Mathematical
Inequalities
& Applications
Volume 17, Number 4 (2014), 1529–1541 doi:10.7153/mia-17-112

VOLUME INEQUALITIES FOR ORLICZ MEAN ZONOIDS

CHANGMIN DU, GANGSONG LENG AND DONGMENG XI

(*Communicated by M. A. Hernandez Cifre*)

Abstract. In this paper, a more general mean zonoid called Orlicz mean zonoid $\bar{Z}_\phi K$ of a convex body K is introduced. Using shadow systems of convex bodies, we give a sharp lower estimate for the volume ratio of $\bar{Z}_\phi K$ and K, and a sharp upper estimate for the volume product of $\bar{Z}_\phi^* K$ and K.

1. Introduction

The Orlicz Brunn-Minkowski theory emerged following three remarkable papers which established the Orlicz projection inequality [21], the Orlicz centroid inequality [22] and the even Orlicz Minkowski problem [12]. Analogous to the way that Orlicz spaces generalize L_p spaces, this theory represents an extension of the evolving L_p Brunn-Minkowski theory. This extension is motivated by asymmetric concepts within the L_p Brunn-Minkowski theory developed by Ludwig [15], Haberl and Schuster [10, 11], Ludwig and Reitzner [16]. Recently, Gardner, Hug and Weil [7] have given the systematic description of the Orlicz Brunn-Minkowski theory. Despite various studies are issued within this theory, there are still lots of works to be considered (see e.g. [9, 13, 14, 17, 18, 19, 20, 32, 33]). The aim of this paper is to go on in this way to establish volume inequalities for the Orlicz mean zonoids.

A zonotope is the Minkowski sum of segments (see e.g. [4, 27, 28]). Zonoids are defined as limits of zonotopes in the Hausdorff metric. A zonoid Z can be denoted by

$$h_Z(u) = \frac{1}{2}\int_{S^{n-1}} |u\cdot v|\,d\mu(v),$$

for all $u \in S^{n-1}$, where μ is an even measure on the unit sphere S^{n-1}. The extremal problems about geometric mean values are studied by Pfiefer [24, 25]. We focus now our attention on an interesting paper [34], in which Zhang defined the mean zonoid $\bar{Z}K$ of a convex body K by

$$h_{\bar{Z}K}(u) = \frac{1}{|K|^2}\int_K\int_K |u\cdot(x-y)|\,dx\,dy$$

Mathematics subject classification (2010): 52A20, 52A40.
Keywords and phrases: Convex body, Orlicz mean zonoid, shadow systems.
This research is supported by the National Natural Science Foundation of China (10971128), the National Natural Science Foundation of China (11271244) and Shanghai Leading Academic Discipline Project (S30104).

Paper MIA-17-112

for all $u \in S^{n-1}$. In fact, the body $\bar{Z}K$ is a zonoid.

In this paper, we are interested in the Orlicz mean zonoids. We consider an even convex function $\phi : \mathbb{R} \to [0,\infty)$ such that $\phi(0) = 0$. This means that ϕ must be decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. We require that ϕ is strictly increasing on $[0,\infty)$. The class of such ϕ will be denoted by $\mathscr{C}$. If K is a convex body with volume $|K|$, and $\phi \in \mathscr{C}$, then we define the Orlicz mean zonoid $\bar{Z}_\phi K$ of K by

$$h_{\bar{Z}_\phi K}(u) = \inf\left\{\lambda > 0 : \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{u\cdot(x-y)}{\lambda}\right)dxdy \leqslant 1\right\}, \tag{1}$$

for all $u \in \mathbb{R}^n$. Taking $\phi(\cdot) = |\cdot|$, $\bar{Z}_\phi K$ is just $\bar{Z}K$. Taking $\phi(\cdot) = |\cdot|^p$ for $p \geqslant 1$, we obtain $\bar{Z}_p K$. It is easy to see that $\bar{Z}_\infty K = DK$ for $p = \infty$, where DK is the difference body of K (i.e., $DK = K + (-K)$). By Jensen's inequality, we have

$$\bar{Z}_p K \subset \bar{Z}_q K \subset DK,$$

for all $1 \leqslant p \leqslant q$. From the definition of L_p radial mean bodies [5] and L_p centroid bodies [19], we can get that L_p mean zonoids are just the L_p centroid bodies of L_{n+p} radial mean bodies with a dilatation.

Using shadow systems of convex bodies, Campi and Gronchi (see e.g. [1, 2, 3, 4]) got volume inequalities for L_p zonotopes and related results. Inspired by their works, Wang, Leng and Huang [32] obtained volume inequalities for Orlicz zonotopes. Following all of their works, we will prove the following theorems.

THEOREM 1. *If $\phi \in \mathscr{C}$ and K is a convex body in $\mathbb{R}^n$, then the volume ratio*

$$|\bar{Z}_\phi K|/|K| \tag{2}$$

is minimized if and only if K is an ellipsoid.

Taking $\phi(\cdot) = |\cdot|$, Theorem 1 is just Theorem 7 in [34].

A Blaschke-Santaló type inequality is given by the next theorem.

THEOREM 2. *If $\phi \in \mathscr{C}$ and K is a convex body in $\mathbb{R}^n$, then*

$$|\bar{Z}_\phi^* K||K| \leqslant |\bar{Z}_\phi^* E||E|, \tag{3}$$

equality holds if and only if K is an ellipsoid.

This paper is organized as follows: In Section 2, we present some notation and existing results needed in this paper. In Section 3, we obtain some properties for Orlicz mean bodies. The proofs of the main theorems are given in Section 4.

2. Background and notation

In this section we present the terminology and notation we shall use throughout. For general references the reader may wish to consult the books of Gardner [6], Gruber [8], and Schneider [29].

A convex body K in Euclidean n-space $\mathbb{R}^n$ is a compact convex set with non-empty interiors. $|K|$ is defined as the volume of K. Denote by $\mathscr{K}^n$ the class of convex bodies. A convex body K is uniquely determined by its support function defined by

$$h_K(x) = \max_{y\in K} x\cdot y, \text{ for all } x\in\mathbb{R}^n,$$

where $x\cdot y$ is the standard inner product of x and y. If the origin is an interior point of K, then its polar body K^* is defined by

$$K^* = \{x\in\mathbb{R}^n : x\cdot y \leqslant 1 \text{ for all } y\in K\}.$$

The Hausdorff metric $\delta(\cdot,\cdot)$ on $\mathscr{K}^n$ is

$$\delta(K,L) = \max_{u\in S^{n-1}} |h_K(u)-h_L(u)|,$$

for $K,L\in\mathscr{K}^n$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. For $v\in S^{n-1}$, $v^\perp = \{x\in\mathbb{R}^n : x\cdot v = 0\}$. Denote by $K|v^\perp$ and $x|v^\perp$ the orthogonal projection of the convex body K and the vector x onto $v^\perp$ respectively. $|K|v^\perp|_{n-1}$ defines the $(n-1)$-dimensional volume of $K|v^\perp$.

Denote by $GL(n)$ the group of linear transformations. For $\psi\in GL(n)$, ψ^T is the transpose of ψ. Each affine transformation consists of a linear transformation followed by a translation.

The technique used in this paper is that of shadow systems developed by Rogers and Shephard (see [26] and [30]). A shadow system along the unit direction v is a family of convex sets $K_t\in\mathbb{R}^n$ that can be defined by

$$K_t = \operatorname{conv}\{z+\alpha(z)tv : z\in A\subset\mathbb{R}^n\},$$

where A is an arbitrary bounded set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

A particular type of shadow system called parallel chord movements along the unit direction v is a family of convex bodies $K_t\in\mathbb{R}^n$ defined by

$$K_t = \{z+\beta(z|v^\perp)tv : z\in K, 0\leqslant t\leqslant 1\}, \tag{4}$$

where K is a convex body in $\mathbb{R}^n$ and β is a continuous real function on $v^\perp$. Notice that $K_t|v^\perp$ is independent of t. By Fubini's theorem, $|K_t|$ is also independent of t.

If the speed function β of the movement is an affine function (that is, $\beta(x) = x\cdot u+k$, for some vector u and real constant k), then it is easy to see that K_t is an affine image of K for every t in the range of the movement.

Fix a direction v and let

$$K = \{x+yv\in\mathbb{R}^n : x\in K|v^\perp, y\in\mathbb{R}, f(x)\leqslant y\leqslant g(x)\},$$

where f and $-g$ are convex functions on $K|v^{\perp}$. The parallel chord movement with speed function $\beta(x) = -(f(x)+g(x))$ is such that $K_0 = K$, $K_1 = K^v$, the reflection of K with respect to hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^{\perp}$.

The following lemmas about shadow systems of convex bodies are important in this paper.

LEMMA 1. [1] *Let $\{H_t : 0 \leqslant t \leqslant 1\}$ be a one-parameter family of convex bodies such that $H_t|v^{\perp}$ is independent of t. Assume that the bodies H_t are defined by*

$$H_t = \{x + yv : x \in H_t|v^{\perp}, y \in \mathbb{R}, f_t(x) \leqslant y \leqslant g_t(x)\}, \ \ 0 \leqslant t \leqslant 1,$$

for suitable functions f_t, g_t. Then $\{H_t : 0 \leqslant t \leqslant 1\}$ is a shadow system along the direction v if and only if for every $x \in H_0|v^{\perp}$,

(i) g_t and $-f_t$ are convex functions of the parameter t in $[0,1]$,

(ii) $f_{\lambda t_1 + (1-\lambda)t_2}(x) \leqslant \lambda g_{t_1}(x) + (1-\lambda) f_{t_2}(x) \leqslant g_{\lambda t_1 + (1-\lambda)t_2}(x)$, for every $t_1, t_2, \lambda \in [0,1]$.

LEMMA 2. [30] *The volume of a shadow system is a convex function of the parameter t.*

LEMMA 3. [3] *Let K be a convex body in $\mathbb{R}^n$. If K_t is a shadow system of origin symmetric convex bodies in $\mathbb{R}^n$, then $|K_t^*|^{-1}$ is a convex function of t.*

3. Description of results

Since a function $\phi \in \mathscr{C}$ is strictly increasing on $[0,\infty)$, it follows that the function

$$\lambda \mapsto \frac{1}{|K|^2} \int_K \int_K \phi\left(\frac{u \cdot (x-y)}{\lambda}\right) dxdy$$

is strictly decreasing on $[0,\infty)$. In view of the definition (1), we have

LEMMA 4. *If $\phi \in \mathscr{C}$, $K \in \mathscr{K}^n$ and $u_0 \in \mathbb{R}^n \setminus \{0\}$, then*

$$h_{\bar{Z}_\phi K}(u_0) = \lambda_0 \quad \Leftrightarrow \quad \frac{1}{|K|^2} \int_K \int_K \phi\left(\frac{u_0 \cdot (x-y)}{\lambda_0}\right) dxdy = 1. \tag{5}$$

The following lemma shows that $\bar{Z}_\phi K$ defined in (1) is a convex body.

LEMMA 5. *If $\phi \in \mathscr{C}$ and $K \in \mathscr{K}^n$, then the function $h_{\bar{Z}_\phi K}$ is the support function of a convex body $\bar{Z}_\phi K$.*

Proof. Because a sublinear function can uniquely determine a support function of a convex body, we only need to prove $h_{\bar{Z}_\phi K}$ is a sublinear function. By the definition (1), for all $u \in \mathbb{R}^n$ and $c > 0$, it is easy to see

$$h_{\bar{Z}_\phi K}(cu) = ch_{\bar{Z}_\phi K}(u). \tag{6}$$

For all $u_1, u_2 \in \mathbb{R}^n$, by the convexity of the function ϕ, we have

$$\begin{aligned}\phi\left(\frac{(u_1+u_2)\cdot(x-y)}{\lambda_1+\lambda_2}\right) &= \phi\left(\frac{u_1\cdot(x-y)}{\lambda_1+\lambda_2}+\frac{u_2\cdot(x-y)}{\lambda_1+\lambda_2}\right)\\ &\leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\left(\frac{u_1\cdot(x-y)}{\lambda_1}\right)+\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\left(\frac{u_2\cdot(x-y)}{\lambda_2}\right).\end{aligned}$$

Let $h_{\bar{Z}_\phi K}(u_i) = \lambda_i,\ i = 1,2$. By lemma 4, integrating both sides of the above inequality, we get

$$\frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{(u_1+u_2)\cdot(x-y)}{\lambda_1+\lambda_2}\right)dxdy \leqslant 1.$$

This is

$$h_{\bar{Z}_\phi K}(u_1+u_2) \leqslant \lambda_1+\lambda_2.$$

It means

$$h_{\bar{Z}_\phi K}(u_1+u_2) \leqslant h_{\bar{Z}_\phi K}(u_1)+h_{\bar{Z}_\phi K}(u_2). \tag{7}$$

Hence, we complete the proof from (6) and (7). □

The next lemma demonstrates that $|\bar{Z}_\phi K|/|K|$ is affine invariant.

LEMMA 6. *If $\phi \in \mathscr{C}$, $K \in \mathscr{K}^n$ and $\psi \in GL(n)$, then*

$$\bar{Z}_\phi(\psi K+\gamma) = \psi \bar{Z}_\phi K.$$

for any $\gamma \in \mathbb{R}^n$.

Proof. Let $\psi \in GL(n)$, it follows (1) that

$$\begin{aligned}&h_{\bar{Z}_\phi(\psi K+\gamma)}(u)\\ &= \inf\left\{\lambda > 0 : \frac{1}{|\psi K+\gamma|^2}\int_{\psi K+\gamma}\int_{\psi K+\gamma}\phi\left(\frac{u\cdot(x-y)}{\lambda}\right)dxdy \leqslant 1\right\}\\ &= \inf\left\{\lambda > 0 : \frac{1}{|\psi K|^2}\int_K\int_K \phi\left(\frac{u\cdot(\psi x'+\gamma-\psi y'-\gamma)}{\lambda}\right)|\psi|^2dx'dy' \leqslant 1\right\}\\ &= \inf\left\{\lambda > 0 : \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{\psi^T u\cdot(x'-y')}{\lambda}\right)dx'dy' \leqslant 1\right\}\\ &= h_{\bar{Z}_\phi K}(\psi^T u). \quad \square\end{aligned}$$

Associated with each $\phi \in \mathscr{C}$, c_ϕ is defined by

$$c_\phi = \min\{c > 0 : \phi(c) \geqslant 1\}.$$

Thus $\phi(c_\phi) = 1$. In order to get the continuity of the operator $\bar{Z}_\phi$, the following lemma is needed.

LEMMA 7. *If $\phi \in \mathscr{C}$, $K \in \mathscr{K}^n$ and $u \in S^{n-1}$, then*

$$\frac{|K|u^\perp|_{n-1}}{|K|c_\phi} \leqslant h_{\bar{Z}_\phi K}(u) \leqslant \frac{D_K}{c_\phi},$$

where D_K denotes the longest chord length of K.

Proof. Suppose $u_0 \in S^{n-1}$ such that $h_{\bar{Z}_\phi K}(u_0) = \lambda_0$. From the definition of c_ϕ, Jensen's inequality and Lemma 4, we get

$$\begin{aligned}
\phi(c_\phi) = 1 &= \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{u_0\cdot(x-y)}{\lambda_0}\right)dxdy \\
&\geqslant \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{(u_0\cdot(x-y))_+}{\lambda_0}\right)dxdy \\
&\geqslant \phi\left(\frac{1}{|K|^2}\int_K\int_K \frac{(u_0\cdot(x-y))_+}{\lambda_0}dxdy\right) \\
&= \phi\left(\frac{1}{|K|^2}\int_K\int_{K-y} \frac{(u_0\cdot z)_+}{\lambda_0}dzdy\right) \\
&= \phi\left(\frac{1}{|K|^2\lambda_0}\int_K |(K-y)|u_0^\perp|_{n-1}dy\right) \\
&= \phi\left(\frac{1}{|K|^2\lambda_0}\int_K |K|u_0^\perp|_{n-1}dy\right) \\
&= \phi\left(\frac{|K|u_0^\perp|_{n-1}}{|K|\lambda_0}\right),
\end{aligned}$$

where $(u_0\cdot(x-y))_+ = \max\{u_0\cdot(x-y), 0\}$. Since ϕ is increasing on $[0,\infty)$, then we obtain the lower bound of $h_{\bar{Z}_\phi K}$:

$$\frac{|K|u_0^\perp|_{n-1}}{|K|c_\phi} \leqslant \lambda_0.$$

In order to get the upper bound, we also use the monotonicity of ϕ on $[0,\infty)$.

$$\begin{aligned}
\phi(c_\phi) = 1 &= \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{u_0\cdot(x-y)}{\lambda_0}\right)dxdy \\
&\leqslant \frac{1}{|K|^2}\int_K\int_K \left(\max_{x,y\in K} \phi\left(\frac{u_0\cdot(x-y)}{\lambda_0}\right)\right)dxdy
\end{aligned}$$

$$\leqslant \frac{1}{|K|^2}\int_K\int_K \phi\left(\frac{D_K}{\lambda_0}\right)dxdy$$
$$= \phi\left(\frac{D_K}{\lambda_0}\right).$$

Thus

$$\lambda_0 \leqslant \frac{D_K}{c_\phi}. \quad \square$$

The continuity of the operator $\bar{Z}_\phi$ on $\mathscr{K}^n$ can be obtained as follows.

LEMMA 8. *If* $\phi \in \mathscr{C}$, $K_i \in \mathscr{K}^n$ *and* $K_i \to K \in \mathscr{K}^n$, *then* $\bar{Z}_\phi K_i \to \bar{Z}_\phi K$.

Proof. Suppose $u_0 \in S^{n-1}$, we will show that

$$h_{\bar{Z}_\phi K_i}(u_0) \to h_{\bar{Z}_\phi K}(u_0).$$

Let

$$h_{\bar{Z}_\phi K_i}(u_0) = \lambda_i,$$

and by Lemma 7,

$$\frac{|K_i|u^\perp|_{n-1}}{|K_i|c_\phi} \leqslant \lambda_i \leqslant \frac{D_{K_i}}{c_\phi}.$$

Since $K_i \to K$, there exist real numbers a, b such that $0 < a \leqslant \lambda_i \leqslant b < \infty$, for all large enough i.

We will show that the bounded sequence $\{\lambda_i\}$ converges to $h_{\bar{Z}_\phi K}(u_0)$, and it only remains to be shown that every convergent subsequence of $\{\lambda_i\}$ converges to $h_{\bar{Z}_\phi K}(u_0)$. Denote an arbitrary convergent subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well. Assume

$$\lambda_i \to \lambda_*.$$

Obviously, $a \leqslant \lambda_* \leqslant b$. Let $\tilde{K}_i = \lambda_i^{-1}K_i$. Since $\lambda_i^{-1} \to \lambda_*^{-1}$, we get

$$\tilde{K}_i \to \lambda_*^{-1}K.$$

Lemma 6 and the fact $h_{\bar{Z}_\phi K_i}(u_0) = \lambda_i$ show that $h_{\bar{Z}_\phi \tilde{K}_i}(u_0) = 1$; that is,

$$\frac{1}{|\tilde{K}_i|^2}\int_{\tilde{K}_i}\int_{\tilde{K}_i} \phi\Big(u_0 \cdot (x-y)\Big)dxdy = 1,$$

for all i. Since $\tilde{K}_i \to \lambda_*^{-1}K$, the above formula implies

$$\frac{1}{|\lambda_*^{-1}K|^2}\int_{\lambda_*^{-1}K}\int_{\lambda_*^{-1}K} \phi\Big(u_0 \cdot (x-y)\Big)dxdy = 1.$$

Thus we have

$$h_{\bar{Z}_\phi \lambda_*^{-1} K}(u_0) = 1.$$

By Lemma 6, we get

$$h_{\bar{Z}_\phi K}(u_0) = \lambda_*.$$

This shows that $h_{\bar{Z}_\phi K_i}(u_0) \to h_{\bar{Z}_\phi K}(u_0)$ as desired.

Since the support functions $h_{\bar{Z}_\phi K_i} \to h_{\bar{Z}_\phi K}$ pointwise on S^{n-1}, they converge uniformly and hence

$$\bar{Z}_\phi K_i \to \bar{Z}_\phi K. \quad \square$$

As described in Section 2, we know that if $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v, then $K_t|v^\perp$ is independent of t. The next lemma shows that $(\bar{Z}_\phi K_t)|v^\perp$ has the same property.

LEMMA 9. *If $\phi \in \mathscr{C}$, and $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v, then the orthogonal projection of $\bar{Z}_\phi K_t$ onto $v^\perp$ is independent of t.*

Proof. For any $u \in \mathbb{R}^n$, we have

$$\begin{aligned}
&h_{\bar{Z}_\phi K_t}(u) \\
&= \inf\left\{\lambda > 0 : \frac{1}{|K_t|^2}\int_{K_t}\int_{K_t}\phi\left(\frac{u\cdot(x-y)}{\lambda}\right)dxdy \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{|K_0|^2}\int_{K_0}\int_{K_0}\phi\left(\frac{u\cdot(z_1+\beta(z_1|v^\perp)tv-z_2-\beta(z_2|v^\perp)tv)}{\lambda}\right)dz_1dz_2 \leqslant 1\right\} \\
&= \inf\left\{\lambda > 0 : \frac{1}{|K_0|^2}\int_{K_0}\int_{K_0}\phi\left(\frac{u\cdot(z_1-z_2)+(\beta(z_1|v^\perp)-\beta(z_2|v^\perp))tu\cdot v}{\lambda}\right)dz_1dz_2 \leqslant 1\right\}.
\end{aligned}$$

Then $h_{\bar{Z}_\phi K_t}(u) = h_{\bar{Z}_\phi K}(u)$ for all $u \in v^\perp$. $\square$

In order to obtain Theorem 1, we also need the following theorems which will be proved in the next section.

THEOREM 3. *If $\phi \in \mathscr{C}$, and $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v with speed function β, then $\bar{Z}_\phi K_t$ is a shadow system along the same direction v.*

THEOREM 4. *If $\phi \in \mathscr{C}$, and $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v with speed function β, then the volume of $\bar{Z}_\phi K_t$ is a strictly convex function of t unless β is affine.*

4. The proofs of main theorems

Proof of Theorem 3. By Lemma 9, the orthogonal projection of $\bar{Z}_\phi K_t$ onto $v^\perp$ is independent of t. Next, we only need to verify that $\bar{Z}_\phi K_t$ satisfies conditions (i) and (ii) of Lemma 1.

Since $\bar{Z}_\phi K_t$ is an origin symmetric convex body for every $t \in [0,1]$, it can be written as

$$\bar{Z}_\phi K_t = \{x + yv : x \in \bar{Z}_\phi K_0 | v^\perp, -g_t(-x) \leqslant y \leqslant g_t(x)\},$$

where $g_t(\cdot)$ is a suitable concave function defined on $\bar{Z}_\phi K_0 | v^\perp$.

Since $z \in \bar{Z}_\phi K_t$ if and only if $z \cdot u \leqslant h_{\bar{Z}_\phi K_t}(u)$ for every $u \in \mathbb{R}^n$, we have

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : (x + \lambda v) \cdot u \leqslant h_{\bar{Z}_\phi K_t}(u), \ \forall\, u \in \mathbb{R}^n\} \\ &= \sup\{\lambda \in \mathbb{R} : \lambda v \cdot u \leqslant h_{\bar{Z}_\phi K_t}(u) - x \cdot u, \ \forall\, u \in \mathbb{R}^n\}. \end{aligned}$$

Since the inner product and support function are both homogeneous of degree 1, we only need to consider the vectors v such that $|u \cdot v| = 1$. Due to the fact that vectors u with a non-positive inner product with v provide no bounds for λ, we get

$$\begin{aligned} g_t(x) &= \sup\{\lambda \in \mathbb{R} : \lambda \leqslant h_{\bar{Z}_\phi K_t}(\omega + v) - x \cdot (\omega + v), \ \forall\, \omega \in v^\perp\} \\ &= \inf_{\omega \in v^\perp} \{h_{\bar{Z}_\phi K_t}(\omega + v) - x \cdot \omega\}. \end{aligned} \tag{8}$$

Then, the convexity of g_t with respect to t is stated as follows.

We first show that if $u_1, u_2 \in v^\perp, t_1, t_2 \in [0,1]$, then

$$h_{\bar{Z}_\phi K_{\frac{t_1+t_2}{2}}}(u_1 + u_2 + 2v) \leqslant h_{\bar{Z}_\phi K_{t_1}}(u_1 + v) + h_{\bar{Z}_\phi K_{t_2}}(u_2 + v). \tag{9}$$

In fact, let

$$h_{\bar{Z}_\phi K_{t_1}}(u_1 + v) = \lambda_1, h_{\bar{Z}_\phi K_{t_2}}(u_2 + v) = \lambda_2. \tag{10}$$

The convexity of ϕ implies that

$$\begin{aligned} &\phi\left(\frac{(u_1+u_2+2v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))\frac{t_1+t_2}{2}(u_1+u_2+2v)\cdot v}{\lambda_1+\lambda_2}\right) \\ &= \phi\left(\frac{(u_1+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_1 + (u_2+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_2}{\lambda_1+\lambda_2}\right) \\ &= \phi\left(\frac{(u_1+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_1(u_1+v)\cdot v + (u_2+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_2(u_2+v)\cdot v}{\lambda_1+\lambda_2}\right) \\ &\leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\phi\left(\frac{(u_1+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_1(u_1+v)\cdot v}{\lambda_1}\right) \\ &\qquad + \frac{\lambda_2}{\lambda_1+\lambda_2}\phi\left(\frac{(u_2+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_2(u_2+v)\cdot v}{\lambda_2}\right). \end{aligned}$$

Integrating both sides and using (10), we get (9).

It follows from (8) and (9) that

$$\begin{aligned}
&2g_{\frac{t_1+t_2}{2}}(x)\\
&= \inf_{\omega\in v^{\perp}}\{h_{\bar{Z}_\phi K_{\frac{t_1+t_2}{2}}}(2(\omega+v))-x\cdot 2\omega\}\\
&= \inf_{u_1,u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{\frac{t_1+t_2}{2}}}(u_1+u_2+2v)-x\cdot(u_1+u_2)\}\\
&\leqslant \inf_{u_1,u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_1}}(u_1+v)+h_{\bar{Z}_\phi K_{t_2}}(u_2+v)-x\cdot(u_1+u_2)\}\\
&= \inf_{u_1\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_1}}(u_1+v)-x\cdot u_1\}+\inf_{u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_2}}(u_2+v)-x\cdot u_2\}\\
&= g_{t_1}(x)+g_{t_2}(x).
\end{aligned}$$

Thus, the condition (i) of Lemma 1 is verified.

Assume $\theta\in[0,1]$, let

$$h_{\bar{Z}_\phi K_{t_1}}(-\theta u_1-\theta v)=\mu_1,\ h_{\bar{Z}_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2+v)=\mu_2. \tag{11}$$

Then we get

$$\begin{aligned}
&\phi\left(\frac{(u_2-\theta u_1+(1-\theta)v)\cdot(z_1-z_2)+(\beta(z_1|v^{\perp})-\beta(z_2|v^{\perp}))t_2(u_2-\theta u_1+(1-\theta)v)\cdot v}{\mu_1+\mu_2}\right)\\
&=\phi\left(\frac{(u_2+v)\cdot(z_1-z_2)+(-\theta u_1-\theta v)\cdot(z_1-z_2)-(\beta(z_1|v^{\perp})-\beta(z_2|v^{\perp}))((1-\theta)t_2+\theta t_1-\theta t_1)}{\mu_1+\mu_2}\right)\\
&\leqslant\frac{\mu_2}{\mu_1+\mu_2}\phi\left(\frac{(u_2+v)\cdot(z_1-z_2)+(\beta(z_1|v^{\perp})-\beta(z_2|v^{\perp}))((1-\theta)t_2+\theta t_1)(u_2+v)\cdot v}{\mu_2}\right)\\
&\qquad+\frac{\mu_1}{\mu_1+\mu_2}\phi\left(\frac{(-\theta u_1-\theta v)\cdot(z_1-z_2)+(\beta(z_1|v^{\perp})-\beta(z_2|v^{\perp}))t_1(-\theta u_1-\theta v)\cdot v}{\mu_1}\right).
\end{aligned}$$

Integrating both sides and using (11), we get

$$h_{\bar{Z}_\phi K_{t_2}}(u_2-\theta u_1+(1-\theta)v)\leqslant h_{\bar{Z}_\phi K_{t_1}}(-\theta u_1-\theta v)+h_{\bar{Z}_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2+v).$$

Thus, we have

$$\begin{aligned}
&(1-\theta)g_{t_2}(x)\\
&=\inf_{u\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_2}}((1-\theta)(u+v))-x\cdot u\}\\
&=\inf_{u_1,u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_2}}(u_2-\theta u_1+(1-\theta)v)-x\cdot(u_2-\theta u_1)\}\\
&\leqslant\inf_{u_1,u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_1}}(-\theta u_1-\theta v)+h_{\bar{Z}_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2+v)-x\cdot(u_2-\theta u_1)\}\\
&=\inf_{u_1\in v^{\perp}}\{h_{\bar{Z}_\phi K_{t_1}}(-\theta u_1-\theta v)-x\cdot(-\theta u_1)\}+\inf_{u_2\in v^{\perp}}\{h_{\bar{Z}_\phi K_{\theta t_1+(1-\theta)t_2}}(u_2+v)-x\cdot u_2\}\\
&=\theta g_{t_1}(-x)+g_{(1-\theta)t_2+\theta t_1}(x).
\end{aligned}$$

This is the first inequality of (ii). Interchanging t_1 with t_2, θ with $(1-\theta)$ and x with $-x$, we obtain the second inequality. □

Proof of Theorem 4. By Theorem 3 and Lemma 2, we get that the volume of $\bar{Z}_\phi K_t$ is a convex function of t. Fubini's theorem and the symmetry of $\bar{Z}_\phi K_t$ imply

$$|\bar{Z}_\phi K_t| = \int_{(\bar{Z}_\phi K_0)|v^\perp} [g_t(x)+g_t(-x)]dx = 2\int_{(\bar{Z}_\phi K_0)|v^\perp} g_t(x)dx. \tag{12}$$

The convexity of the volume of $\bar{Z}_\phi K_t$ with respect to t easily follows from that of $g_t(x)$ with respect to t.

Suppose that

$$2|\bar{Z}_\phi K_{\frac{t_1+t_2}{2}}| = |\bar{Z}_\phi K_{t_1}| + |\bar{Z}_\phi K_{t_2}| \tag{13}$$

for some $t_1, t_2 \in [0,1]$. From the convexity of g_t with respect to t, (12) and (13) we obtain

$$2g_{\frac{t_1+t_2}{2}}(x) = g_{t_1}(x) + g_{t_2}(x) \tag{14}$$

for almost every $x \in (\bar{Z}_\phi K_0)|v^\perp$. Take $x \in relint((\bar{Z}_\phi K_0)|v^\perp)$. There exist $u_1, u_2 \in v^\perp$ such that

$$\begin{aligned} & g_{t_1}(x) + g_{t_2}(x) \\ &= h_{\bar{Z}_\phi K_{t_1}}(u_1+v) - x\cdot u_1 + h_{\bar{Z}_\phi K_{t_2}}(u_2+v) - x\cdot u_2 \\ &\geqslant 2h_{\bar{Z}_\phi K_{\frac{t_1+t_2}{2}}}\left(\frac{u_1+u_2}{2}+v\right) - 2x\cdot\frac{u_1+u_2}{2} \\ &\geqslant 2g_{\frac{t_1+t_2}{2}}(x). \end{aligned} \tag{15}$$

The first inequality of (15) follows from (9), and the second follows from (8). The equation (14) guarantees the equality in (15) and (9).

Since ϕ is strictly convex and the equality condition for the Jensen's inequality holds, the equality in (9) implies

$$\begin{aligned} &\frac{(u_1+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_1}{\lambda_1} \\ =& \frac{(u_2+v)\cdot(z_1-z_2) + (\beta(z_1|v^\perp) - \beta(z_2|v^\perp))t_2}{\lambda_2} \end{aligned} \tag{16}$$

for every $z_1, z_2 \in K_0$, due to the continuity of β.

Taking $z_1 = z_1' + \lambda_1' v, z_2 = z_2' + \lambda_2' v$ in (16), where $z_1', z_2' \in K_0|v^\perp$, then by differentiating with respect to the parameter λ_1' or λ_2', we get that $\lambda_1/\lambda_2 = 1$. Hence, β is an affine function. □

Proof of Theorem 1. If $\{K_t : 0 \leqslant t \leqslant 1\}$ is a parallel chord movement along the unit direction v with speed function $\beta(x) = -(f(x)+g(x))$. It follows from Theorem 4 that the volume of $\bar{Z}_\phi K_t$ is a convex function of the parameter t. Thus,

$$|\bar{Z}_\phi K_{1/2}| \leqslant \frac{1}{2}|\bar{Z}_\phi K_0| + \frac{1}{2}|\bar{Z}_\phi K_1|. \tag{17}$$

It is easy to verify that $\bar{Z}_\phi(K^v) = (\bar{Z}_\phi K)^v$ for every unit direction v by Lemma 6. Since $K_0 = K, K_1 = K^v$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^\perp$, the volume of the Orlicz mean zonoid does not increase after a Steiner symmetrization. It is well known that every convex body can be transformed into a ball through a sequence of suitable Steiner symmetrizations (see e.g. [31]). By Lemma 8, the ratio $|\bar{Z}_\phi K|/|K|$ is continuous in the Hausdorff metric. Therefore it attains its minimum value when K is a ball.

Next, we characterize all the minimizers. If the speed function β of the movement is an affine function, then K_t is an affine image of K, for every t in the range of the movement. It is well known that (see e.g. [23]) if K is not an ellipsoid, then there exists a direction v such that the Steiner symmetral of K along the direnction v is not an image of K under an affine transformation. However, Lemma 6 shows that $|\bar{Z}_\phi K|/|K|$ is affine invariant. Thus, by Theorem 4, we know that $|\bar{Z}_\phi K|/|K|$ is minimized if and only if K is an ellipsoid. □

Proof of Theorem 2. By Lemma 6, we see that $|\bar{Z}^*_\phi K||K|$ is affine invariant. Thus, following step by step the proof of Theorem 1 together with Lemma 3, we conclude the proof. □

Acknowledgement. The authors are grateful to the anonymous referees for their careful reading and many valuable suggestions, such that the paper is greatly improved.

REFERENCES

[1] S. CAMPI AND P. GRONCHI, *The L^p-Busemann-Petty centroid inequality*, Adv. Math. **167** (2002), 128–141.

[2] S. CAMPI AND P. GRONCHI, *On the reverse L^p-Busemann-Petty centroid inequality*, Mathematika **49** (2002), 1–11.

[3] S. CAMPI AND P. GRONCHI, *On volume product inequalities for convex sets*, Proc. Amer. Math. Soc. **134** (2006), 2393–2402.

[4] S. CAMPI AND P. GRONCHI, *Volume inequalities for L^p-zonotopes*, Mathematika **53** (2006), 71–80.

[5] R. J. GARDNER AND G. ZHANG, *Affine inequalities and radial mean bodies*, Amer. J. Math. **120** (1998), 505–528.

[6] R. J. GARDNER, *Geometric Tomography, 2nd edition*, Encyclopedia of Mathematics and its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.

[7] R. J. GARDNER, D. HUG AND W. WEIL, *The Orlicz-Brunn-Minkowski theory: A general framework, additions, and inequalities*, J. Differential Geom., accepted.

[8] P. M. GRUBER, *Convex and Discrete Geometry*, Grundlehren Math. Wiss., vol. 336, Springer, Berlin, 2007.

[9] C. HABERL, *L_p intersection bodies*, Adv. Math. **217** (2008), 2599–2624.

[10] C. HABERL AND F. SCHUSTER, *General L_p affine isoperimetric inequalities*, J. Differential Geom. **83** (2009), 1–26.

[11] C. HABERL AND F. SCHUSTER, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal. **257** (2009), 641–658.

[12] C. HABERL, E. LUTWAK, D. YANG AND G. ZHANG, *The even Orlicz Minkowski problem*, Adv. Math. **224** (2010), 2485–2510.

[13] Q. HUANG AND B. HE, *On the Orlicz Minkowski problem for polytopes*, Discrete Comput. Geom. **48** (2012), 281–297.

[14] A. LI AND G. LENG, *A new proof of the Orlicz Busemann-Petty centroid inequality*, Proc. Amer. Math. Soc. **139** (2011), 1473–1481.

[15] M. LUDWIG, *Minkowski valuations*, Trans. Amer. Math. Soc. **357** (2005), 4191–4213.
[16] M. LUDWIG AND M. REITZNER, *A classification of $SL(n)$ invariant valuations*, Ann. Math. **172** (2010), 1219–1267.
[17] E. LUTWAK, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.
[18] E. LUTWAK, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.
[19] E. LUTWAK, D. YANG AND G. ZHANG, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.
[20] E. LUTWAK, D. YANG AND G. ZHANG, *L_p John ellipsoids*, Proc. London Math. Soc. **90** (2005), 497–520.
[21] E. LUTWAK, D. YANG AND G. ZHANG, *Orlicz projection bodies*, Adv. Math. **223** (2010), 220–242.
[22] E. LUTWAK, D. YANG AND G. ZHANG, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365–387.
[23] C. M. PETTY, *Ellipsoids, Convexity and its Applications* (P. M. Gruber and J. M. Wills, eds.), Birkhäuser, Basel, 1983, pp. 264–276.
[24] R. E. PFIEFER, *The Extrema of Geometric Mean Values*, PhD dissertation, Department of Mathematics, University of California, Davis, CA, 1982.
[25] R. E. PFIEFER, *Maximum and minimum sets for some geometric mean values*, J. Theoret. Probab. **3** (1990), 169–179.
[26] C. A. ROGERS AND G. C. SHEPHARD, *Some extremal problems for convex bodies*, Mathematika **5** (1958), 93–102.
[27] R. SCHNEIDER, *Random hyperplanes meeting a convex body*, Z. Wahr. Verw. Gebiete **61** (1982), 379–387.
[28] R. SCHNEIDER AND W. WEIL, *Zonoids and related topics, Convexity and its Applications*, Birkhäuser, Basel, 1983, pp. 296–317.
[29] R. SCHNEIDER, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge Univ. Press, Cambridge, 1993.
[30] G. C. SHEPHARD, *Shadow systems of convex bodies*, Israel J. Math. **2** (1964), 229–236.
[31] J. STEINER, *Einfacher Beweis der isoperimetrischen Hauptsätze*, J. Reine Angew Math. **18** (1838), 281–296, and Gesammelte Werke Vol. 2, 77–91, Reimer, Berlin, 1882.
[32] G. WANG, G. LENG AND Q. HUANG, *Volume inequalities for Orlicz zonotopes*, J. Math. Anal. Appl. **391** (2012), 183–189.
[33] M. WEBERNDORFER, *Shadow systems of asymmetric L_p zonotopes*, Adv. Math. **240** (2013), 613–635.
[34] G. ZHANG, *Restricted chord projection and affine inequalities*, Geom. Dedicata **39** (1991), 213–222.

(Received April 15, 2013)

Changmin Du
Department of Mathematics
Shanghai University
Shanghai, 200444, China
e-mail: adu860110@163.com

Gangsong Leng
Department of Mathematics
Shanghai University
Shanghai, 200444, China
e-mail: gleng@staff.shu.edu.cn

Dongmeng Xi
Department of Mathematics
Shanghai University
Shanghai, 200444, China
e-mail: dongmeng.xi@live.com