors are grateful to the referees for their useful comments.

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(Received February 4, 2013)

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